Cauchy-Reimann form of biduals

- Bidual numbers have an equivalent CR form consisting of a matrix of all real coefficients.
- In this form there are no imaginary numbers and matrix operations can be used to manipulate the bidual numbers.

$$x_0 + x_1\epsilon_1 + x_2\epsilon_2 + x_{12}\epsilon_{12} = \begin{pmatrix} x_0 & 0 & 0 & 0 \\ x_1 & x_0 & 0 & 0 \\ x_2 & 0 & x_0 & 0 \\ x_{12} & x_2 & x_1 & x_0 \end{pmatrix}$$

Operations using CR form - Multiplication

- Consider the multiplication of two bidual numbers using the CR form

$$x^* = x_0 + x_1\epsilon_1 + x_2\epsilon_2 + x_{12}\epsilon_{12} \qquad y^* = y_0 + y_1\epsilon_1 + y_2\epsilon_2 + y_{12}\epsilon_{12}$$

$$x^* * y^* = \begin{pmatrix} x_0 & 0 & 0 & 0 \\ x_1 & x_0 & 0 & 0 \\ x_2 & 0 & x_0 & 0 \\ x_{12} & x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 & 0 & 0 & 0 \\ y_1 & y_0 & 0 & 0 \\ y_2 & 0 & y_0 & 0 \\ y_{12} & x_2 & y_1 & y_0 \end{pmatrix}$$

$$= \begin{pmatrix} x_0 y_0 & 0 & 0 & 0 \\ x_1 y_0 + a_0 y_1 & x_0 y_0 & 0 & 0 \\ x_2 y_0 + a_0 y_2 & 0 & x_0 y_0 & 0 \\ x_0 y_{12} + x_1 y_2 + x_2 y_1 + x_{12} y_0 & x_1 y_0 + x_0 y_1 & x_2 y_0 + x_0 y_2 & x_0 y_0 \end{pmatrix}$$

Operations using CR form - Division

- Consider a bidual number divided by a bidual number using the CR form

$$x^* = x_0 + x_1\epsilon_1 + x_2\epsilon_2 + x_{12}\epsilon_{12} \qquad y^* = y_0 + y_1\epsilon_1 + y_2\epsilon_2 + y_{12}\epsilon_{12}$$

$$\frac{x^*}{y^*} = \begin{pmatrix} x_0 & 0 & 0 & 0 \\ x_1 & x_0 & 0 & 0 \\ x_2 & 0 & x_0 & 0 \\ x_{12} & x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 & 0 & 0 & 0 \\ y_1 & y_0 & 0 & 0 \\ y_2 & 0 & y_0 & 0 \\ y_{12} & x_2 & y_1 & y_0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} x_0/y_0 & 0 & 0 & 0 \\ x_1/y_0 - x_0 y_1/y_0^2 & x_0/y_0 & 0 & 0 \\ x_2/y_0 - x_0 y_2/y_0^2 & 0 & x_0/y_0 & 0 \\ x_{12}/y_0 - x_2 y_1/y_0^2 - x_1 y_2/y_0^2 - x_0(2y_0 y_1 y_2 - y_{12} y_0^2)/y_0^4 & x_1/y_0 - x_0 y_1/y_0^2 & x_2/y_0 - x_0 y_2/y_0^2 & x_0/y_0 \end{pmatrix}$$

Functions of Bidual numbers using the CR form

- Functions of bidual numbers can be computed using the CR form if one uses **functions of matrices**.

$$x^* = x_0 + x_1\epsilon_1 + x_2\epsilon_2 + x_{12}\epsilon_{12}$$

$$\exp(x^*) = \exp[\begin{pmatrix} x_0 & 0 & 0 & 0 \\ x_1 & x_0 & 0 & 0 \\ x_2 & 0 & x_0 & 0 \\ x_{12} & x_2 & x_1 & x_0 \end{pmatrix}]$$

$$= \begin{pmatrix} e^{x_0} & 0 & 0 & 0 \\ x_1 e^{x_0} & e^{x_0} & 0 & 0 \\ x_2 e^{x_0} & 0 & e^{x_0} & 0 \\ (x_1 x_2 + x_{12})e^{x_0} & x_1 e^{x_0} & x_2 e^{x_0} & e^{x_0} \end{pmatrix}$$

Computing derivatives using Bidual numbers using the CR form – general univariate example

- To use bidual numbers to compute derivatives with the CR form, perturb x along ϵ_1 and ϵ_2, i.e.,

$$x^* = x_0 + h_1\epsilon_1 + h_2\epsilon_2$$

$$[x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ h_1 & x_0 & 0 & 0 \\ h_2 & 0 & x_0 & 0 \\ 0 & h_2 & h_1 & x_0 \end{bmatrix}$$

$$f([x^*]) = f\left(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ h_1 & x_0 & 0 & 0 \\ h_2 & 0 & x_0 & 0 \\ 0 & h_2 & h_1 & x_0 \end{bmatrix}\right) =$$

$$\begin{bmatrix} f(x_0) & 0 & 0 & 0 \\ h_1\partial f(x_0)/\partial x & f(x_0) & 0 & 0 \\ h_2\partial f(x_0)/\partial x & 0 & f(x_0) & 0 \\ h_1h_2\partial^2 f(x_0)/\partial x^2 & h_2\partial f(x_0)/\partial x & h_1\partial f(x_0)/\partial x & f(x_0) \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11}$$

$$\partial f(x_0)/\partial x = \frac{1}{h_1} f([x^*])_{21} = \frac{1}{h_2} f([x^*])_{31}$$

$$\partial^2 f(x_0)/\partial x^2 = \frac{1}{h_1h_2} f([x^*])_{41}$$

Computing derivatives using Bidual numbers using the CR form – recommended univariate example

- It is recommended that we use $h_1 = h_2 = 1$ and the method simplifies to,

$$x^* = x + \epsilon_1 + \epsilon_2 \qquad [x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}$$

$$f([x^*]) = f\left(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}\right) =$$

$$\begin{bmatrix} f(x_0) & 0 & 0 & 0 \\ \partial f(x_0)/\partial x & f(x_0) & 0 & 0 \\ \partial f(x_0)/\partial x & 0 & f(x_0) & 0 \\ \partial^2 f(x_0)/\partial x^2 & \partial f(x_0)/\partial x & \partial f(x_0)/\partial x & f(x_0) \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11}$$

$$\partial f(x_0)/\partial x = f([x^*])_{21} = f([x^*])_{31}$$

$$\partial^2 f(x_0)/\partial x^2 = f([x^*])_{41}$$

Closed-form example: $f(x) = x^2$

- Compute the derivatives of $f(x) = x^2$ using the CR form for bidual numbers.

$$f(x) = x^{*2}$$

$$f(x^*) = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}^2 = \begin{bmatrix} x_0^2 & 0 & 0 & 0 \\ 2x & x_0^2 & 0 & 0 \\ 2x & 0 & x_0^2 & 0 \\ 2 & 2x & 2x & x_0^2 \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11} = x_0^2$$

$$\partial f(x_0)/\partial x = f([x^*])_{21} = 2x$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = f([x^*])_{41} = 2$$

Closed-form example: $f(x) = x^3$

- Compute the derivatives of $f(x) = x^3$ using the CR form for bidual numbers.

$$f(x) = x^{*3}$$

$$f(x^*) = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}^3 = \begin{bmatrix} x_0^2 & 0 & 0 & 0 \\ 3x^2 & x_0^2 & 0 & 0 \\ 3x^2 & 0 & x_0^2 & 0 \\ 6x & 3x^2 & 3x^2 & x_0^2 \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11} = x_0^2$$

$$\frac{\partial f(x_0)}{\partial x} = f([x^*])_{21} = 3x^2$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = f([x^*])_{41} = 6x$$

Closed-form example: $f(x) = e^x$

- Compute the derivatives of $f(x) = e^x$ using the CR form for bidual numbers.

$$f(x) = e^{x^*}$$

$$f(x^*) = \exp\left(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}\right) = \begin{bmatrix} e^{x_0} & 0 & 0 & 0 \\ e^{x_0} & e^{x_0} & 0 & 0 \\ e^{x_0} & 0 & e^{x_0} & 0 \\ e^{x_0} & e^{x_0} & e^{x_0} & e^{x_0} \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11} = e^{x_0}$$

$$\partial f(x_0)/\partial x = f([x^*])_{21} = e^{x_0}$$

$$\partial^2 f(x_0)/\partial x^2 = e^{x_0}$$

Closed-form example: $f(x) = \sin(x)$

- Compute the derivatives of $f(x) = \cos(x)$using the CR form for bidual numbers.

$$f(x) = \sin(x^*)$$

$$f(x^*) = \sin(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}) = \begin{bmatrix} \sin x_0 & 0 & 0 & 0 \\ \cos x_0 & \sin x_0 & 0 & 0 \\ \cos x_0 & 0 & \sin x_0 & 0 \\ -\sin x_0 & \cos x_0 & \cos x_0 & \sin x_0 \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11} = \sin(x_0)$$

$$\frac{\partial f(x_0)}{\partial x} = f([x^*])_{21} = \cos(x_0)$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = -\sin(x_0)$$

Closed-form example: $f(x) = \cos(x)$

- Compute the derivatives of $f(x) = \cos(x)$ using the CR form for bidual numbers.

$$f(x) = \cos(x^*)$$

$$f(x^*) = \cos\left(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}\right) = \begin{bmatrix} \cos x_0 & 0 & 0 & 0 \\ -\sin x_0 & \cos x_0 & 0 & 0 \\ -\sin x_0 & 0 & \cos x_0 & 0 \\ -\cos x_0 & -\sin x_0 & -\sin x_0 & \cos x_0 \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11} = \cos(x_0)$$

$$\frac{\partial f(x_0)}{\partial x} = f([x^*])_{21} = -\sin(x_0)$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = -\cos(x_0)$$

Closed-form example: $f(x) = \ln(x)$

- Compute the derivatives of $f(x) = \ln(x)$ using the CR form for bidual numbers.

$$f(x) = \ln(x^*)$$

$$f(x^*) = \ln\left(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}\right) = \begin{bmatrix} \ln(x_0) & 0 & 0 & 0 \\ 1/x_0 & \ln(x_0) & 0 & 0 \\ 1/x_0 & 0 & \ln(x_0) & 0 \\ -1/x_0^2 & 1/x_0 & 1/x_0 & \ln(x_0) \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11} = \ln(x_0)$$

$$\frac{\partial f(x_0)}{\partial x} = f([x^*])_{21} = 1/x_0$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = -1/x_0^2$$

Computing derivatives using Bidual numbers using the CR form – general multivariate example

- To use bidual numbers to compute mixed derivatives, perturb x along ϵ_1 and y along ϵ_2, i.e.,

$$x^* = x_0 + h_1\epsilon_1 \qquad [x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ h_1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & h_1 & x_0 \end{bmatrix}$$

$$y^* = y_0 + h_2\epsilon_2 \qquad [y^*] = \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ h_2 & 0 & y_0 & 0 \\ 0 & h_2 & 0 & y_0 \end{bmatrix}$$

$$f([x^*, y^*]) = f\left(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ h_1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & h_1 & x_0 \end{bmatrix}, \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ h_2 & 0 & y_0 & 0 \\ 0 & h_2 & 0 & y_0 \end{bmatrix}\right) =$$

$$\begin{bmatrix} f(x_0) & 0 & 0 & 0 \\ h_1\partial f(x_0)/\partial x & f(x_0) & 0 & 0 \\ h_2\partial f(x_0)/\partial x & 0 & f(x_0) & 0 \\ h_1 h_2 \partial^2 f(x_0)/\partial x \partial y & h_2\partial f(x_0)/\partial x & h_1\partial f(x_0)/\partial x & f(x_0) \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11}$$

$$\frac{\partial f(x_0)}{\partial x} = \frac{1}{h_1} f([x^*],[y^*])_{21}$$

$$\frac{\partial f(x_0)}{\partial y} = \frac{1}{h_2} f([x^*],[y^*])_{31}$$

$$\frac{\partial^2 f(x_0)}{\partial x \partial y} = \frac{1}{h_1 h_2} f([x^*])_{41}$$

Computing derivatives using Bidual numbers using the CR form – recommended multivariate example

- The recommendation is to use $h_1 = h_2 = 1$.

$$x^* = x + h_1\epsilon_1 \qquad [x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix}$$

$$y^* = y + h_2\epsilon_2 \qquad [y^*] = \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}$$

$$f([x^*, y^*]) = f\left(\begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix}, \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}\right) =$$

$$\begin{bmatrix} f(x_0) & 0 & 0 & 0 \\ \partial f(x_0)/\partial x & f(x_0) & 0 & 0 \\ \partial f(x_0)/\partial y & 0 & f(x_0) & 0 \\ \partial^2 f(x_0)/\partial x^2 & \partial f(x_0)/\partial y & \partial f(x_0)/\partial x & f(x_0) \end{bmatrix}$$

$$f(x_0) = f([x^*])_{11}$$

$$\frac{\partial f(x_0)}{\partial x} = f([x^*],[y^*])_{21}$$

$$\frac{\partial f(x_0)}{\partial y} = f([x^*],[y^*])_{31}$$

$$\frac{\partial^2 f(x_0)}{\partial x \partial y} = f([x^*])_{41}$$

Closed-form example: $f(x) = xy$

- Compute the derivatives of $f(x) = xy$ using the CR form for bidual numbers.

$$x^* = x_0 + h_1\epsilon_1 \qquad [x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix} \qquad y^* = y_0 + h_2\epsilon_2 \qquad [y^*] = \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}$$

$$f(x) = x^* y^*$$

$$f(x^*) = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix} \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix} = \begin{bmatrix} x_0^2 y_0^2 & 0 & 0 & 0 \\ 2x_0 y_0^2 & x_0^2 y_0^2 & 0 & 0 \\ 2x_0^2 y_0 & 0 & x_0^2 y_0^2 & 0 \\ 4x_0 y_0 & 2x_0^2 y_0 & 2x_0 y_0^2 & x_0^2 y_0^2 \end{bmatrix}$$

$$f(x_0, y_0) = f([x^* y^*])_{11} = x_0 y_0$$

$$\frac{\partial f(x_0, y_0)}{\partial x} = f([x^*, y^*])_{21} = 2x_0 y_0^2; \quad \frac{\partial f(x_0, y_0)}{\partial y} = f([x^*, y^*])_{31} = 2x_0^2 y_0$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = 4x_0 y_0$$

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Closed-form example: $f(x) = x^2y^2$

- Compute the derivatives of $f(x) = x^2y^2$ using the CR form for bidual numbers.

$$x^* = x_0 + h_1\epsilon_1 \qquad [x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix} \qquad y^* = y_0 + h_2\epsilon_2 \qquad [y^*] = \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}$$

$$f(x) = {x^*}^2{y^*}^2$$

$$f(x^*) = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 1 & 0 & x_0 & 0 \\ 0 & 1 & 1 & x_0 \end{bmatrix}^2 \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}^2 = \begin{bmatrix} x_0^2y_0^2 & 0 & 0 & 0 \\ 2x_0y_0^2 & x_0^2y_0^2 & 0 & 0 \\ 2x_0^2y_0 & 0 & x_0^2y_0^2 & 0 \\ 4x_0y_0 & 2x_0^2y_0 & 2x_0y_0^2 & x_0^2y_0^2 \end{bmatrix}$$

$$f(x_0, y_0) = f([x^*y^*])_{11} = x_0y_0$$

$$\frac{\partial f(x_0, y_0)}{\partial x} = f([x^*, y^*])_{21} = 2x_0y_0^2; \quad \frac{\partial f(x_0, y_0)}{\partial y} = f([x^*, y^*])_{31} = 2x_0^2y_0$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = 4x_0y_0$$

Closed-form example: $f(x) = x \sin y$

- Compute the derivatives of $f(x) = x \sin y$ using the CR form for bidual numbers.

$$x^* = x_0 + h_1\epsilon_1 \qquad [x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix} \qquad y^* = y_0 + h_2\epsilon_2 \qquad [y^*] = \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}$$

$$f(x) = x^* \sin y^*$$

$$f(x^*) = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix} \sin\left(\begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}\right) = \begin{bmatrix} x_0 \sin(y_0) & 0 & 0 & 0 \\ \sin(y_0) & x_0 \sin(y_0) & 0 & 0 \\ x_0 \sin(y_0) & 0 & x_0 \sin(y_0) & 0 \\ \cos(y_0) & x_0 \sin(y_0) & \sin(y_0) & x_0 \sin(y_0) \end{bmatrix}$$

$$f(x_0, y_0) = f([x^* y^*])_{11} = x_0 y_0$$

$$\frac{\partial f(x_0, y_0)}{\partial x} = f([x^*, y^*])_{21} = \sin(y_0)\,; \qquad \frac{\partial f(x_0, y_0)}{\partial y} = f([x^*, y^*])_{31} = x_0 \sin(y_0)$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = \cos(y_0)$$

Closed-form example: $f(x) = xe^y$

- Compute the derivatives of $f(x) = xe^x$ using the CR form for bidual numbers.

$$x^* = x_0 + h_1\epsilon_1 \qquad [x^*] = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix} \qquad y^* = y_0 + h_2\epsilon_2 \qquad [y^*] = \begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}$$

$$f(x) = xe^x$$

$$f(x^*) = \begin{bmatrix} x_0 & 0 & 0 & 0 \\ 1 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 1 & x_0 \end{bmatrix} \exp\left[\begin{bmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \\ 1 & 0 & y_0 & 0 \\ 0 & 1 & 0 & y_0 \end{bmatrix}\right] = \begin{bmatrix} x_0^2y_0^2 & 0 & 0 & 0 \\ 2x_0y_0^2 & x_0^2y_0^2 & 0 & 0 \\ 2x_0^2y_0 & 0 & x_0^2y_0^2 & 0 \\ 4x_0y_0 & 2x_0^2y_0 & 2x_0y_0^2 & x_0^2y_0^2 \end{bmatrix}$$

$$f(x_0, y_0) = f([x^*y^*])_{11} = x_0y_0$$

$$\frac{\partial f(x_0,y_0)}{\partial x} = f([x^*,y^*])_{21} = 2x_0y_0^2; \qquad \frac{\partial f(x_0,y_0)}{\partial y} = f([x^*,y^*])_{31} = 2x_0^2y_0$$

$$\frac{\partial^2 f(x_0)}{\partial x^2} = 4x_0y_0$$

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