Automatic Differentiation Using Complex and Hypercomplex Variables

Using CTSE with special functions

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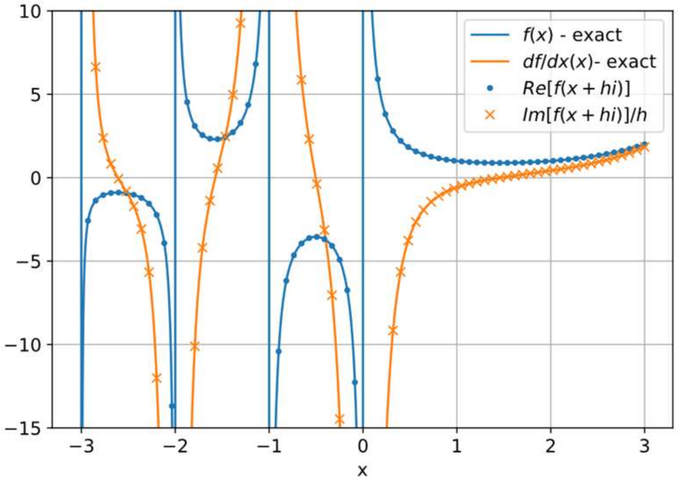
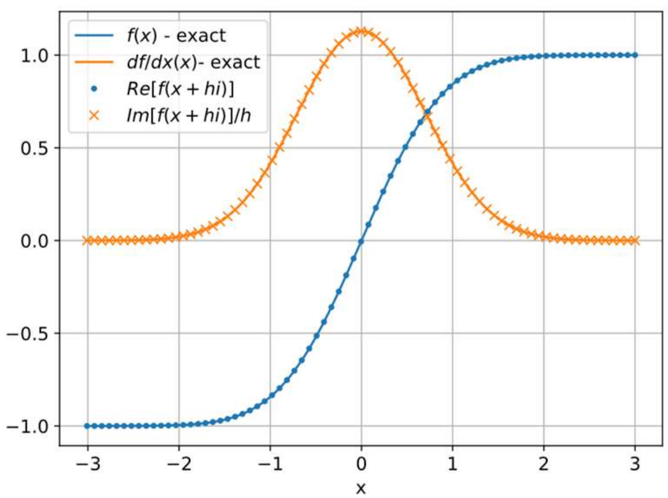
July, 2023

Numerical evaluation of special functions

- CTSE can be easily used to numerically compute the derivative of special functions such as the Error and complementary Error functions (Erf, Erfc), Gamma function, Beta function, inverse trigonometric functions, elliptic integrals, and many others as long as they have complex variable versions.



Re and Im plots of $\Gamma(x + ih)$ with $h = 10^{-10}$



Re and Im plots of $Erf(x + ih)$ with $h = 10^{-10}$

Numerical evaluation of special functions

- For functions that do not have a complex-variable version operator overloading can be used to enable a complex version such that CTSE can be used. Note, the complex version would only be useful for CTSE, not complex analysis in general. The only requirement is to locate the derivative as the imaginary function,

$$f(x_0 + ih) = f(x_0) + \frac{df(x_0)}{dx} ih$$

- Example, consider the simple case of the cosine function. If the complex variable version did not exist, then a complex variable could be defined as

$$\cos(x_0 + ih) = \cos(x_0) - \sin(x_0)\, hi$$

- Then

$$\frac{1}{h} Im(\cos(x^*)) = -\sin(x_0)$$

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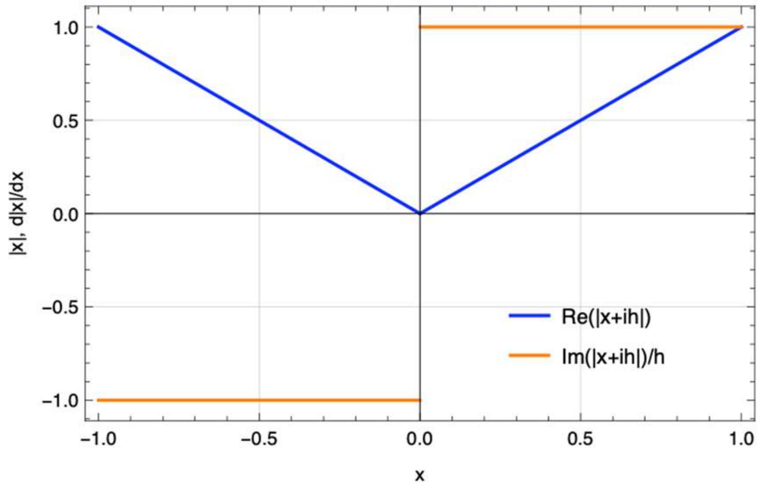
Numerical evaluation of special functions

- Example, consider the complex operator $Abs(x)$. CTSE does not work on this operator since the Abs operator applied to a complex number yields a real result, $Abs(a+bi) = \sqrt{a^2+b^2}$. Therefore, we can define the overloaded Abs for a complex argument of $x^* = x_0 + ih$ as

$$Abs(x_0 + ih) = \sqrt{x_0^2 + \frac{x_0}{\sqrt{x_0^2}} hi}$$



Re and Im plots of $\mathrm{Abs}(x+ih)$ with $h = 10^{-10}$

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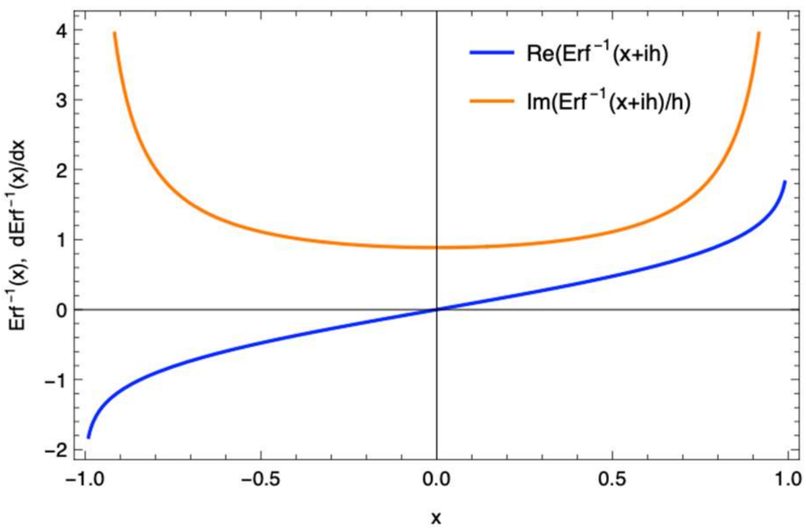
Numerical evaluation of special functions

- As a second example, consider the inverse error function, $InverseErf(x)$. CTSE does not work on this operator since there isn't a complex variable version of this function.

$$InverseErf(x_0 + ih) = InverseErf(x_0) + \frac{\sqrt{\pi}}{2} e^{InverseErf(x_0)^2} hi$$



Re and Im plots of $\mathrm{InverseErf}(x_0 + ih)$ with $h = 10^{-10}$