April 2016

2-D Hexagonal Quaternion Fourier Transform in Color Image Processing

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Outline

- Abstract
- Introduction
- Quaternion Numbers
  - Definition and properties of quaternions
  - Color image models: RGB color space
- Tensor representation
  - Gray-scale images
  - Color images in quaternion algebra
- Hexagonal 2D QDFT
- Conclusion
- References

Abstract

In this paper, we present a novel concept of the quaternion discrete Fourier transform on the two-dimensional hexagonal lattice, which we call the two-dimensional hexagonal quaternion discrete Fourier transform (2-D HQDFT). The concept of the right-side 2D HQDFT is described and the left-side 2-D HQDFT is similarly con sidered.

We analyze and present a new approach in processing the color images in the frequency domain, which is based on the tensor representation of color images.

- In color tensor representation on the hexagonal lattice, three components of the image in the RGB space are described by one dimensional signal in the quaternion algebra. The representation is effective and allows us to process the color image by 1-D quaternion signals which can be processed separately.
- The 2-D HQDFT can be calculated by a set of 1-D quaternion discrete Fourier transforms (QDFT) of the splitting-signals.
- The tensor transform-based 2-D QDFT is simple to apply and design, which makes it very practical in color image processing in the frequency domain.

Inroduction – Quanterions in Imaging

- The quaternion can be considered 4-dimensional generation of a complex number with one real part and three imaginary parts.

  Any quaternion may be represented in a hyper-complex form

$$q = (a + bi) + (c + di)j = a + bi + cj + dk = a + (bi + cj + dk),$$

  where a, b, c, and d are real numbers and i, j, and k are three imaginary units with the following multiplication laws:

$$ij = -ji = k,\ \ jk = -kj = i,$$

$$ki = -ik = -j,\ \ i^2 = j^2 = k^2 = ijk = -1.$$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

- The commutativity does not hold in quaternion algebra, i.e., $q_1q_2 \neq q_2q_1$.
- A unit pure quaternion is $\mu = i\mu_i + j\mu_j + k\mu_k$ such that $|\mu| = 1$, $\mu^2 = -1$.

  For instance, the number $\mu=(i+j+k)/\sqrt{3}$, $\mu=(i+j)/\sqrt{2}$, $\mu=(i-k)/\sqrt{2}$, and $\mu=-k$.
- The exponential number is defined as (when $\mu^2 = -1$)

$$\exp(\mu x) = \cos(x) + \mu \sin(x) = \cos(x) + i\mu_i \sin(x) + j\mu_j \sin(x) + k\mu_k \sin(x).$$

RGB Model for Color Images

- A discrete color image $f_{n,m}$ in the RGB color space can be transformed into imaginary part of quaternion numbers form by encoding the red, green, and blue components of the RGB value as a pure quaternion (with zero real part):

$$f_{n,m} = 0 + (r_{n,m}i + g_{n,m}j + b_{n,m}k)$$

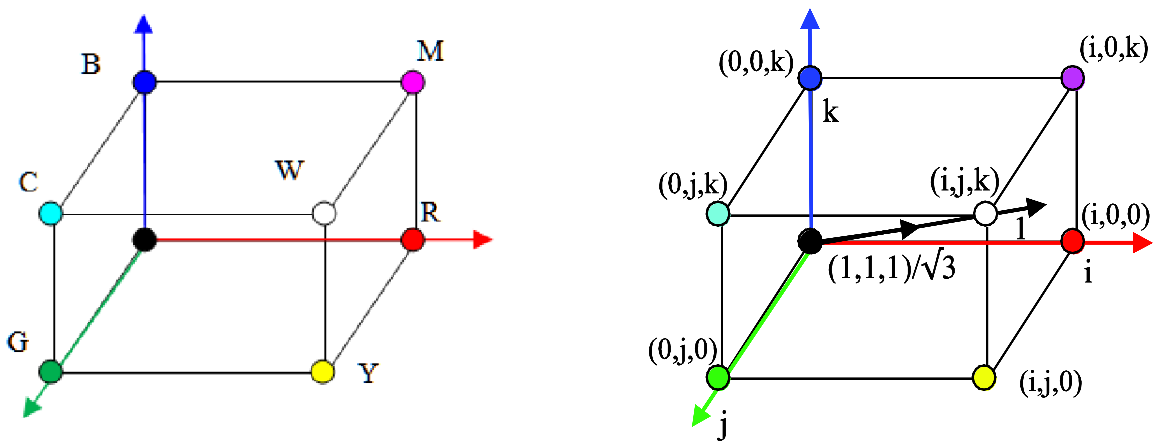


Figure 1: Transformation of the RBG color cube into the quaternion space.

- The advantage of using quaternion based operations to manipulate color information in an image is that we do not have to process each color channel independently, but rather, treat each color triple as a whole unit.

Model of color images

- A quaternion number has four components, a real part and three imaginary parts, which naturally coincides with the three components, R(ed), G(reen), and B(lue) of a color pixel for 2-D images. Therefore, a discrete color image $f_{n,m}$ in the RGB color space can be transformed into imaginary part of quaternion numbers form by encoding the red, green, and blue components of the RGB value as a pure quaternion (with zero real part):

$$q_{n,m} = 0 + i(r_{n,m} + jg_{n,m} + kb_{n,m}).$$

In quaternion imaging, each color triple is treated as a whole unit38 and new operations and methods in quaternion space may result in effective methods in color image processing, such as enhancement and filtration. We also can consider the quaternion image with the real part equal to the gray-image as

$$q_{n,m} = f_{n,m} + i(r_{n,m} + jg_{n,m} + kb_{n,m}),$$

where $f_{n,m}$ is the gray-scale image calculated by $f_{n,m} = (r_{n,m} + g_{n,m} + b_{n,m})/3$, or the image describing the brightness of the color image

$$f_{n,m} = 0.30r_{n,m} + 0.11g_{n,m} + 0.59b_{n,m}.$$

The right-side 2-D QDFTs

- The generalized quaternion Fourier transform of the quaternion image $q_{n,m}$ is defined as

$$Q_{p,s} = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} q_{n,m}\varphi_{p,s;\mu}(n,m), \quad p,s = 0:(N-1).$$

where the basic functions of the transform are

$$\varphi_{p,s;\mu}(n,m) = W_{\mu}^{np+ms} = \exp\left(-\mu\frac{2\pi}{N}(np+ms)\right), \quad n,m = 0:(N-1),$$

$$\exp\left(-\mu\frac{2\pi}{N}t\right) = \cos\left(\frac{2\pi}{N}t\right) - \mu\sin\left(\frac{2\pi}{N}t\right) \quad t = 0,1,\ldots,(N-1),$$

The 2-D QDFT is parameterized, i.e., is based on the choice of the unit pure quaternion μ. Such quaternion numbers are located on the unit sphere $\mu_1^2+\mu_2^2+\mu_3^2=1$ in the 3-D space.

- The special cases when μ =i,j, and k lead to the traditional complex 2-D DFT. The diagonal in the color cube in the RGB model corresponds to the gray-level and the value of μ equal $(i+j+k)/\sqrt{3}$ can be considered as a vector $(1,1,1)/\sqrt{3}$ in this direction.
- It should be noted, that the right-side 2-D QDFT is not separable, since the multiplication of quaternion does not posses the commutativity and for the basic functions we have

$$\varphi_{p,s;\mu}(n,m) \neq \varphi_{p;\mu}(n)\varphi_{s;\mu}(m) \qquad \text{for many } (p,s).$$

Tensor Representation of the regular 2-D DFT

Let $f_{n,m}$ be the gray-scale image of size $N{\times}N$.

- The tensor representation of the image $f_{n,m}$ is the 2D-frequency-and-1D-time representation when the image is described by a set of 1-D splitting-signals each of length N

$$\boxed{\chi : \{f_{n,m}\} \to \{f_{T_{p,s}} = \{f_{p,s,t}; t = 0 : (N-1)\}\}_{(p,s)\in J_{N,N}}.}$$

The components of the signals are the ray-sums of the image along the parallel lines

$$f_{p,s,t} = \sum_{(n,m)\in X} \{f_{n,m};\ np + ms = t \bmod N\}.$$

The components of the spliting-signal $\{f_{p,s,t} = f_{p,s}(t);\ t=0:(N-1)\}$ are numbered by the frequency-point (p,s) and time t.

The set of splitting-signals defines uniquely the image, in other words, the image can be calculatded by splitting-signals.

Tensor Representation of the regular 2-D DFT

Each splitting-signals $\{f_{p,s,t} = f_{p,s}(t);\ t=0:(N-1)\}$ defines 2-D DFT at N frequency-points of the set

$$T_{p,s} = \{(kp \bmod N, ks \bmod N);\ k = 0 : (N-1)\}$$

on the cartesian lattice

$$X = X_{N,N} = \{(n,m);\ n,m = 0,1,\dots,(N-1)\}$$

Each such set contains N points in the Cartesian lattice if (p,s) is not (0,0). The subsets cover the lattice with minimum number of intersections. For instance, when N is a power of two, this set contains 3N/2 generators and can be defined as

$$J_{N,N} = \{(1,s); s = 0 : (N-1)\} \cup \{(2p,1); p = 0 : (N/2-1)\}.$$

The 1-D DFT of the splitting-signal equals the 2-D DFT of the image $f_{n,m}$ at N frequency-points of the set $T_{p,s}$, i.e.,

$$\boxed{F_{kp \bmod N, ks \bmod N} = \sum_{t=0}^{N-1} f_{p,s,t} W_N^{kt}, \quad k = 0 : (N-1).}$$

Example: Tensor Representation of the right-side 2-D DFT

1-D splitting-signal of the tensor representation of the image 1024×1024

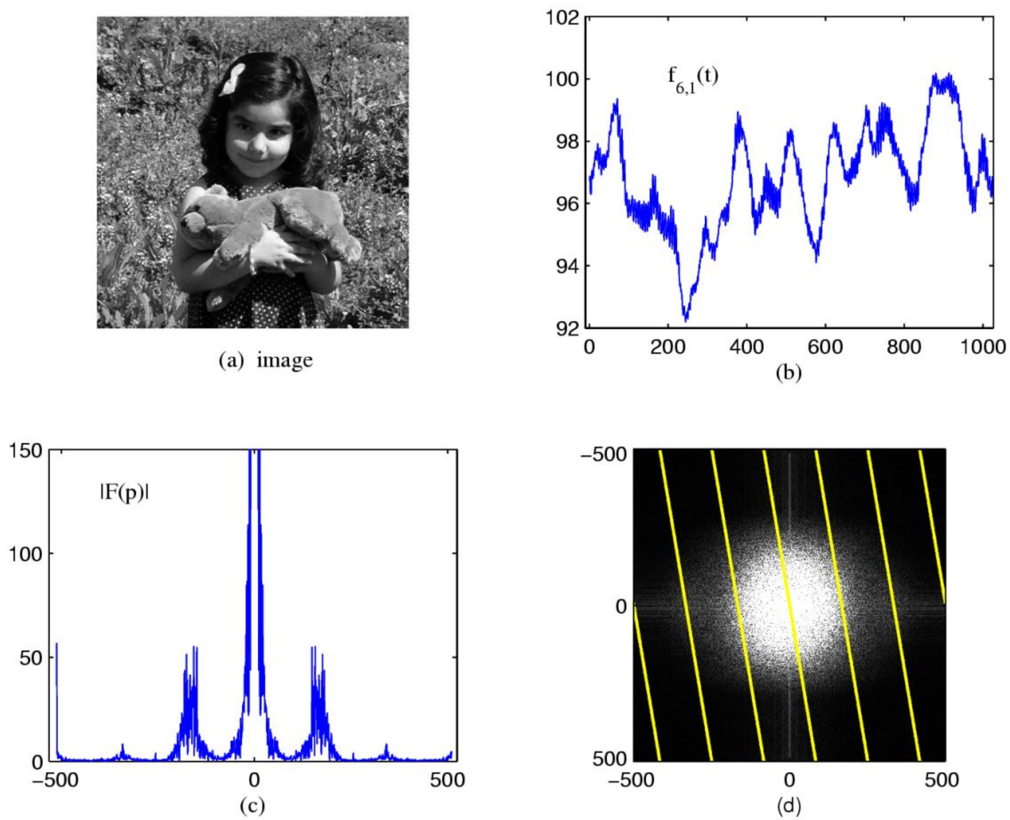


Figure 2. (a) The image, (b) the splitting-signal for the frequency-point (6,1), (c) the 1-D DFT of the splitting-signal, and (d) the 2-D DFT of the image with the frequency-points of the subset $T_{6,1}$. (The DFTs are shown in the absolute scale and shifted to the middle.).

Example: Tensor Representation of the right-side 2-D DFT

In the N prime case, the number of such splitting-signals equals (N+1) of length N each.

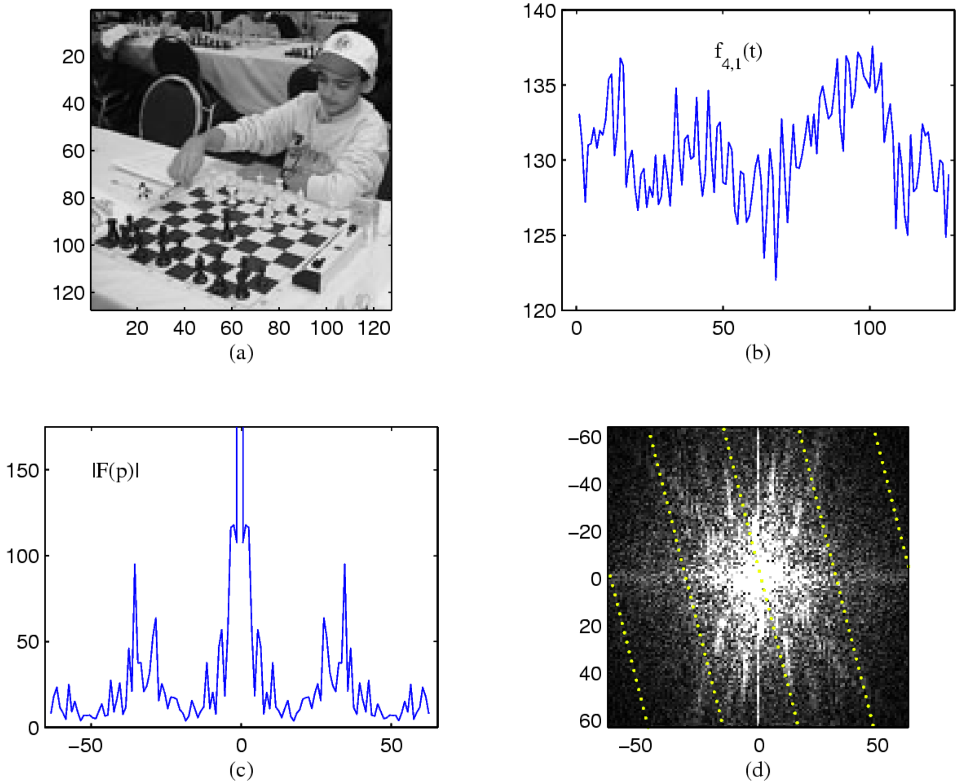
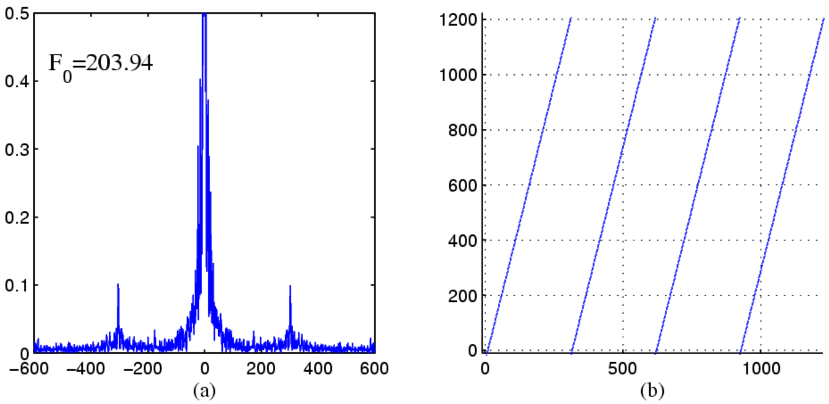


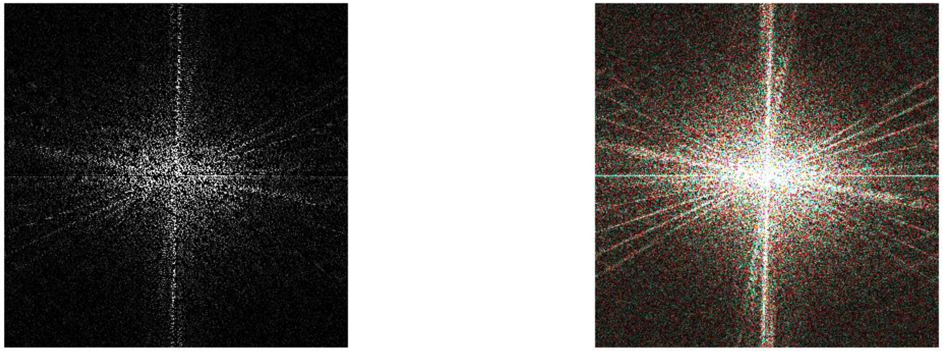
Figure 3. (a) The image 127×127, (b) the splitting-signal for the frequency-point (4,1), (c) its 1-D DFT, and (d) the 2-D DFT of the image with the frequency-points of the subset $T_{4,1}$. (The DFTs are shown in the absolute scale and shifted to the middle.).

Example: Tensor Representation of the 2-D QDFT



$\mu=(i+2j+k)/\sqrt{6}$

Figure 4. (a) The 1-D QDFT the quaternion splitting-signal $q_{1,4,t}$ (in absolute scale), and (b) the location of 1223 frequency-points of the set $T_{1,4}$ on the Cartesian grid, wherein this 1-D QDFT equals the 2-D QDFT of the quaternion image of size 1223×1223.



(a) (b)

Figure 5. (a) The real part and (b) the imaginary part of the 2-D QDFT of the 2-D color-in-quaternion girl image.

The 2-D Quaternion Hexagonal DFT

- Consider the hexagonal lattice of size 2N×N:

$$X_{2N,N} = \{(p+[s],s);\ p=0:(2N-1),\ s=0:(N-1)\},$$

where $[s]=(1-(-1)^s)/4$, for all integer s=0:(N−1). For even s, [s]=0, and [s]=0.5 for odd s. Each second row of knots in the lattice is shifted by 0.5.

The 2-D HDFT of the quaternion image $q_{n,m}$ is defined as

$$Q_{p+[s],s} = \sum_{n=0}^{2N-1}\sum_{m=0}^{N-1} q_{n+[m],m}\varphi_{p+[s],s;\mu}(n+[m],m), \quad (p+[s],s)\in X_{2N,N}.$$

The kernel of the transform is composed by the quaternion exponential functions

$$\varphi_{p+[s],s;\mu}(n+[m],m) = W_\mu^{(n+[m])(p+[s])+ms}, \qquad (n+[m],m),\ (p+[s],s)\in X_{2N,N}.$$

The basic functions are not separable and the traditional row-column method cannot be applied directly for computing 2-D HDFT. The lattice of size 2N×N is the fundamental period of the 2-D DHFT not the hexagonal lattice of size N×N, i.e., $Q_{(p+2N)+[s+N],s+N} = Q_{p+[s],s}$ for any integer p and s.

Tensor Representation of the 2-D Quaternion HDFT

The quaternion "image", i.e., the set of four images in the quaternion space,

$$q_{n+[m],m} = f_{n+[m],m} + ir_{n+[m],m} + jg_{n+[m],m} + kb_{n+[m],m},$$

where $f_{n+[m],m}$, $r_{n+[m],m}$, $g_{n+[m],m}$, and $b_{n+[m],m}$ are images on the lattice, which are not necessary to be related to the red, green, and blue components of the image in the RGB color model.

We define the following subsets of frequency-points on the hexagonal lattice:

$$T_{p+[s],s} = \left\{(\overline{k(p+[s])}, \overline{ks}),\ k = 0 : (2N-1)\right\},$$

where the operation $\bar{l}$ denotes the l mod(2N) or l mod(N), depending of the dimension.

$$T_{1+[0],0} = T_{1,0} = \{\ (0,0), (1,0), (2,0), (3,0), (4,0), \ldots, (12,0), (13,0)\ \},$$

$$T_{1+[1],1} = T_{1.5,1} = \left\{ \begin{array}{l} (0,0), (1.5,1), (3,2), (4.5,3), (6,4), (7.5,5), (9,6), \\ (10.5,0), (12,1), (13.5,2), (1,3), (2.5,4), (4,5), (5.5,6) \\ (7,0), (8.5,1), (10,2), (11.5,3), (13,4), (0.5,5), (2,6), \\ (3.5,0), (5,1), (6.5,2), (8,3), (9.5,4), (11,5), (12.5,6) \end{array} \right\}$$

$$T_{2+[1],1} = T_{2.5,1} = \left\{ \begin{array}{l} (0,0), (2.5,1), (5,2), (7.5,3), (10,4), (12.5,5), (1,6), \\ (3.5,0), (6,1), (8.5,2), (11,3), (13.5,4), (2,5), (4.5,6) \\ (7,0), (9.5,1), (12,2), (0.5,3), (3,4), (5.5,5), (8,6), \\ (10.5,0), (13,1), (1.5,2), (4,3), (6.5,4), (9,5), (11.5,6) \end{array} \right\}$$

Tensor Representation of the 2-D QHDFT

Given frequency-point (p+[s],s) in the hexagonal lattice $X_{2N,N}$ and integer 4t in [0,4N−1], we consider in the lattice the subsets of knots on the parallel lines,

$$V_t = V_{p+[s],s,t} = \{(n+[m],m);\ (n+[m])(p+[s]) + ms = t \bmod N\}$$

and the following components of the splitting-signal generated by the frequency:

$$q_{p+[s],s,t} = \sum_{V_{p+[s],s,t}} q_{n+[m],m}, \quad 4t \in [0, 4N-1].$$

of length N, 2N, or 4N, depending on the frequency-point (p+[s],s) and values of N. The index t varies from 0 through N with step Δ=1/4 or 1/2. This 4-D signal is calculated component-wise as

$$\begin{aligned} q_{p+[s],s,t} &= f_{p+[s],s,t} + i(r_{p+[s],s,t}) + j(g_{p+[s],s,t}) + k(b_{p+[s],s,t}) \\ &= \sum_{(n,m)\in V_t} f_{n,m} + i\left(\sum_{(n,m)\in V_t} r_{n,m}\right) + j\left(\sum_{(n,m)\in V_t} g_{n,m}\right) + k\left(\sum_{(n,m)\in V_t} b_{n,m}\right) \end{aligned}$$

The Tensor Representation of The Quaternion Image on the Hehagonal Lattice

Statement: The 1-D quaternion DFT of the splitting-signal $q_{n+[m],m}$ equals the 2-D quaternion HDFT of the image at the frequency-points of the subset $T_{p+[s],s}$, i.e.,

$$Q_{\overline{k(p+[s])},\overline{ks}} = \sum_{4t=0}^{N-1} q_{p+[s],s,t} W_{\mu}^{kt}$$

where k is an integer.

If we construct an irreducible covering of hexagonal lattice $X_{2N,N}$ composed of the subsets T,

$$\sigma = \left(T_{p+[s],s}\right)_{(p+[s],s)\in J}$$

for a certain subset of frequency-points J of the latice $X_{2N,N}$, then the 2-D QHDFT will be split by the 1-D DFTs of the quaternion splitting-signals.

The Tensor Representation of The Quaternion Image

- In the $N=2^r$ case, the number of elements of subsets $T_{p1+[s],s}$ shows that the 2-D QHDFT is split by the 1-D QDFT which has the following order M:

$$M = \begin{cases} 2N, & \text{if } p \text{ is odd, } s \text{ is even} \\ 4N, & \text{if } s \text{ is odd} \\ N/2^n, & \text{if } p \text{ and } s \text{ are even, } \quad g.c.d.(p,s) = 2^n. \end{cases}$$

For instance, when N=4, we can consider the following set of generators:

$$J = \{(1,0), (0+[1],1), (1+[1],1), (2,2), (0,2), (4,2)\}.$$

The subset $T_{1,0}$ has eight elements, and therefore the 8-point QDFT is used for the quaternion splitting-signal $\{q_{1,0,t};\, t = 0, 0.5, 1, 1.5, \ldots, 3.5\}$.
The next two subsets, $T_{0.5,1}$ and $T_{1.5,1}$ have 16 elements each. Therefore, the 16-point QDFT is used for each of the corresponding quaternion splitting-signals to fill the 2-D 8×4-point QHDFT at frequency-points of these subsets. At frequency-points of subsets $T_{2,2}$, $T_{0,2}$, and $T_{4,2}$ the 8×4-point QHDFT is calculated by the the 4, 4, and 4-point QDFTs, respectively.

The Tensor Representation of The Quaternion Image

- N=4 case. The subsets T with these generators are the following:

$$
\begin{aligned}
T_{1+[0],0} &= T_{1,0} = \{ (0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0)\},\\
T_{0+[1],1} &= T_{0.5,1} = \left\{ \begin{array}{l} (0,0),(0.5,1),(1,2),(1.5,3),(2,0),(2.5,1),(3,2),(3.5,3),\\ (4,0),(4.5,1),(5,2),(5.5,3),(6,0),(6.5,1),(7,2),(7.5,3) \end{array} \right\},\\
T_{1+[1],1} &= T_{1.5,1} = \left\{ \begin{array}{l} (0,0),(1.5,1),(3,2),(4.5,3)(6,0),(7.5,1),(1,2),(2.5,3)\\ (4,0),(5.5,1),(7,2),(0.5,3)(2,0),(3.5,1),(5,2),(6.5,3) \end{array} \right\},\\
T_{2+[2],2} &= T_{2,2} = \{ (0,0),(2,2),(4,0),(6,2) \},\\
T_{0+[2],2} &= T_{0,2} = \{ (0,0),(0,2) \},\\
T_{4+[2],2} &= T_{4,2} = \{ (0,0),(4,2) \}.
\end{aligned}
$$

One can notice, that the subsets have intersections at many frequency-points (p+[s], s) with s = 0 and 2. The redundancy of the tensor algorithm at these intersection can be removed by the method of paired transforms which is described in [1,2]. For the N = 8 case, such set is

$$
J = \{(p+[1],1);\ p=0:3\} \cup \{(1,0),(1,4),(2,0),(6,2)\} \cup \{(4p,2);\ p=0:3\}.
$$

Conclusion

- We presented a new concept of the 2-D quaernion DFT on the hexagonal lattice.
- The 2-D right-side hexagonal quaternion discrete Fourier transform (HQDFT) is described in the tensor representation in the quaternion algebra wherein the color image can be transformed from different color models, such as RGB and XYZ.
- The color and quaternion images on the hexagonal lattice can uniquely be described by a set of quaternion splitting-signals which allow to calculate the 2-D right-side HQDFT by a minimum number of 1-D right-side QDFTs.
- The tensor representation is revealing the structure of both right- and let-side 2-D HQDFT and allows for transferring the processing of color and quaternion images through 1-D splitting-signals.
- The concept of the 2-D left-side hexagonal quaternion DFT is similarly defined and can be described by the same tensor representation.

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