Automatic Differentiation Using Complex and Hypercomplex Variables

Applying CTSE to closed-form examples

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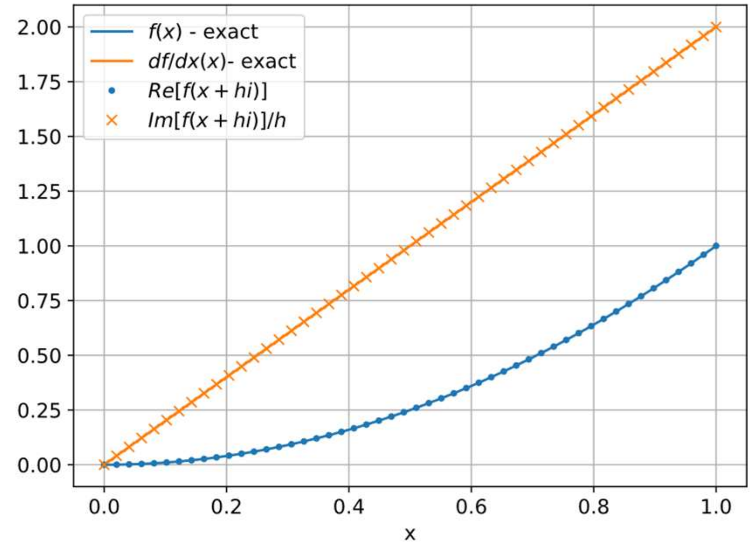
Closed-form example: $f(x) = x^2$

- The use of CTSE to compute derivatives can be demonstrated using some closed-form examples.

$$f(x) = x^2$$
$$f(x + ih) = (x + ih)^2 = x^2 - h^2 + 2ihx$$

$$\frac{df}{dx} = \frac{Im(f(x + ih))}{h} = \frac{2hx}{h} = 2x$$

In this example we see that CTSE is exact.



$h = 10^{-10}$

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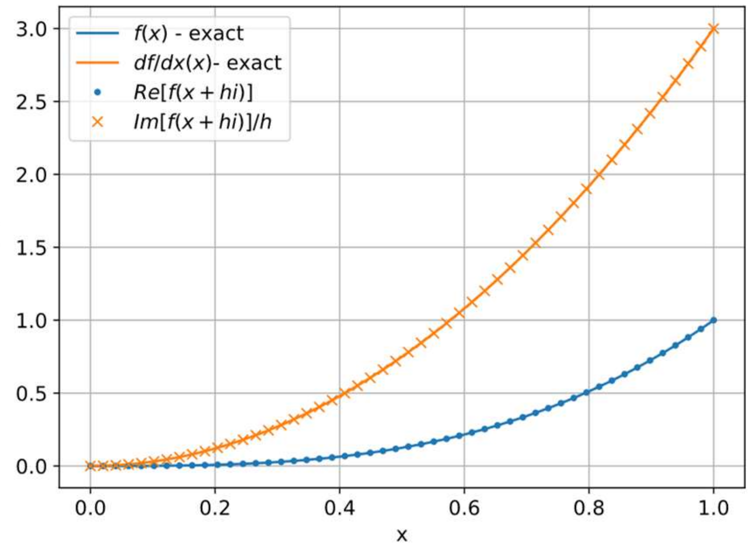
Closed-form example: $f(x) = x^3$

$$f(x) = x^3$$

$$f(x + ih) = (x + ih)^3 = (x^3 - 3h^2 x) + i(3hx^2 - h^3)$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{Im(f(x + ih))}{h} = \lim_{h \to 0} \frac{3hx^2 - h^3}{h} = \lim_{h \to 0} 3x^2 - h^2 = 3x^2$$

CTSE is exact in the limit as $h \to 0$.



$h = 10^{-10}$

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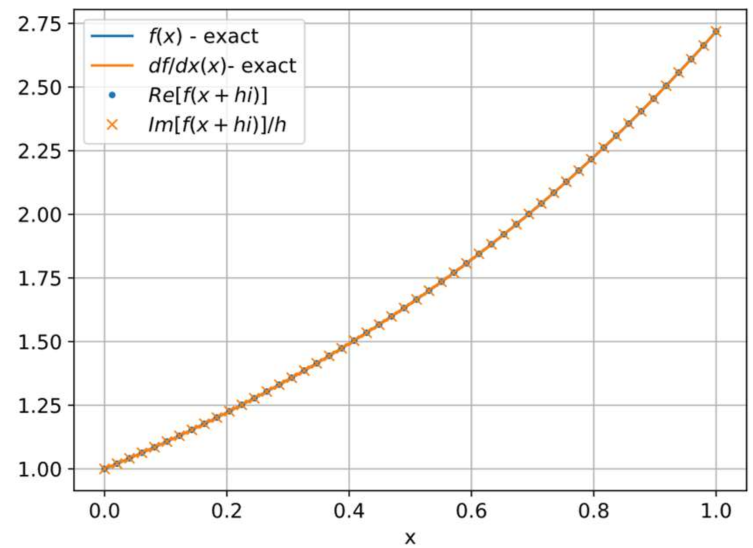
Closed-form example: $f(x) = e^x$

$$f(x) = e^x$$

$$f(x + ih) = e^{x+ih} = e^x e^{ih} = e^x\big(\cos(h) + i sin(h)\big)$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{Im(f(x + ih))}{h} = \lim_{h \to 0} e^x \frac{\sin(h)}{h} = e^x$$

CTSE is exact in the limit as $h \to 0$ since $\lim_{h \to 0} \frac{\sin(h)}{h} = 1.$



$h = 10^{-10}$

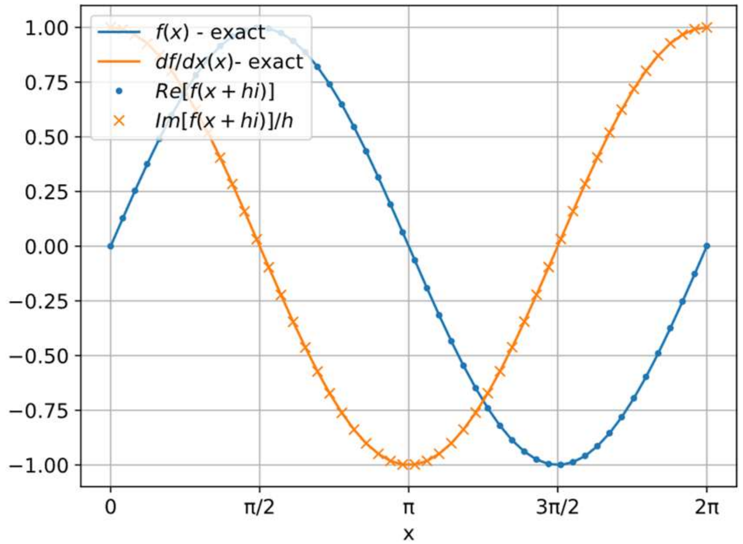
Closed-form example $f(x) = \sin(x)$

$$f(x) = \sin(x)$$

$$f(x + ih) = \sin(x + ih) = \sin(x)\cos(ih) + \cos(x)\sin(ih) =$$
$$= \sin(x)\cosh(h) + i cos(x)\sinh(h)$$

$$\frac{df}{dx} = \lim_{h\to 0}\frac{Im\big(f(x + ih)\big)}{h} = \lim_{h\to 0}\cos(x)\frac{\sinh(h)}{h} = \cos(x)$$

CTSE is exact in the limit as $h \to 0$ since $\lim_{h\to 0}\frac{\sinh(h)}{h} = 1.$



$h = 10^{-10}$

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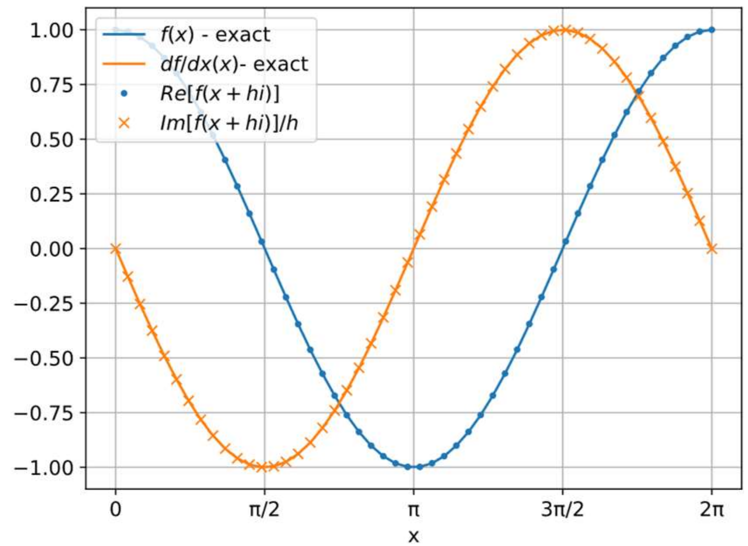
Closed-form example: $f(x) = \cos(x)$

$$f(x) = \cos(x)$$
$$f(x + ih) = \cos(x + ih) = \cos(x)\cos(ih) - \sin(x)\sin(ih) =$$
$$= \cos(x)\cosh(h) - i\sin(x)\sinh(h)$$

$$\frac{df}{dx} = \lim_{h\to 0}\frac{Im\big(f(x+ih)\big)}{h} = \lim_{h\to 0} -\sin(x)\frac{\sinh(h)}{h} = -\sin(x)$$

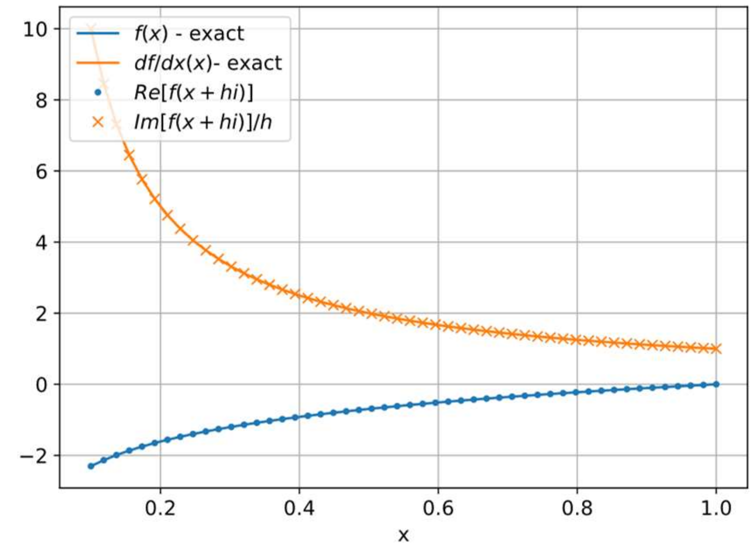
CTSE is exact in the limit as $h \to 0$ since $\lim_{h\to 0}\frac{\sinh(h)}{h} = 1$.



$h = 10^{-10}$

Closed-form example: $f(x) = \ln(x)$

$$f(x) = \ln(x)$$
$$f(x + ih) = \ln(x + ih)$$
Using the fact that $x + ih = (x^2 + h^2)e^{i\theta}$:
$$\ln(x + ih) = \ln(x^2 + h^2) + i\theta$$
$$\Rightarrow$$
$$\frac{Im(f(x+ih))}{h} = \frac{\theta}{h}$$
$$\theta = \arctan\left(\frac{x}{h}\right)$$
$$\frac{df}{dx} = \lim_{h\to 0} \frac{\arctan\left(\frac{h}{x}\right)}{h} = \frac{1}{x}$$



$h = 10^{-10}$

CTSE is exact in the limit as $h \to 0$ since $\lim_{h\to 0} \frac{\arctan\left(\frac{h}{x}\right)}{h} = \frac{1}{x}$.

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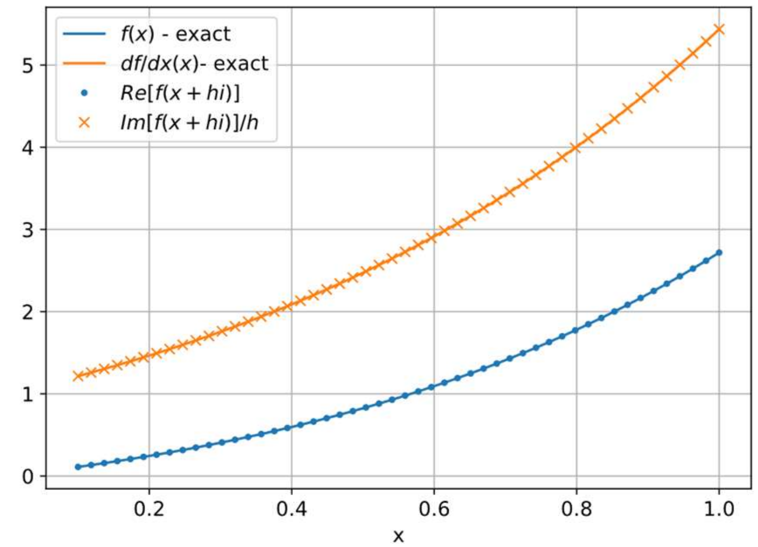
Closed-form example: $f(x) = xe^x$

$$f(x) = xe^x$$
$$f(x+ih) = (x+ih)e^{(x+ih)} = (x+ih)e^x e^{ih} =$$
$$(x+ih)e^x(\cos(h) + i\sin(h)) =$$
$$e^x(x\cos(x) - h\sin(h)) + e^x(x\sin(h) + h\cos(h))i$$

$$\frac{df}{dx} = \lim_{h\to 0}\frac{Im\big(f(x+ih)\big)}{h} = \lim_{h\to 0}\frac{e^x(x\sin(h) + h\cos(h))}{h}$$
$$= \lim_{h\to 0} e^x(x\sin(h)/h + \cos(h)) = e^x(x+1)$$



$h = 10^{-1}$

CTSE is exact in the limit as $h \to 0$ since $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$ and the $\lim_{h\to 0}\cos(h) = 1$.

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CTSE using Excel

- CTSE can be employed with Excel - although somewhat cumbersome and only with single precision accuracy. In all cases, complex functions must be used.

Function	Result
Complex(a,b) or “a+bi”	$a + bi$
Imreal(complex(a,b))	a
Imaginary(complex(a,b))	b
Imsum(c1,c2,…)	Sum of 2 or more complex numbers $(c_1 + c_2 + \cdots)$
Imsub(c1,c2)	Subtraction of 2 complex numbers $(c_1 - c_2)$
Improduct(c1,c2,…)	Multiplication of 2 or more complex numbers $(c_1 c_2 \ldots)$
Imdiv(c1,c2)	Complex division of 2 complex numbers, (c_1/c_2)
Imsin(c1)	Sine of a complex number $(\sin(c_1))$
Imcos(c1)	Cosine of a complex number $(\cos(c_1))$
Imexp(c1)	Exponential of a complex number $(\exp(c_1))$
Imln(c1)	Natural log of a complex number $(\ln(c_1))$

CTSE examples using Excel

- Several examples are shown below. In all cases $x = 1.5 + ih$ with $h = 10^{-10}$, that is $x^* = 1.5 + 10^{-1}\ i$ and x^* is located in cell A1.

Simulated Spreadsheet

Function	Excel command	dfdx-CTSE	dfdx-exact
X^2	= Improduct(A1,A1)	3.0000	3.0000
X^3	= Improduct(A1,A1,A1)	6.7500	6.7500
Sin(x)	= Imsin(A1)	0.0707	0.0707
Cos(x)	= Imcos(A1)	-0.9975	-0.9975
Exp(x)	= Imexp(A1)	4.4817	4.4817
In(x)	= Imln(A1)	0.6667	0.6667
x exp(x)	= Improduct(A1,Imexp(a1))	11.2042	11.2042
Sin(x^2)	= Imsin(Improduct(A1,A1))	-1.8845	-1.8845

Microsoft Excel Worksheet

Click here to open the Excel example.