Functions of Bidual numbers

- Functions of bidual numbers can be developed using the Taylor series expansion as shown below.

$$f(x^*) \approx f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x^* - x_0)^k$$

where for bidual numbers $n = 2$

$$f(x^*) \approx f(x_0) + f^{(k)}(x_0)(x^* - x_0) + \frac{1}{2} f^{(2)}(x_0)(x^* - x_0)^2$$

Functions of Bidual numbers

- Notice that the function f is only applied to the real coefficient x_0.

$$f(x^*) \approx f(x_0) + f^{(k)}(x_0)(x^* - x_0) + \frac{1}{2} f^{(2)}(x_0)(x^* - x_0)^2$$

$$(x^* - x_0) = bidual(0, x_1, x_2, x_{12})$$

and

$$(x^* - x_0)^2 = bidual(0, x_1, x_2, x_{12})^2 = bidual(0,0,0,2x_1 x_2)$$

Apply the specific form of the Taylor series to each function.

Exponential of Bidual numbers

- The Taylor series for the exponential function can be written as

$$e^{x^*} \approx e^{x_0}[1 + \sum_{k=1}^{2} \frac{1}{k!}(x^* - x_0)^k] =$$

$$e^{x_0}\left[1 + (x^* - x_0) + \frac{1}{2}(x^* - x_0)^2\right] =$$

$$e^{x_0}\left[1 + bidual(0, x_1, x_2, x_{12}) + \frac{1}{2} bidual(0,0,0,2x_1x_2)\right] =$$

$$e^{x_0} + x_1 e^{x_0}\epsilon_1 + x_2 e^{x_0}\epsilon_2 + (x_1x_2 + x)e^{x_0}\epsilon_{12}$$

Note, if $x_1 = x_2 = 1\ and\ x_{12} = 0$

$$Exp(a_0, 1,1,0) = e^{x_0} + e^{x_0}\epsilon_1 + e^{x_0}\epsilon_2 + e^{x_0}\epsilon_{12}$$

Log of Bidual numbers

- The Taylor series for the natural log function can be written as

$$\log(x^*) \approx \log(x_0) + \sum_{k=1}^{2} \frac{(-1)^{k-1}}{k x_0^k}(x^* - x_0)^k] =$$

$$\log(x_0) + \frac{x - x_0}{x_0} - \frac{(x - x_0)^2}{2x_0^2} =$$

$$\log(x_0) + \frac{bidual(0, x_1, x_2, x_{12})}{x_0} - \frac{bidual(0,0,0,2x_1 x_2)}{2x_0^2}$$

$$log(x_0) + \frac{x_1}{x_0}\epsilon_1 + \frac{x_2}{x_0}\epsilon_2 + \frac{(x_0 x_{12} - x_1 x_2)}{x_0^2}\epsilon_{12}$$

Note, if $x_1 = x_2 = 1 \; and \; x_{12} = 0$

$$\log(x_0, 1,1,0) = \log(x_0) + \frac{1}{x_0}\epsilon_1 + \frac{1}{x_0}\epsilon_2 - \frac{1}{x_0^2}\epsilon_2]$$

Sine of Bidual numbers

- The Taylor series for the sine function can be written as

$$\sin(x^*) \approx \sin(x_0) + \cos(x_0)(x^* - x_0) - \frac{1}{2}\sin(x_0)(x^* - x_0)^2 =$$

$$\sin(x_0) + \cos(x_0)\, bidual(0, x_1, x_2, x_{12}) - \frac{1}{2}\sin(x_0)\, bidual(0,0,0,2x_1x_2) =$$

$$sin(x_0) + a_1 cos(x_0)\epsilon_1 + a_2\, cos(x_0)\epsilon_2 + (a_{12}\, cos(x_0) - a_1 a_2\, sin(x_0))\epsilon_{12}$$

Note, if $x_1 = x_2 = 1\ and\ x_{12} = 0$

$$\sin(x_0, 1,1,0) = sin(x_0) + cos(x_0)\epsilon_1 + cos(x_0)\epsilon_2 - sin(x_0)\epsilon_{12}$$

Cosine of Bidual numbers

- The Taylor series for the cosine function can be written as

$$\cos(x^*) \approx \cos(x_0) - \sin(x_0)(x^* - x_0) - \frac{1}{2}\cos(x_0)(x^* - x_0)^2 =$$

$$\cos(x_0) - \sin(x_0)\, bidual(0, x_1, x_2, x_{12}) - \frac{1}{2}\cos(x_0)\, bidual(0,0,0,2x_1x_2) =$$

$$cos(x_0) - x_1\, sin(x_0)\epsilon_1 - x_2\, sin(x_0)\epsilon_1 + (-x_{12}\, sin(x_0) - x_1\, x_2\, cos(x_0))\epsilon_{12}$$

Note, if $x_1 = x_2 = 1\ and\ x_{12} = 0$

$$\cos(x_0, 1,1,0) = cos(x_0) - sin(x_0)\epsilon_1 - sin(x_0)\epsilon_1 - cos(x_0)\epsilon_{12}$$

Bidual raised to a bidual number

- A bidual number raised to a bidual number can be evaluated in terms of the exponential and logarithmic functions

Use the formula $x^y = e^{y ln(x)}$

This formula can be implemented with bidual numbers using the previously defined exponential and logarithmic functions.

$$a^{*\mathrm{b}^*} = e^{b^* ln(a^*)}$$

Bidual number a^* raised to a real number n

$$(a^*)^n = a_0^n + n a_0^{n-1} a_1 \epsilon_1 + n a_0^{n-1} a_1 \epsilon_2 + n a_0^{n-2}\left((n-1) a_1 a_2 + a_0 a_{12}\right) \epsilon_{12}$$

Real number r raised to a bidual number a^*

$$r^a = r^{a_0} + r^{a_0} \log(r) a_1 \epsilon_1 + r^{a_0} \log(r) a_2 \epsilon_2 + r^{a_0} \log(r)(\log(r)\, a_1 a_2 + a_{12})\ \epsilon_{12}$$

Derived functions of Bidual numbers

- Many functions can be derived in terms previously defined functions

$$\tan(a^*) = \frac{\sin(a^*)}{\cos(a^*)} =$$
$$\tan(a_0) + a_1 \sec(a_0)^2\, \epsilon_1 + a_2 \sec(a_0)^2\, \epsilon_2 + \sec(a_0)^2\,(a_{12} + 2a_1 a_2 \tan(a_0))\, \epsilon_{12}$$

Note: $\tan(bidual(x_0, 1,1,0)) = \tan(x_0) + \sec(x_0)^2\, \epsilon_1 + \sec(x_0)^2\, \epsilon_2 + 2\sec(x_0)^2 \tan(x_0)\, \epsilon_{12}$

$$\text{sqrt}(x^*) = (x^*)^{1/2} =$$
$$\sqrt{x_0} + \frac{x_1}{2\sqrt{x_0}}\epsilon_1 + \frac{x_2}{2\sqrt{x_0}}\epsilon_1 + \frac{-x_1 x_2 + 2x_{12}x_0}{4x_0^{3/2}}\epsilon_{12}$$

Note: $sqrt(bidual(x_0, 1,1,0)) = \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}\epsilon_1 + \frac{1}{2\sqrt{x_0}}\epsilon_1 + \frac{-1}{4x_0^{3/2}}\epsilon_1$