Fibonacci Thresholding: Signal Representation and Morphological Filters

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Introduction

- Thresholding is one of the most popular techniques among all existing segmentation methods due to its simplicity, robustness, and accuracy.
- The cross sections, or threshold sets describing 1-D and 2-D real-valued functions, signals and images, play very important role in nonlinear filtering and mathematical morphology.
- In this paper, the traditional horizontal cross sections are considered and a general concept of weighted thresholding is described. Examples of the Fibonacci thresholding are given.

General threshold superposition

Let *f(x)* be a multi-level non-negative semi-continuous function with integer values of the interval [*0*,*L*], *L*>*1*.

Consider a monotonic non-decreasing (or, non-increasing) function *k(α)* ≥*1* in a set of the non-negative integers *α*.

Definition*:* The threshold *k(α)*-valued signals of *f(x)* at amplitude levels *α* ϵ [*0*,*L*] are calculated by

$$\bar{f}_\alpha(x) = k(\alpha) f_\alpha(x) = \begin{cases} k(a), & \text{if } f(x) \geq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

The representation $f(x) \rightarrow (\bar{f}_1(x), \bar{f}_2(x), ..., \bar{f}_L(x))$

is called *the k-weighted thresholding* of *f(x)*, or simply, *a weighted thresholding*.

The function *f(x)* can be reconstructed by its family of threshold *k(α)*-valued signals as

$$f(x) = \sup\{\alpha;\ \bar{f}_\alpha(x) \neq 0\} = \sum_{\alpha=1}^{L} \frac{\bar{f}_\alpha(x)}{k(\alpha)}, \quad x \in R^n.$$

The following function can uniquely be assigned to the function *f(x)*:

$$\mathcal{K}_f(x) = \sum_{\alpha=1}^{m} \bar{f}_\alpha(x), \quad (m \leq L).$$

The operator $\mathcal{K}:\ f(x) \rightarrow \mathcal{K}_f(x)$ keeps all smooth parts of f(x), increasing its peaks.

$$\mathcal{K}_f(x) \geq f(x),\ (k(\alpha) \geq 1)$$

Canonical representation

The traditional threshold decomposition of the functions *f(x)* can be reduced to the *k*-weighted thresholding, if the signal has the equal binary signals.

$$f(x) = \{0,1,5,3,3,0\} \rightarrow f'(x) = \{0,1,3,2,2,0\}$$

$$f(x) = \sum_{\alpha=1}^{5} f_\alpha(x) \qquad f'(x) = \sum_{\alpha=1}^{3} f'_\alpha(x)$$

$$\begin{aligned} \alpha = 1,\ f_1(x) &= \{0,1,1,1,1,0\} \\ \alpha = 2,\ f_2(x) &= \{0,0,1,1,1,0\} \\ \alpha = 3,\ f_3(x) &= \{0,0,1,1,1,0\} \\ \alpha = 4,\ f_4(x) &= \{0,0,1,0,0,0\} \\ \alpha = 5,\ f_5(x) &= \{0,0,1,0,0,0\} \end{aligned} \qquad \begin{aligned} f'_1(x) &= \{0,1,1,1,1,0\} \\ f'_2(x) &= \{0,0,1,1,1,0\} \\ f'_3(x) &= \{0,0,1,0,0,0\} \end{aligned}$$

$$f(x) = \mathcal{K}_{f'}(x) = f'_1(x) + 2f'_2(x) + 2f'_3(x)$$

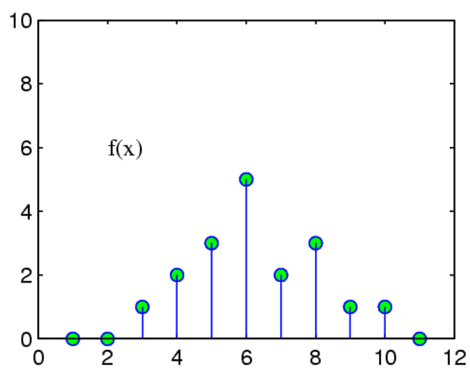
Example: the 5-level signal

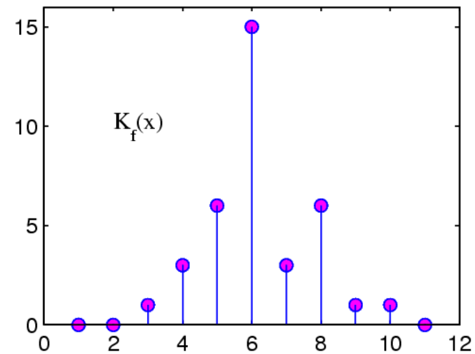
Consider $f(x)=\{\ldots,0,0,1,2,3,5,2,3,1,1,0,\ldots\}$.

$k(\alpha)$ be the arithmetic series of integers,

$k(\alpha)=\alpha=1,2,3,4,5.$

$$
\begin{array}{lclclcl}
f(x) & = & \{0,0,1,2,3,5,2,3,1,1,0\} & & \rightarrow & \mathcal{K}_f(x) & = & \{0,0,1,3,6,15,3,6,1,1,0\} \\
f_1(x) & = & \{0,0,1,1,1,1,1,1,1,1,0\} & \times 1 & = & \bar{f}_1(x) & = & \{0,0,1,1,1,\ 1,\ 1,\ 1,1,1,0\} \\
f_2(x) & = & \{0,0,0,1,1,1,1,1,0,0,0\} & \times 2 & = & \bar{f}_2(x) & = & \{0,0,0,2,2,\ 2,\ 2,\ 2,0,0,0\} \\
f_3(x) & = & \{0,0,0,0,1,1,0,1,0,0,0\} & \times 3 & = & \bar{f}_3(x) & = & \{0,0,0,0,3,\ 3,\ 0,\ 3,0,0,0\} \\
f_4(x) & = & \{0,0,0,0,0,1,0,0,0,0,0\} & \times 4 & = & \bar{f}_4(x) & = & \{0,0,0,0,0,\ 4,\ 0,\ 0,0,0,0\} \\
f_5(x) & = & \{0,0,0,0,0,1,0,0,0,0,0\} & \times 5 & = & \bar{f}_5(x) & = & \{0,0,0,0,0,\ 5,\ 0,\ 0,0,0,0\}.
\end{array}
$$





Examples: the 5-level signal

Consider f(x)={… ,0,0,1,2,3,5,2,3,1,1,0, … }.
Gray scale transformation: $f(x) \rightarrow \mathcal{K}_f(x)$

1. k(α)= 2α-1 =1,3,5,7,9.

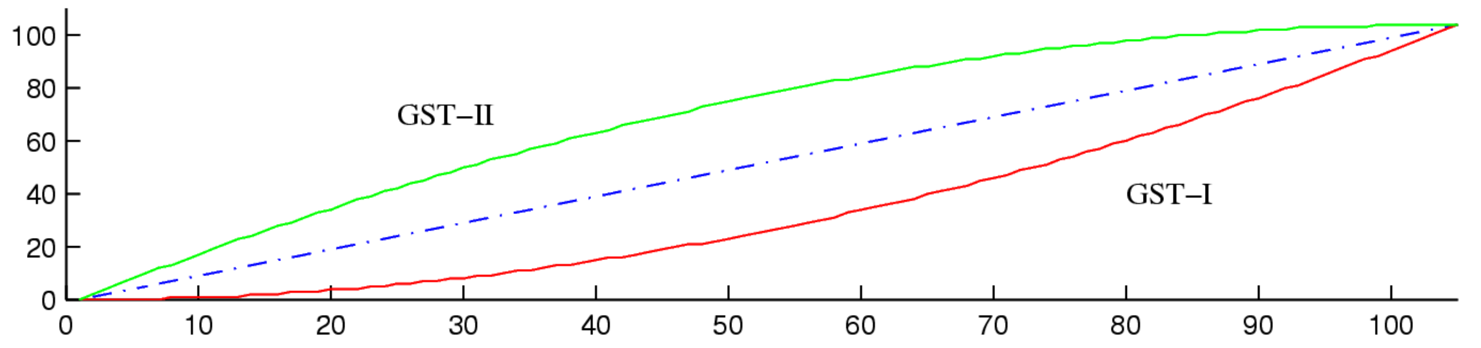
$$
\begin{array}{lclcccl}
f(x) & = & \{0,0,1,2,3,5,2,3,1,1,0\} & & \rightarrow & \mathcal{K}_f(x) & = \{0,0,1,4,9,25,4,9,1,1,0\} \\
f_1(x) & = & \{0,0,1,1,1,1,1,1,1,1,0\} & \times 1 & = & \bar{f}_1(x) & = \{0,0,1,1,1,\ 1,\ 1,\ 1,1,1,0\} \\
f_2(x) & = & \{0,0,0,1,1,1,1,1,0,0,0\} & \times 3 & = & \bar{f}_2(x) & = \{0,0,0,3,3,\ 3,\ 3,\ 3,0,0,0\} \\
f_3(x) & = & \{0,0,0,0,1,1,0,1,0,0,0\} & \times 5 & = & \bar{f}_3(x) & = \{0,0,0,0,5,\ 5,\ 0,\ 5,0,0,0\} \\
f_4(x) & = & \{0,0,0,0,0,1,0,0,0,0,0\} & \times 7 & = & \bar{f}_4(x) & = \{0,0,0,0,0,\ 7,\ 0,\ 0,0,0,0\} \\
f_5(x) & = & \{0,0,0,0,0,1,0,0,0,0,0\} & \times 9 & = & \bar{f}_5(x) & = \{0,0,0,0,0,\ 9,\ 0,\ 0,0,0,0\}
\end{array}
$$

Then, we obtain that $\mathcal{K}_f(x) = f^2(x)$

2. (m=5)

$$k(\alpha) = m + 1 - a: \quad f(x) \rightarrow \mathcal{K}_f(x) = f(x)\left(m - \frac{f(x) - 1}{2}\right)$$

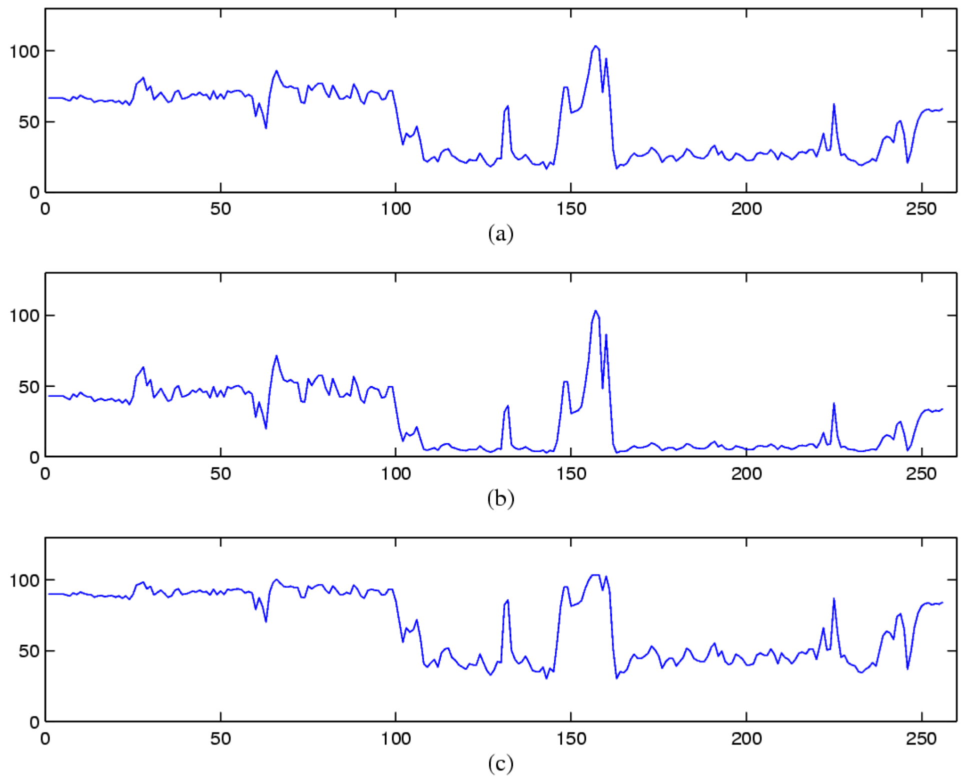
Gray scale transformation $f(x) \to \mathcal{K}_f(x)$



- Fig. 2. Two GSTs of the thresholding (solid lines) and the identical transform (dash line).

- GST-I is for $k(\alpha)= \alpha$, $\alpha=1:104$.
- GST-II is for $k(\alpha)= m-(\alpha-1)=m+1-\alpha$, $(m=104)$.

Gray scale transformation $f(x) \rightarrow \mathcal{K}_f(x)$



$$k(a) = a$$

$$k(a) = (105 - a)$$

Fig. 3. The signal and arithmetical thresholding.

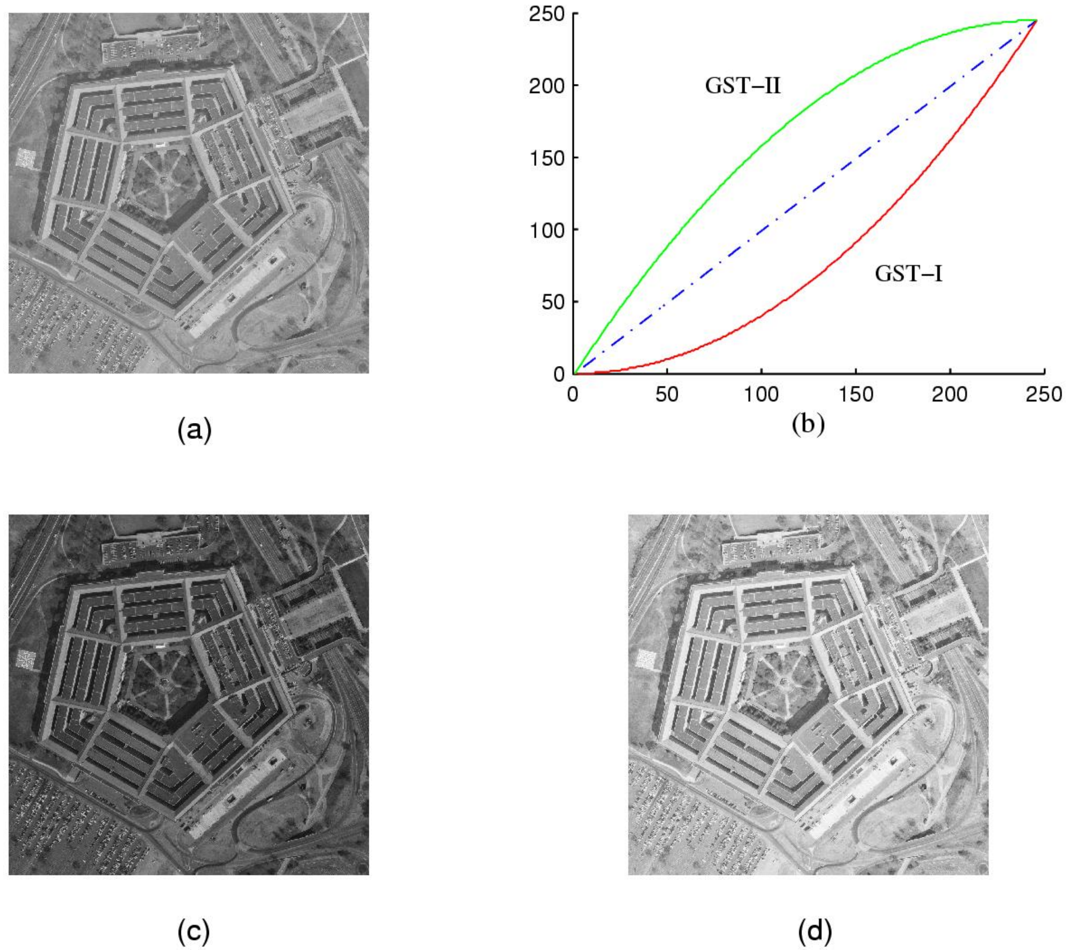
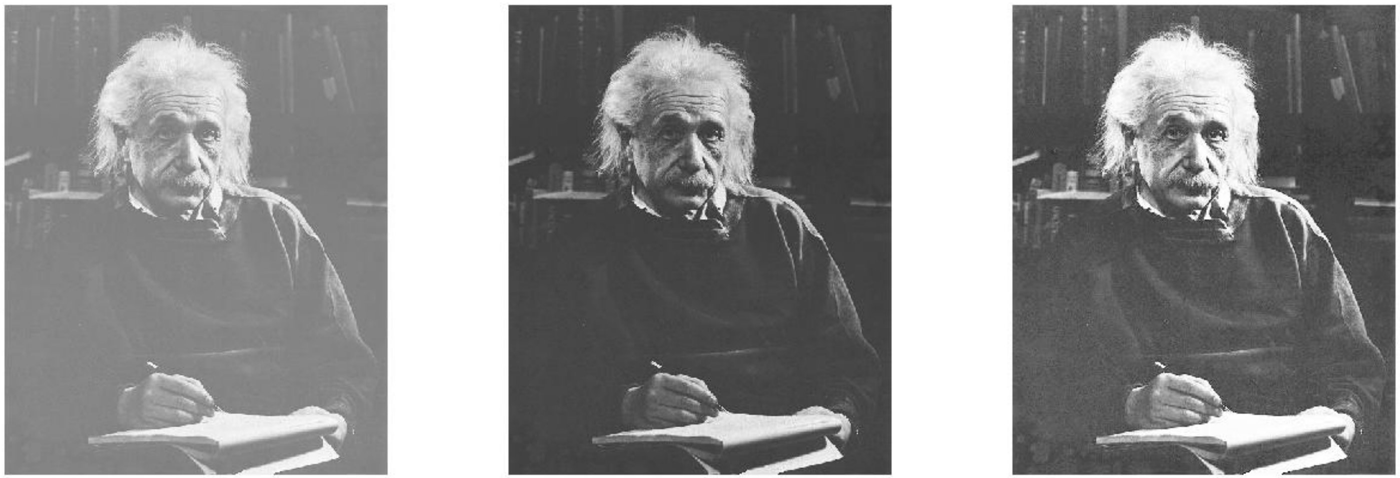


Fig. 4. (a) Image, (b) GSTs, and arithmetical weighted thresholding when (c) $k(\alpha)=\alpha$, and (d) $k(\alpha)=245-\alpha$.

Thresholding and images



(a) (b) GST–I (c) GST–II

Fig. 5. (a) The image and the arithmetical weighted thresholding when (b) k(α)=α, and (c) k(α)=m+1-α, (m=147).

Thresholding and images

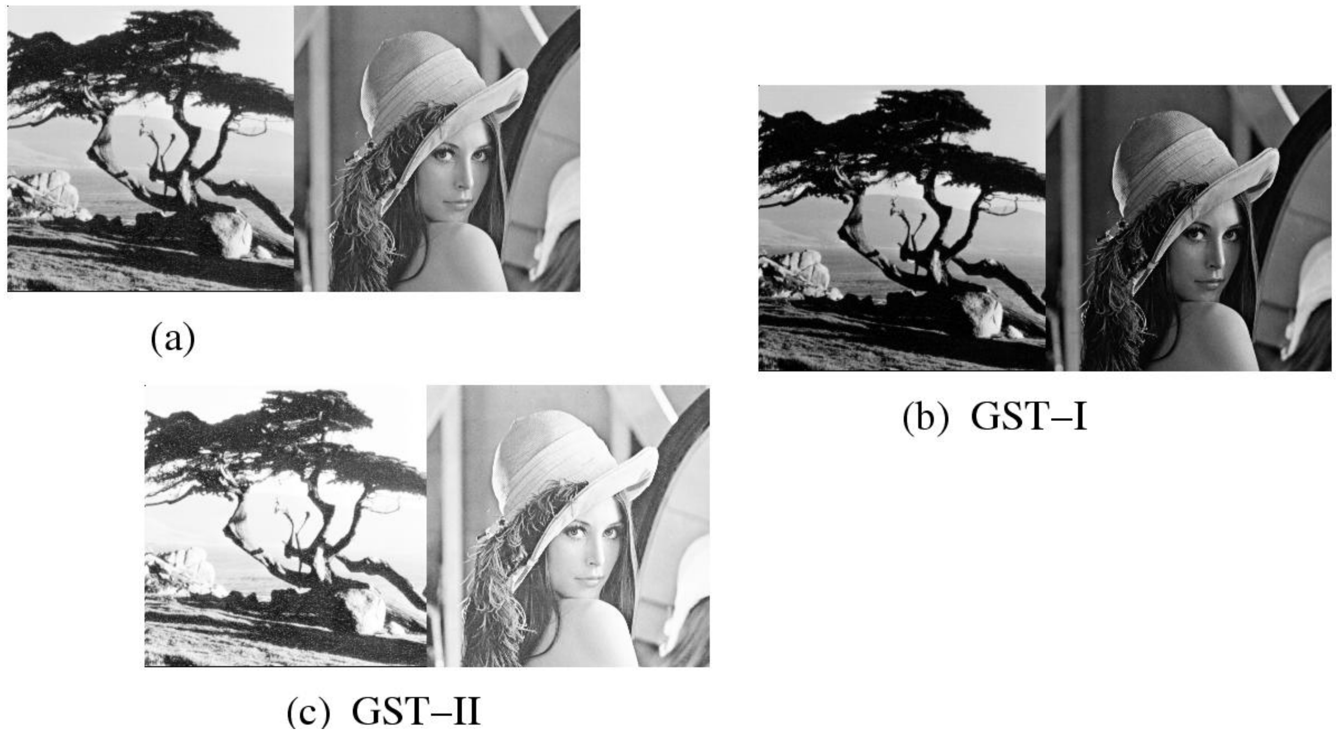


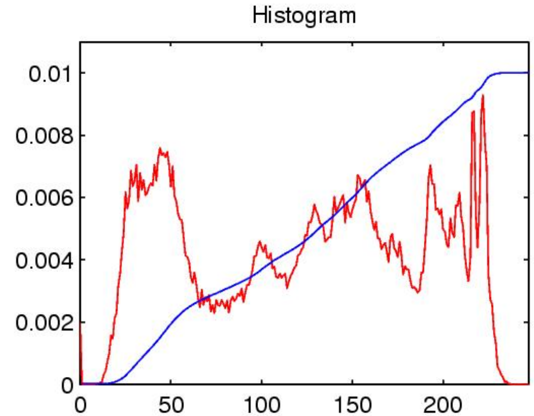
Fig. 6. (a) Tree-and-Lena image and the arithmetical weighted thresholding when (b) $k(\alpha)=\alpha$, and (c) $k(\alpha)=m+1-\alpha$, $(m=242)$.

Probability-type thresholding $k(\alpha)=p_\alpha$, $\alpha=0:(L-1)$

p_a is the probability of the event $\{f(x,y)=a\}$

(a)

$$\mathcal{K}_f(x) = mF(f(x))$$



(b)

(c)

(d)

Fig. 7. (a) Image, (b) histogram and distribution function, (c) weighted thresholding (or HE), and (d) image after processing the 3rd PDF.

High powers of PDF, F(α)

To improve the visibility of details of the scene behind the tree, the high powers of the distribution function can be used.

Fig. 8. Tree image processed by (a) the 2nd and (b) the 4th powers of the distribution function.

Image processed by blocks

The weighed thresholding in general can be applied block-wise. Such processing allows to see many details of the image in different blocks.

Fig. 9. The Lena image (a) processed by blocks (b) of size 64×128 each and (c) of size 8×256 each.

Image processed by blocks

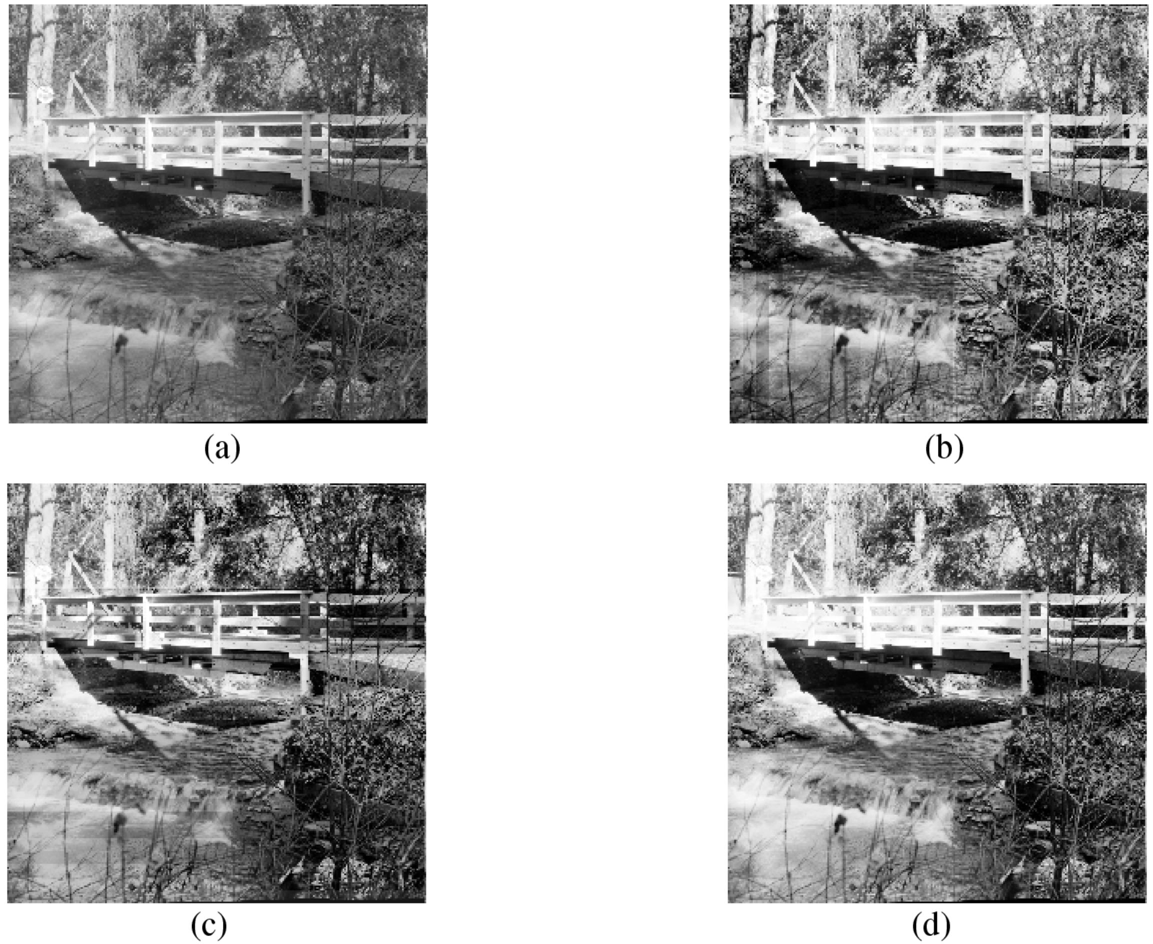


Fig. 10. Image (a) processed by blocks of size (b) 256×8, (c) 8×256, and (d) by the histogram equalization.

Image processed by piecewise linear function

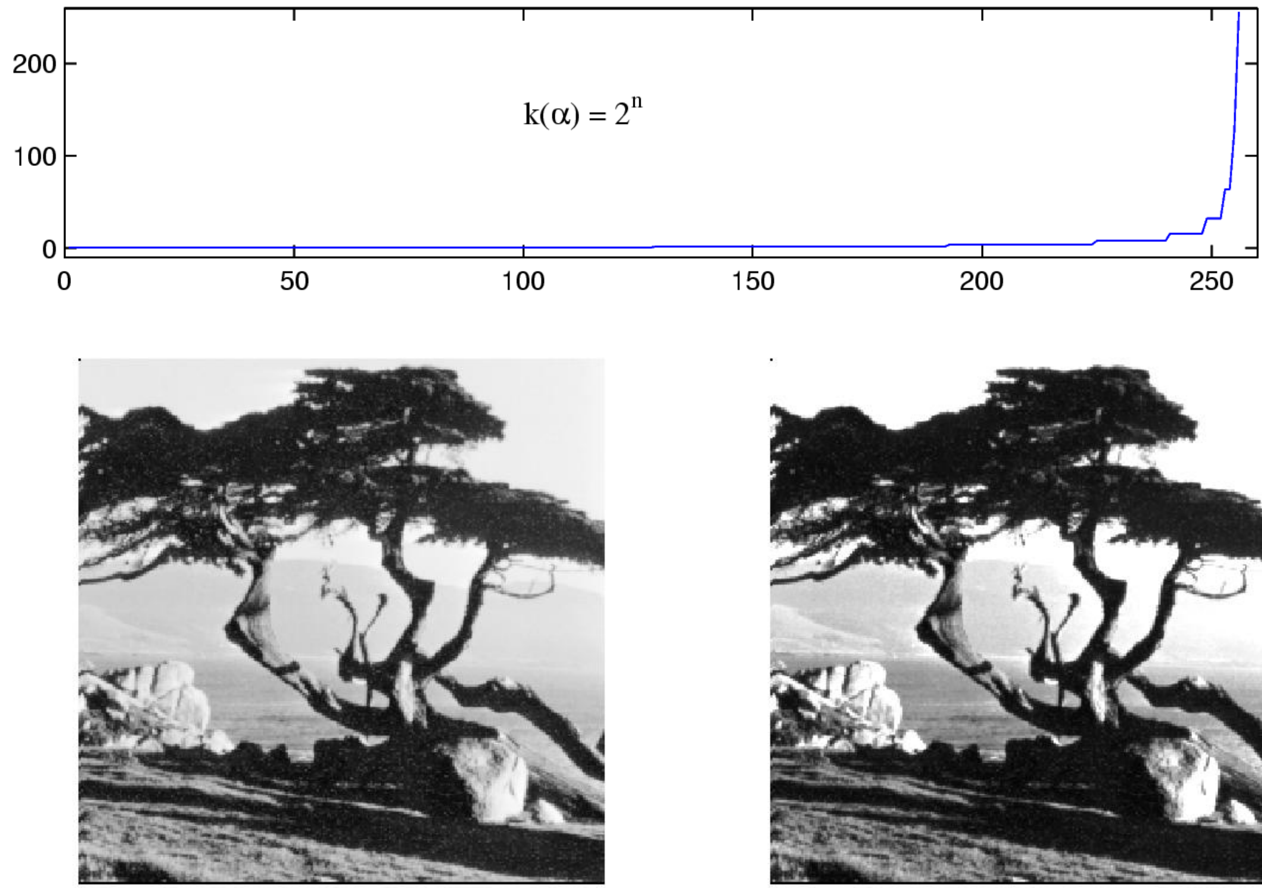


Fig. 11: Exponential function, the original image, and image enhanced by the k(α)-weighted thresholding.

The piecewise integer stepped sequence (2-in-4)

Sequence of numbers G_k is defined as the function with steps satisfying the following condition:

In any four consecutive G_k numbers only two numbers are repeated.

For example, such four numbers are

$$\{ G_{10}, G_{11}, G_{12}, G_{13} \} = \{ 7,7,8,9 \}.$$

The stepped function is monotonic and without high steps. We consider the condition that

$$|G_{k+1} - G_k| = 0, \text{ or } 1, \quad \text{for all } k > 1.$$

The piecewise integer stepped sequence (2-in-4)

Let Φ be the golden ratio, i.e.,

$$\Phi = (1+\sqrt{5})/2 = 1.618033988749894 \ldots$$

The sequence of numbers G_k which begins with integer number m is defined as

$$G_k = [k \log_2(\Phi) - 1.1610] + m, \quad k = 1, 2, 3, \ldots,$$

1	1	2	3	3	4	5	5	6	7	7	8	9	10	10	11
12	12	13	14	14	15	16	17	17	18	19	19	20	21	21	22
23	23	24	25	26	26	27	28	28	29	30	30	31	32	32	33
34	35	35	36	37	37	38	39	39	40	41	41	42	43	44	44
45	46	46	47	48	48	49	50	51	51	52	53	53	54	55	55
56	57	57	58	59	60	60	61	62	62	63	64	64	65	66	66
67	68	69	69	70	71	71	72	73	73	74	75	76	76	77	78
78	79	80	80	81	82	82	83	84	85	85	86	87	87	88	89
89	90	91	91	92	93	94	94	95	96	96	97	98	98	99	100

Fibonacci numbers and thresholding

Consider the Fibonacci numbers F_k that begin with F_1=1,

1,1,2,3,5,8,13,21,34,55,89,144, 233, ...

The Fibonacci numbers are calculated recurrently by the following formula:

$$F_1 = 1,\ \ F_2 = 1,\ \ F_k = F_{k-1} + F_{k-2}, \quad k = 3, 4, 5, \ldots .$$

k	1	2	3	4	5	6	7	8	9	10	11	12
F_k	1	1	2	3	5	8	13	21	34	55	89	144
F_k	233	377	610	987	1597	2584	4181	6765	10946	17711	28657	46368
k	13	14	15	16	17	18	19	20	21	22	23	24

Table II: First 24 Fibonacci numbers

The relation between G_k and F_k numbers is described by

$$G_k = [\log_2 F_k] + 1, \quad k \geq 1.$$

Processing by the G_k and $G_k{}^m$ sequences

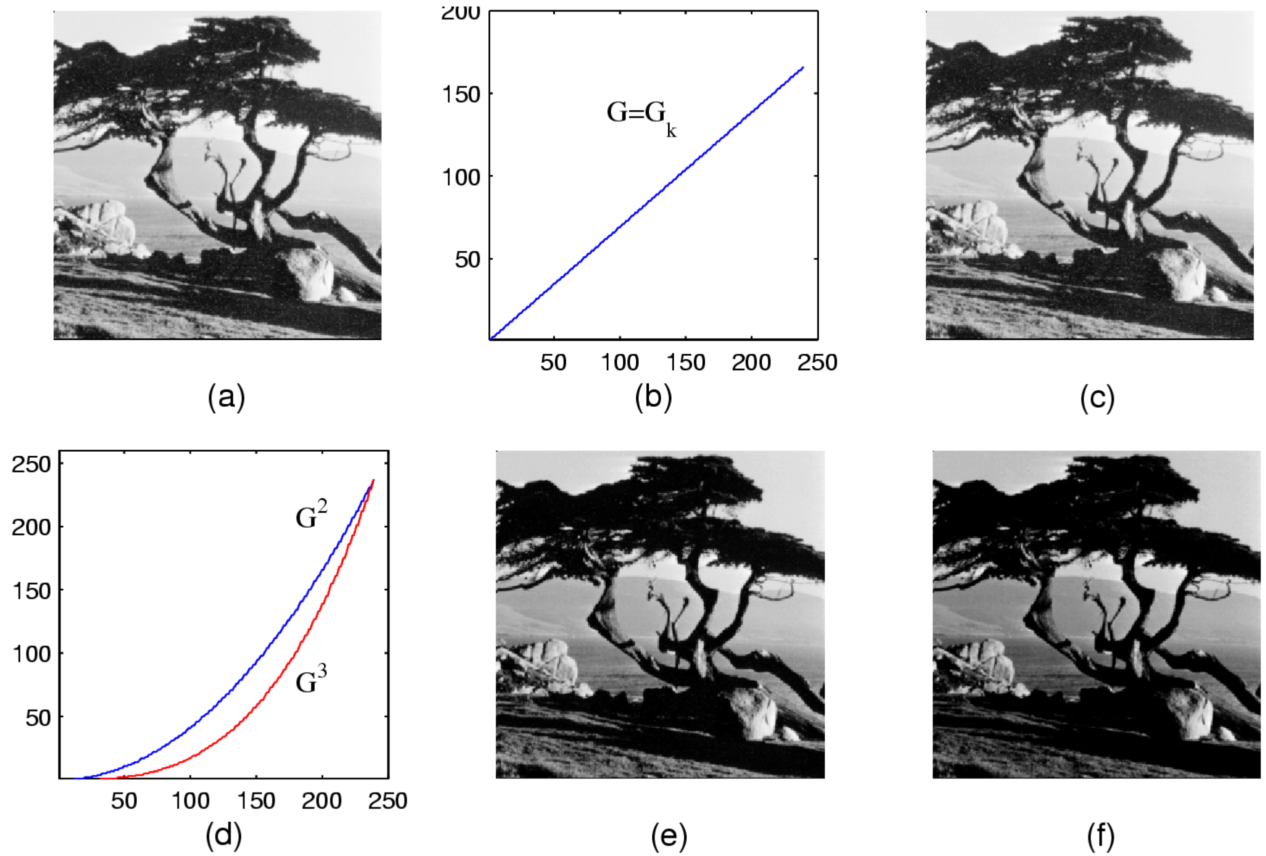


Fig. 12. (a) Tree image, (b) 2-in-4 sequence, (c) image processed by the amplified 2-in-4 sequence, (d) the 2nd and 3rd powers of the 2-in-4 sequence, and images processed by (e) the 2nd and (f) 3rd powers of this sequence.

Example: Thresholding by Fibonacci numbers

Consider $f(x)=\{\ldots,0,0,1,2,3,5,2,3,1,1,0,\ldots\}$.

$k(\alpha)$ be the numbers of Fibonacci series,

$k(\alpha)=1,1,2,3,5,8,13,\ldots$ $\alpha=1,2,3,4,5,6,7,\ldots$.

$$\begin{array}{lcllcll}
f(x) & = & 0,0,1,2,3,5,2,3,1,1,0, & & \rightarrow & \mathcal{K}_f(x) & 0,0,1,2,4,12,2,4,1,1,0, \\
f_1(x) & = & 0,0,1,1,1,1,1,1,1,1,0, & \times 1 & = & \bar{f}_1(x) & 0,0,1,1,1,\ 1,1,\ 1,1,1,0, \\
f_2(x) & = & 0,0,0,1,1,1,1,1,0,0,0, & \times 1 & = & \bar{f}_2(x) & 0,0,0,1,1,\ 1,1,\ 1,0,0,0, \\
f_3(x) & = & 0,0,0,0,1,1,0,1,0,0,0, & \times 2 & = & \bar{f}_3(x) & 0,0,0,0,2,\ 2,0,\ 2,0,0,0, \\
f_4(x) & = & 0,0,0,0,0,1,0,0,0,0,0, & \times 3 & = & \bar{f}_4(x) & 0,0,0,0,0,\ 3,0,\ 0,0,0,0, \\
f_5(x) & = & 0,0,0,0,0,1,0,0,0,0,0, & \times 5 & = & \bar{f}_5(x) & 0,0,0,0,0,\ 5,0,\ 0,0,0,0.
\end{array}$$

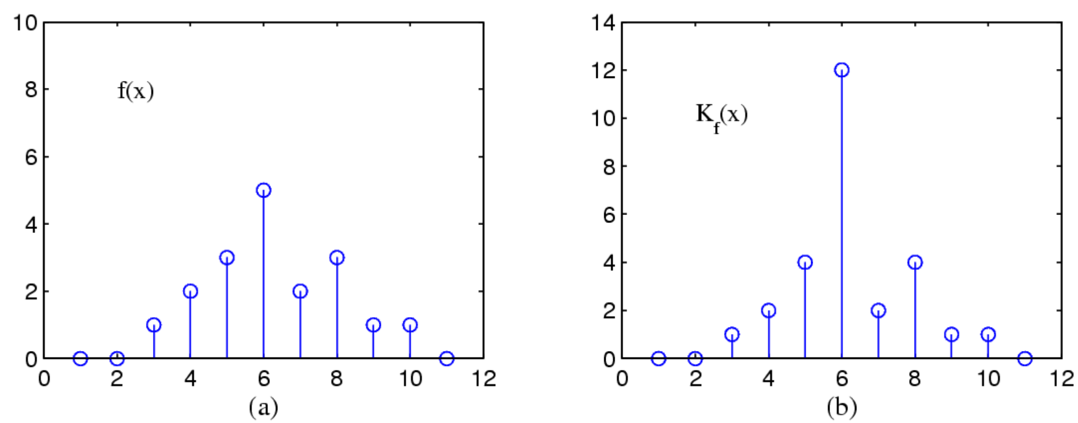


Fig. 13. (a) The signal f(x) and (b) the weighed thresholding $K_f(x)$.

Segmentation by morphological filters

The weighted thresholding can be used for designing new morphological filters in image enhancement, segmentation, and edge detection, together with the traditional thresholding.

For example, the processing of edges of the image, f, when using the difference between the image and its erosion, ($f \ominus B$), by a set B, can also be performed by using the similar difference between the weighted thresholding K_f and its erosion ($K_f \ominus B$).

Segmentation by morphological filters

- The general method of using the weighted thresholding for processing the signal or image can be described by the following algorithm

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Algorithm 4. *Sequence-based morphological image processing*

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1. Input the signal (or image) $f(x)$.
2. Calculate the sequence $k(\alpha)$, when $\alpha = 1 : \max(f)$.
3. Calculate the weighted thresholding $\mathcal{K}_f(x)$.
4. Process the weighted thresholding $M : \mathcal{K}_f(x) \to \mathcal{K}(x)$ by a given morphological filter M.
5. Calculate the corresponding signal (or image) $g(x)$ such that $\mathcal{K}_g(x) = \mathcal{K}(x)$.

$g(x)$ and $\mathcal{K}(x)$ are two output images.

Morphological filters: Gradients

The weighted thresholding is invariant relative the

$$\{(k(\alpha), f_\alpha(x))\} \rightarrow \mathcal{K}_f(x)$$

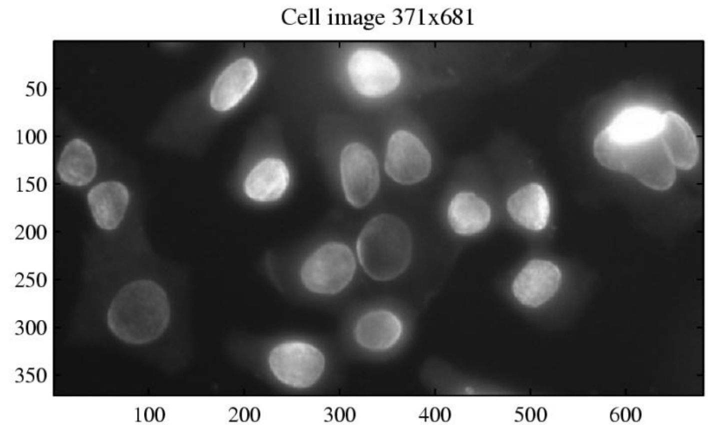
the basic morphological operations, which include the erosion and dilation, opening and closing.

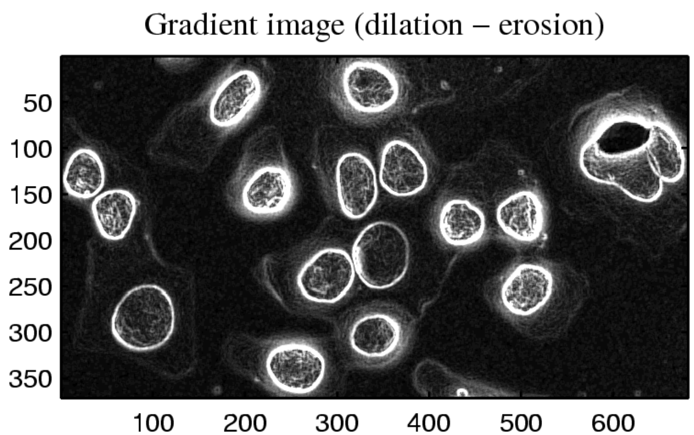
Consider the following three gradients:

$$f \rightarrow G(f) = \begin{cases} EG(f) = f - f \ominus B, \\ DG(f) = f \oplus B - f, \\ SG(f) = f \oplus B - f \ominus B \end{cases}$$

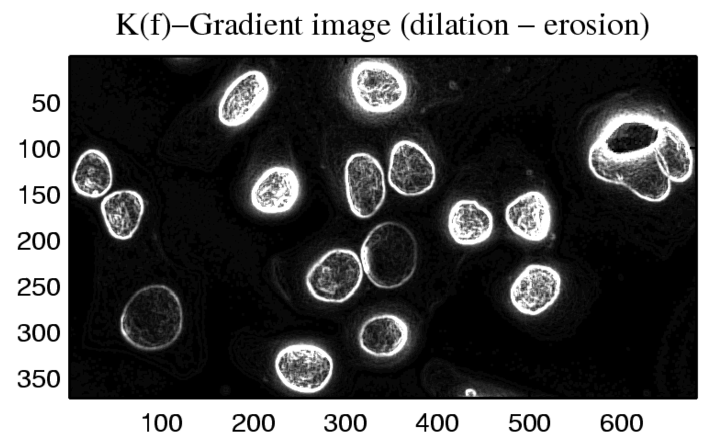
$$\mathcal{K}_f \ominus B = \mathcal{K}_{f \ominus B}, \qquad \mathcal{K}_f \oplus B = \mathcal{K}_{f \oplus B},$$
$$\mathcal{K}_f \circ B = \mathcal{K}_{f \circ B}, \qquad \mathcal{K}_f \bullet B = \mathcal{K}_{f \bullet B}.$$

Weighted thresholding and morphological gradients





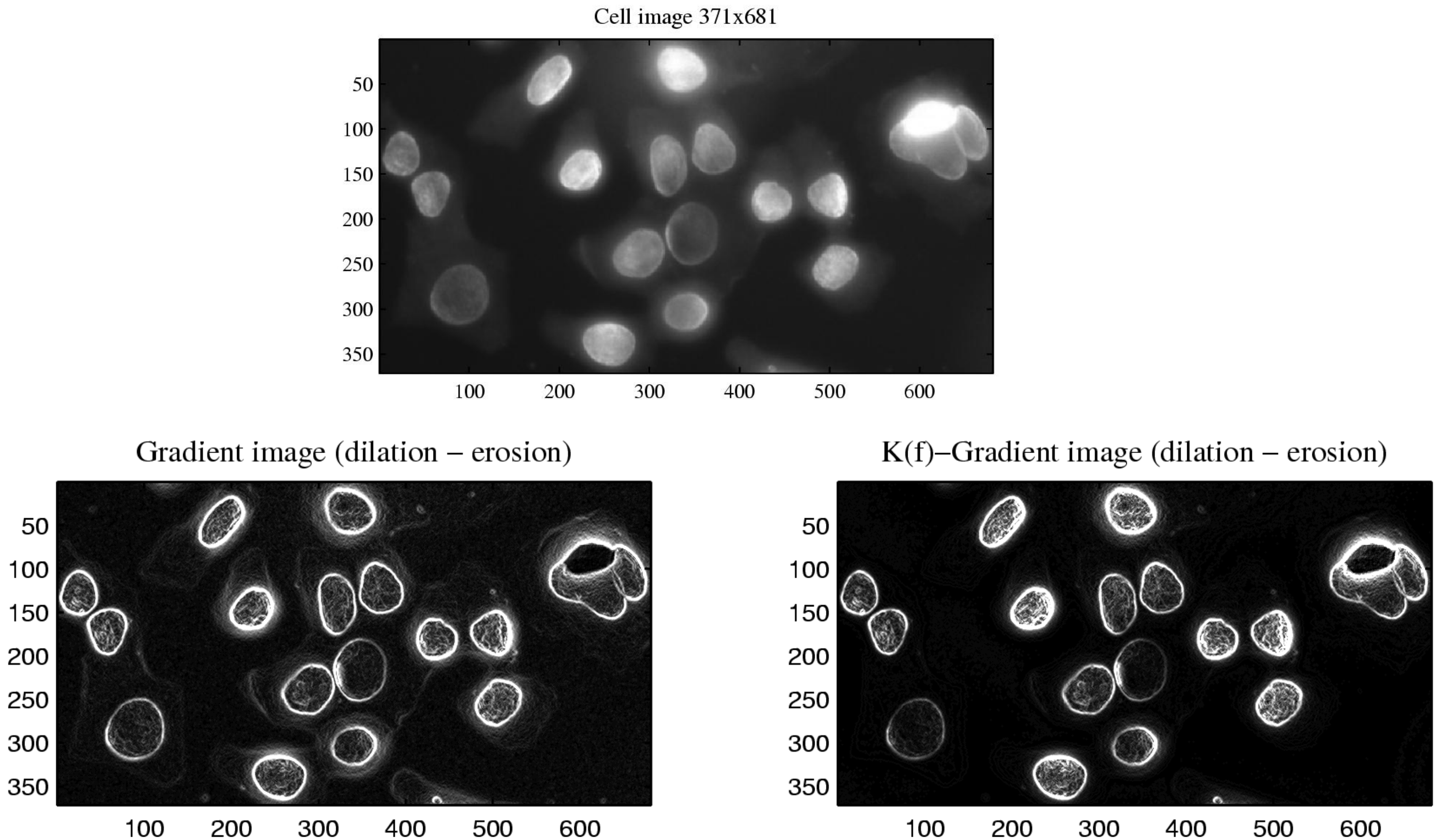
$$(f \oplus B) - (f \ominus B)$$



$$(\mathcal{K}_f \oplus B) - (\mathcal{K}_f \ominus B)$$

Fig. 14. Image and its gradients. (B is the square (3×3), k(α)=1,2,3, …

Weighted thresholding and morphological gradients



$$(f \oplus B) - (f \ominus B)$$

$$(\mathcal{K}_f \oplus B) - (\mathcal{K}_f \ominus B)$$

Fig. 15. Image and its gradients. (B is the disk of radius 1)

Summary

- The general concept of weighted thresholding for signal and image set-theoretical representation was introduced.
- The main advantage of such a representation is that it allows for implementation of nonlinear operations by weighted thresholds, enhancing many geometrical features that are present in the original signals and images by manipulating the weighted coefficients.
- New algorithms can be developed for efficient calculation and representation of nonlinear filters, for image enhancement and segmentation.
- The preliminary study shows that the weighted threshol-ding is very promising in terms of high performance for signal and image processing applications.