New 2-D Discrete Fourier Transforms in Image Processing

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Outline

- Abstract
- Introduction
- Tensor and paired transforms of the image
  - Splitting-signals
  - Complexity of the algorithm
  - Redundancy of the tensor representation
  - Comparison with the existent algorithms
- Conclusion
- References

Abstract

- In this s paper, the concept of the two-dimensional discrete Fourier transformation (2-D DFT) is defined in the general case, when the form of relation between the spatial-points (x,y) and frequency-points (ω_1,ω_2) is defined in the exponential kernel of the transformation by a nonlinear form $L(x, y; \omega_1,\omega_2)$.
- The traditional concept of the 2-D DFT uses the Diaphanous form $x\omega_1+y\omega_2$ and this 2-D DFT is the particular case of the transforms described by this form $L(x,y; \omega_1,\omega_2)$.
- Properties of the general 2-D discrete Fourier transforms are described and examples are given. The special case of the N×N-point 2-D Fourier transforms, when $N=2^r$, $r>1$, is analyzed and effective representation of these transforms is proposed.
- The proposed concept of nonlinear forms can be also applied for other transformations such as Hartley, Hadamard, and cosine transformations.

Tensor Representation of the (N×N) Image

The tensor representation of an image $f_{n,m}$ which is the (2-D)-frequency-and-(1-D)-time representation, the image is described by a set of 1-D splitting-signals of length N each

$$\chi : \{f_{n,m}\} \to \{f_{T_{p,s}} = \{f_{p,s,t}; t = 0 : (N-1)\}\}_{(p,s)\in J_{N,N}}.$$

The components of the signals are the ray-sums of the image along the parallel lines

$$f_{p,s,t} = \sum_{(n,m)\in X} \{f_{n,m};\ np + ms = t \bmod N\}.$$

Each splitting-signals defines 2-D DFT at N frequency-points of the set

$$T_{p,s} = \{(kp \bmod N, ks \bmod N);\ k = 0 : (N-1)\}$$

on the cartesian lattice $X = X_{N,N} = \{(n,m);\ n,m = 0,1,\ldots,(N-1)\}$

$$F_{kp \bmod N, ks \bmod N} = \sum_{t=0}^{N-1} f_{p,s,t} W_N^{kt}, \quad k = 0 : (N-1).$$

Example: 512×512-point 2-D DFT

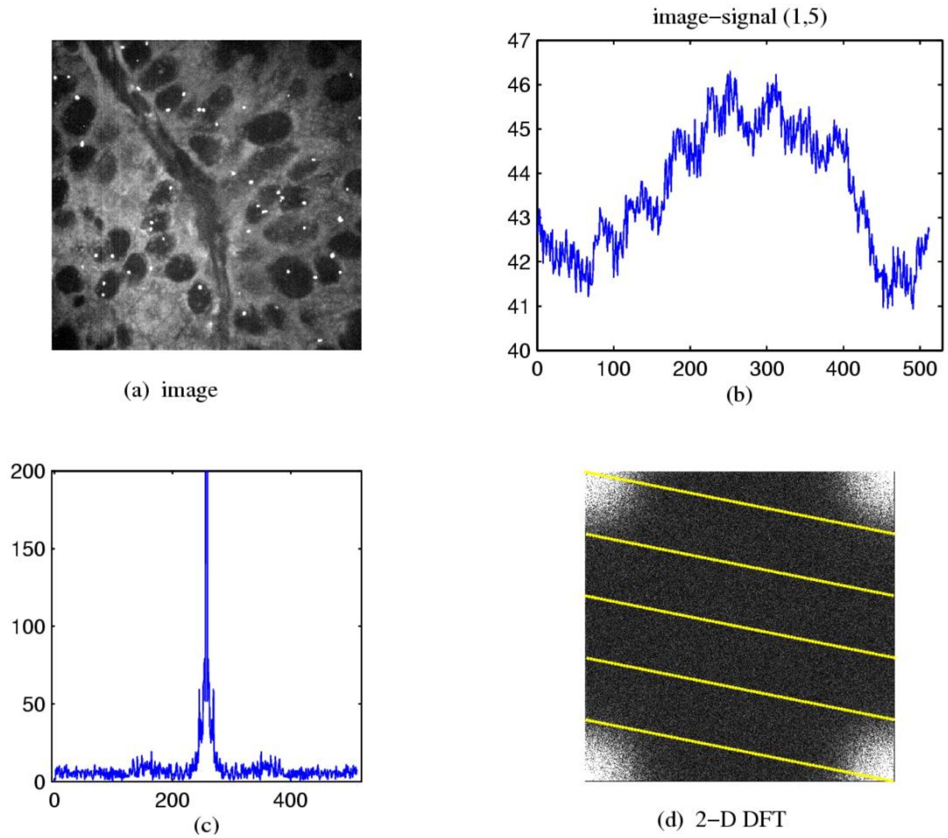

Figure 1. (a) The the maximum image composed from a stack images by fluorescence in situ hybridization (FISH) image, (b) splitting-signal for the frequency-point (p,s)=(1,5), (c) magnitude of the shifted to the middle 1-D DFT of the signal, and (d) the 2-D DFT of the image with the frequency-points of the set $T_{1,5}$.

Set of generators (p,s) for splitting-signals

The $N=2^r$ case is considered.

The set $J_{N,N}$ of frequency-points (p, s), or generators, of the splitting-signals is selected in a way that covers the Cartesian lattice

$$X_{N,N} = \{(p,s);\, p, s = 0 : (N-1)\}$$

with a minimum number of subsets Tp,s.

The set $J_{N,N}$ contains $3N/2$ generators and can be defined as

$$J_{N,N} = \{(1,s); s = 0 : (N-1)\} \cup \{(2p,1); p = 0 : (N/2-1)\}.$$

The tensor representation is unique, and the image can be defined through the 2-D DFT calculated by

$$F_{kp \bmod N, ks \bmod N} = \sum_{t=0}^{N-1} f_{p,s,t} W_N^{kt}, \quad k = 0 : (N-1).$$

The total number of components of 3N/2 splitting-signals equals $N^2+N^2/2$, which exceeds the number of points in the image. Many subsets Tp,s, (p,s) $\in J_{N,N}$, have intersections at frequency-points.

2-D paired representation of images

- To remove the redundancy in the tensor representation, we consider the concept of the paired transform. In paired representation, the image is described by a unique set of splitting-signals of lengths N/2,N/4, …, 2, 1, and 1,

$$\{f_{n,m}\} \rightarrow \{f'_{p,s,2^k t},\ t = 0, 1, ..., (N/2^{k+1} - 1)\}_{(p,s)\in J'_{N,N}},$$

where 2^k =g.c.d.(p,s), and k=r−1 when (p,s)=(0,0). These 1-D signals are generated by the set of (3N−2) generators (p,s).
The components of the 2-D paired transform

$$f'_{p,s,2^k t} = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \chi'_{p,s,2^k t}(n,m) f_{n,m} = f_{p,s,2^k t} - f_{p,s,2^k t+N/2},$$

are calculated by the system of orthogonal paired functions defined as

$$\chi'_{p,s,2^k t}(n,m) = \begin{cases} 1, & \text{if } np + ms = 2^k t \bmod N \\ -1, & \text{if } np + ms = (2^k t + N/2) \bmod N \\ 0, & \text{otherwise,} \end{cases}$$

Set of generators (p,s) in TR of Images

- The set of N^2 triplets of the paired functions is taken equal to the set

$$U_{N,N} = \bigcup_{k=0}^{r-1} \{(p,s,2^k t);\ \ (p,s) \in 2^k J_{N/2^k,N/2^k}\,, \quad t = 0:(N/2^{k+1}-1)\} \cup \{(0,0,0)\}.$$

The number of generators *(p,s)* in the set equals $3N-2$. The splitting-signals in paired representation carry the spectral information of the image at $N/2^k+1$ frequency-points of the following subsets of $T'p,s$:

$$T'_{p,s} = \{((2m+1)p \bmod N, (2m+1)s \bmod N);\ m = 0:N/2^{k+1}-1\},$$

where 2^k =g.c.d.(p,s).

The following equation holds:

$$F_{(2m+1)p \bmod N,(2m+1)s \bmod N} = \sum_{t=0}^{L-1} \left(f'_{p,s,2^k t} W_{2L}^t \right) W_L^{mt}$$

$$m = 0:(L-1), \quad (L = N/2^{k+1}).$$

Example: Paired splitting-signals of image

- Two splitting-signals of the FISH image 512×512 in paired representation:
  - signal $\{f'_{1,5,t};\ t=0:255\}$ is generated by (p,s) = (1,5)
  - signal $\{f'_{2,6,2t};\ t=0:127\}$ is generated by (p,s) = (2,6)=2(1,3)

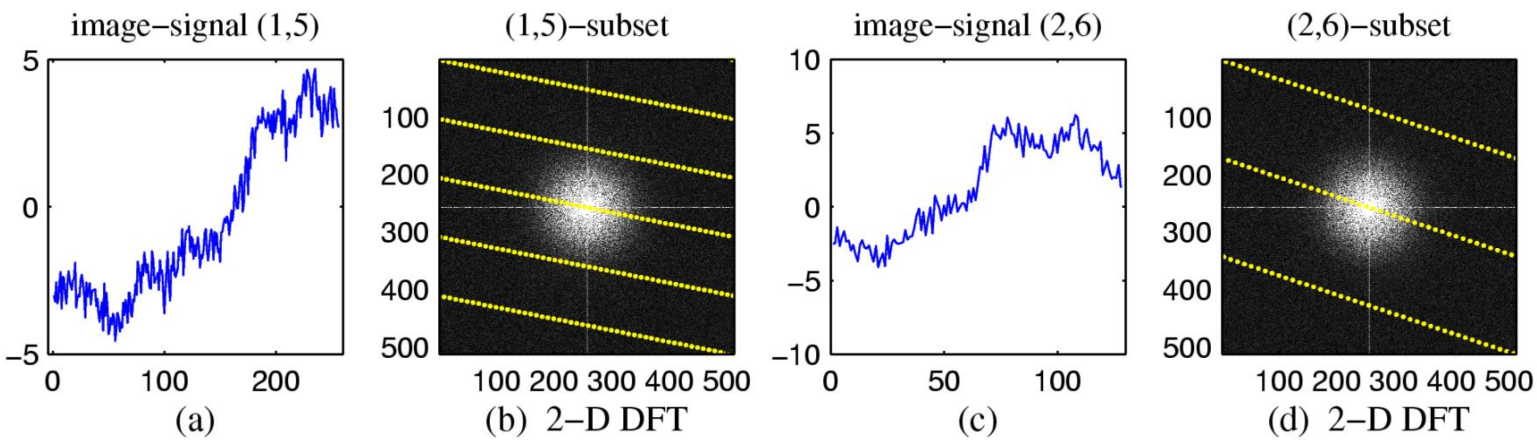

Figure 2. (a) The paired splitting-signal for the frequency-point (1,5) and (b) the 2-D DFT of the image with the frequency-points of the subset $T'_{1,5}$. (c) The paired splitting-signal for the frequency-point (2,6) and (d) the 2-D DFT of the image with the frequency-points of the subset $T'_{2,6}$. (The 2-D DFT of the image and subsets T' are cyclicly shifted to the center.)

2-D Paired Transform is Orthogonal

- The mutual intersections of the subsets $T'_{p,s}$ with generators (p, s) such that

$$(p, s, 2^k t) \in U_{N,N}$$

are empty.

Here, $2^k = g.c.d.(p,s)$ and $k=r-1$ when $(p,s)=(0,0)$.

- Therefore, all splitting-signals in the paired representation carry the spectral information of the image at disjoint subsets of frequency-points, and the paired transformation

$$\{f_{n,m}\} \rightarrow \{f'_{p,s,2^k t},\ t = 0, 1, ..., (N/2^{k+1} - 1)\}_{(p,s)\in J'_{N,N}},$$

is unitary

New Class of Discrete Fourier Transforms

- When considering the 2-D discrete Fourier transformation with the rectangular fundamental period $X_{N,N}$, we take into consideration the following fact:

The kernel W of the transform connects all samples (n_1, n_2) of the image $f_{n1,n2}$ and the frequency-points (p_1, p_2) via the simple form L, namely, the Diophantus form

$$L(n_1,n_2;p_1,p_2)=n_1p_1+n_2p_2, \quad (n_1,n_2),\ (p_1,p_2)\in X_{N,N}$$

considered in the arithmetics modulo N. Analyzing the procedure of construction of the paired transform with respect to the 2-D DFT, where $N=2^r, r>1$, it is not difficult to note that the following three conditions have been used:

1) $L(n_1,n_2;p_1,p_2)=$ maps $X_{N,N}$ onto $X_N=\{0,1,...,N-1\}$,
2) $kL(n_1,n_2;p_1,p_2)=L(n_1,n_2;\overline{kp_1},\overline{kp_2}), \quad k=0:(N-1), \quad (p_1,p_2)\in X_{N,N}$,
3) $W_N(t+N/2)=-W_N(t), \quad t=0:(N/2-1), \qquad W_N(t)=\exp(2\pi i/N)$

The condition #3 indicates only that the paired transform corresponds to the paired representation with respect to the Fourier transform, and for other transforms it can be changed in accordance with their properties. Naturally, the question arises about finding other forms L that satisfy conditions 1) and 2).

Such forms exist and we describe a few one- and two-dimensional examples.

1-D unitaru transforms by forms

In the 1-D case, it is not difficult to notice that the form L in (2) can be taken as

$$L(n;p) = Q_k(n)p, \quad n,p = 0:(N-1),$$

where Q_k ($k \geq 1$) is a polynomial of n of degree k with constant coefficients. Q_k is a permutation of the interval X_N. Such polynomials Q_k exist and can be constructed.

Example 1. (Transform for the polynomial of degree 2)

Consider the polynomial $Q_2(n) = 2n^2 + n$ of degree 2 and the corresponding form

$$L(n;p) = L_2(n;p) = (2n^2 + n)p, \quad n,p = 0:(N-1).$$

The orthogonal transform corresponds to this form, which is denoted by $(\chi'_N; L_2)$

The N=8 example:

$$[\chi'_8; L_2] = \begin{vmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

New Class of Discrete Fourier Transforms

Similarly, with the aid of the described algorithm, the orthogonal transformion $(\chi'_N; L_2)$ is constructed in the general case N = 2r, r > 1, too.

- In order to find the Fourier transform that corresponds to this transformation $(\chi'_N; L_2)$ it is sufficient to look on condition (3). In our case, when *W* is the exponential kernel of the *N*-point DFT, which can be denoted by $(\mathcal{F}_N; L_2)$ and written as

$$F_p = \big((\mathcal{F}_N; L_2) \circ f\big)_p = \sum_{n=0}^{N-1} f_n W(L_2(n;p)) = \sum_{n=0}^{N-1} f_n W^{(2n^2+n)p}$$

$$p = 0 : (N-1).$$

High order forms *L* of the polynomials Q_k in can be considered, too.

Example 2. (Transform for the polynomial of degree *k=3*)

$$Q_3(n) = 2n^3 + n$$

$$L(n;p) = L_3(n;p) = (2n^3 + n)p, \quad n, p = 0 : (N-1).$$

$$F_p = \big((\mathcal{F}_N; L_3) \circ f\big)_p = \sum_{n=0}^{N-1} f_n W(L_3(n;p)) = \sum_{n=0}^{N-1} f_n W^{(2n^3+n)p}$$

New Class of Discrete Fourier Transforms

- **Definition 3.1**. Let p is a point of X_N and let $t \in \{0, , \ldots, N/2-1\}$. The functions

$$\chi'_{p,t}(n) = \chi_{p,t}(n) - \chi_{p,t+N/2}(n)$$

are called one-dimensional paired functions by the form L.

This definition generalizes the concept of the paired functions which are the paired functions by the form

$$L_1(n;p) = np.$$

We call the transformation $(\chi'_N; L)$ which corresponds to the paired functions *the paired transformion by the form* L.

The described N-point transform $(\mathcal{F}_N; L)$ is called the Fourier transformion by the form L.

New Class of 2-D Discrete Fourier Transforms

- New paired functions and unitary transformations $(\chi'_{N,N}; L)$, where $N=2^r$, $r>1$, can be considered in the two-dimensional case.

Let L be the certain form satisfying conditions (1) and (2). For any $(p_1,p_2) \in X_{N,N}$ and point $t \in \{0,1,2, \ldots ,N-1\}$, we define the set of samples

$$V_{p_1,p_2,t} = \{(n_1,n_2);\ (n_1,n_2) \in X_{N,N};\ \underline{L(n_1,n_2;p_1,p_2) = t \bmod N}\}$$

and its characteristic function $\chi_{p_1,p_2,t}(n_1,n_2)$.

Definition 3.2. The function

$$\chi'_{p_1,p_2,t}(n_1,n_2) = \chi_{p_1,p_2,t}(n_1,n_2) - \chi_{p_1,p_2,t+N/2}(n_1,n_2),$$

$$n_1,n_2 = 0 : (N-1),$$

is called *the two-dimensional paired function by the form L*.

2-D Paired transformations and DFTs by forms

Using the same set of triples $U_{N,N}$, the complete system of orthogonal paired functions can be formed. The set of the 2-D paired functions by the form L

$$\chi'_{N,N} = \{\chi'_{p_1,p_2,t};\ T'_{p_1,p_2;L} \in \sigma'_{N,N},\ t \in X_N,\ V_{p_1,p_2,t} \not\equiv \emptyset\}$$

is complete. These functions determine the orthogonal transformation which is called *the paired transformation by the form L* and is denoted as $(\chi'_{N,N}; L)$.

Example 3. (8 × 8-point transform for the polynomial)

$$L_{2,2}(n_1, n_2; p_1, p_2) = (2n_1^2 + n_1)p_1 + (2n_2^2 + n_2)p_2$$

N=8. The mask of the first paired function is

$$[\chi'_{1,1,0}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

2-D Paired transformations by forms

$$L(n_1, n_2; p_1, p_2) = n_1 p_1 + n_2 p_2$$

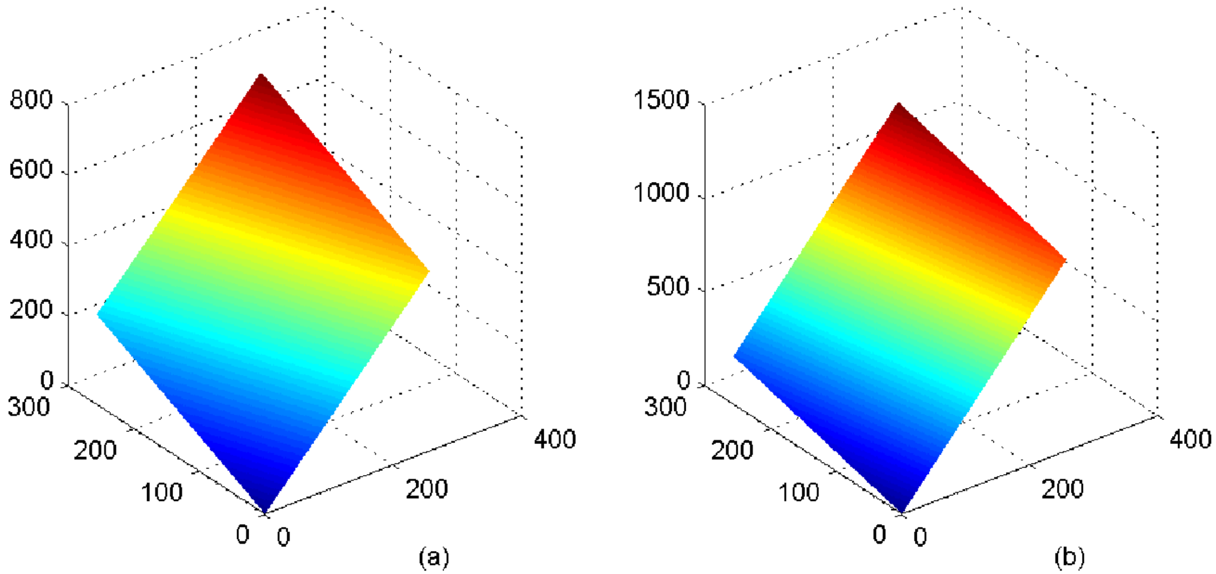

Figure 3. The surface of the Diaphanous form for (a) $(p_1, p_2) = (1, 2)$ and (b) $(p_1, p_2) = (1, 4)$.

Theses surfaces are calculated for the above nonlinear form in arithmetic modulo 256 in parts a and b for the frequency-points (p_1, p_2)=(1,2) and (1,4), respectively.

2-D Paired transformations by forms

$$L_{2,2}(n_1, n_2; p_1, p_2) = (2n_1^2 + n_1)p_1 + (2n_2^2 + n_2)p_2$$

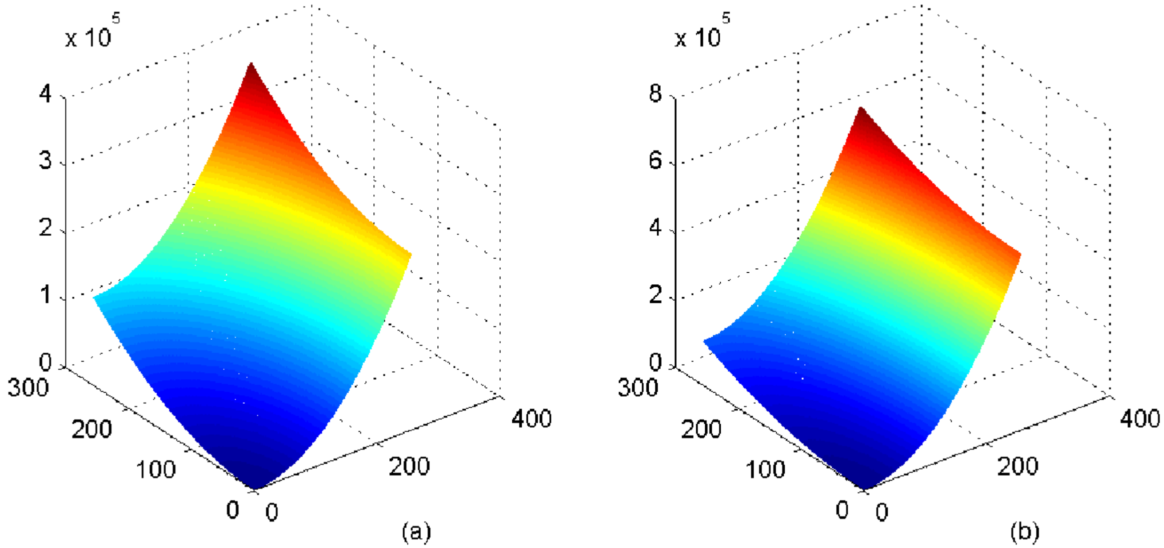

Figure 4. The surface of the nonlinear form for (a) $(p_1, p_2) = (1, 2)$ and (b) $(p_1, p_2) = (1, 4)$.

Theses surfaces are calculated for the above nonlinear form in arithmetic modulo 256 in parts a and b for the frequency-points $(p_1, p_2)=(1,2)$ and (1,4), respectively.

2-D Paired transformations by forms

$$L_{2,2}(n_1, n_2; p_1, p_2) = (2n_1^2 + n_1)p_1 + (2n_2^2 + n_2)p_2$$

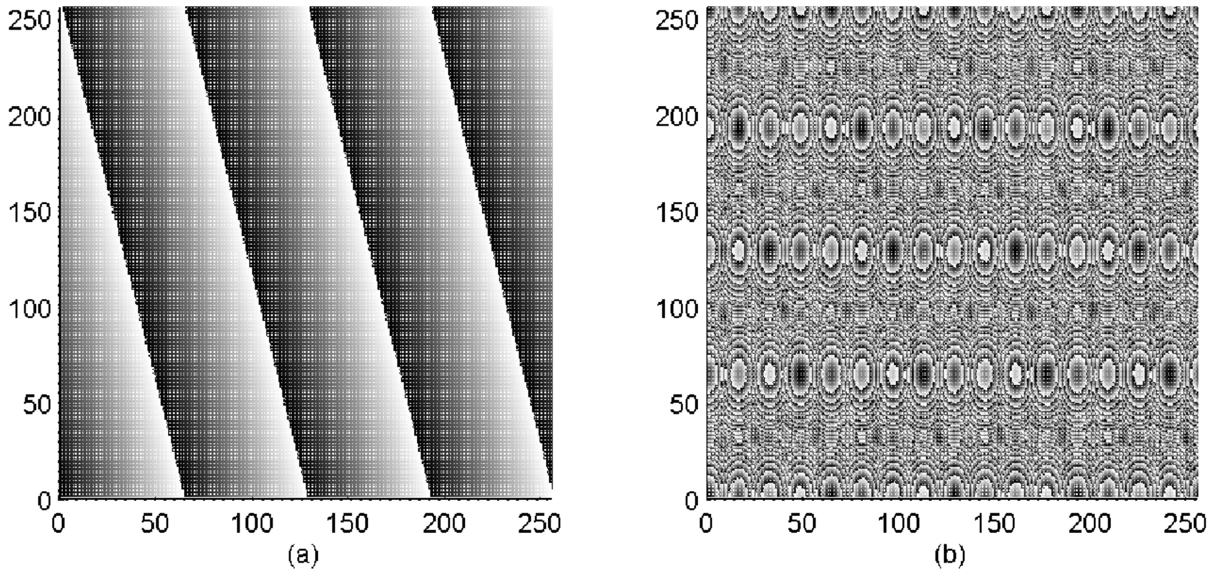

Figure 5. The images of the surfaces of the nonlinear form for (a) $(p_1, p_2) = (1, 2)$ and (b) $(p_1, p_2) = (1, 4)$.

The surfaces are calculated for the above nonlinear form in arithmetic modulo 256 in parts a and b for the frequency-points (p_1, p_2)=(1,2) and (1,4), respectively.

2-D discrete Fourier transforms by forms

Definition 3.2. The two-dimensional $N \times N$-point discrete transform written in the form

$$F_{p_1,p_2} = \left((\mathcal{F}_{N,N}, L) \circ f\right)_{p_1,p_2} = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2} W(L_k(n_1, n_2; p_1, p_2))$$

$$= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2} W^{L_k(n_1,n_2;p_1,p_2)}, \quad p_1, p_2 = 0 : (N-1),$$

is called *the two-dimensional Fourier transform by the form L* and is denoted by $(\mathcal{F}_{N,N}; L)$

When L is of the form

$$L(n_1, n_2; p_1, p_2) = Q^1(n_1)p_1 + Q^2(n_2)p_2$$

the 2-D DFT by this form is

$$F_{p_1,p_2} = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2} W^{Q^1(n_1)p_1 + Q^2(n_2)p_2}, \quad p_1, p_2 = 0 : (N-1).$$

2-D discrete Fourier transforms by forms $(\mathcal{F}_{N,N}; L)$

Example 4. The two-dimensional $N{\times}N$-point DFT by forms

$$\begin{aligned} L_{2,2}(n_1,n_2;p_1,p_2) &= (2n_1^2+n_1)p_1+(2n_2^2+n_2)p_2 \\ L_{2,3}(n_1,n_2;p_1,p_2) &= (2n_1^2+n_1)p_1+(2n_2^3+n_2)p_2. \end{aligned}$$

The correspondinf 2-D DFT by this forms are

$$F_{p_1,p_2} = \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} f_{n_1,n_2} W^{(2n_1^2+n_1)p_1+(2n_2^2+n_2)p_2}$$

$$F_{p_1,p_2} = \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} f_{n_1,n_2} W^{(2n_1^2+n_1)p_1+(2n_2^3+n_2)p_2}.$$

Image and its 2-D DFT by a form L

$$L_{2,2}(n_1, n_2; p_1, p_2) = (2n_1^2 + n_1)p_1 + n_2 p_2$$

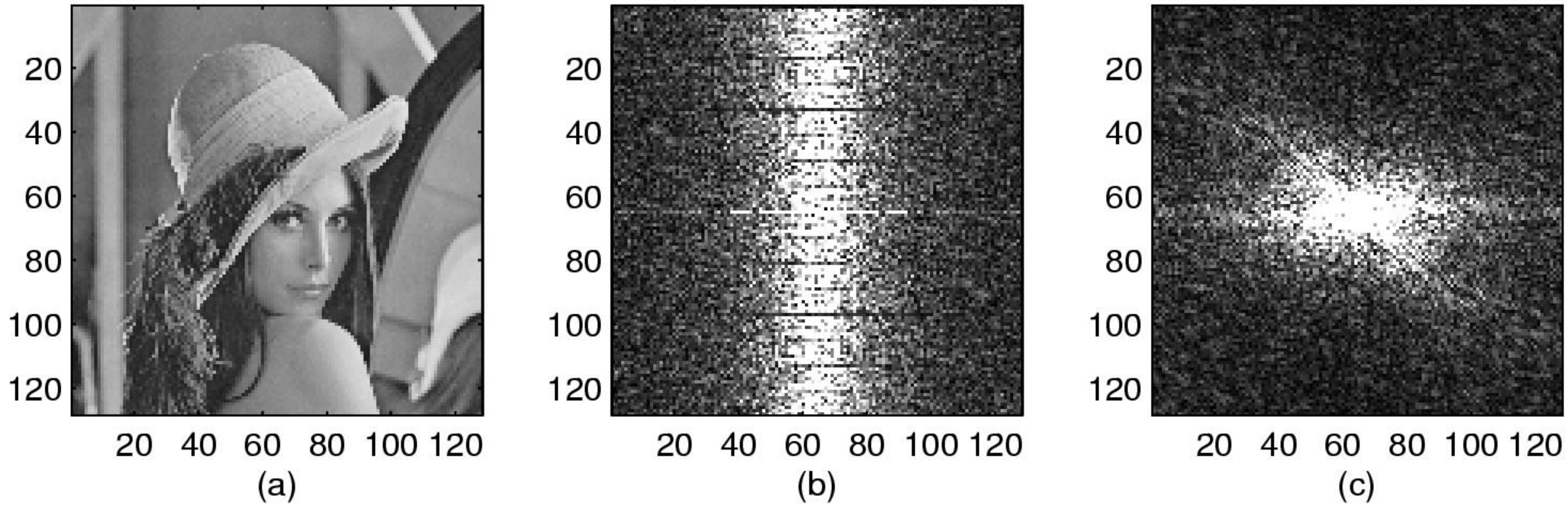

Figure 6. (a) "Lena" image resampled to the size 128×128, (b) 2-D DFT by the form $L_{2,2}$ in the absolute scale and shifted to the center, and (c) the traditional 2-D DFT.

Conclusion

A new concept of the two-dimensional Fourier transforms which generalizes the traditional 2-D DFT is pre-sented. The 2-D DFT is defined in the general case, when the form of relation between the spatial points (x,y) and frequency points (ω_1,ω_2) is defined in the exponential kernel of the transformation by a nonlinear form $L(x,y;\omega_1,\omega_2)$.

The traditional concept of the 2-D DFT is defined for the Diophantus form $x\omega_1+y\omega_2$ and this 2-D DFT is the particular case of the Fourier transforms described by such forms $L(x,y;\omega_1,\omega_2)$. The specialcase of the N×N-point 2-D Fourier transforms, when $N=2^r$, $r>1$, is analyzed and effective representation of these transforms is proposed. Together with the traditional 2-D DFT, the proposed 2-D DFTs can be used in image processing in image filtration and image enhancement..

References

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THANK YOU VERY MUCH!

QUESTIONS, PLEASE?