Fast Heap Transform-Based QR-Decomposition of Real and Complex Matrices: Algorithms and Codes

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Introduction

In this paper, we describe a new look on the application of Givens rotations to the QR-decomposition problem, which is similar to the method of Householder transformations. We apply the concept of the discrete heap transform, or signal-induced unitary transforms which had been introduced by Grigoryan (2006) and used in signal and image processing.

Both cases of real and complex nonsingular matrices are considered and examples of performing QR-decomposition of square matrices are given.

The proposed method of QR-decomposition for the complex matrix is novel and differs from the known method of complex Givens rotation and is based on analytical equations for the heap transforms.

Many examples illustrated the proposed heap transform method of QR-decomposition are given, algorithms are described in detail, and MATLAB-based codes are included.

Introduction

Methods of QR-decomposition (or factorization) of a nonsingular matrix into a unitary (or orthogonal in the case of real matrices) matrix and a triangular matrix are well known in mathematics. QR-decomposition is used in many applications in computing and data analysis. For example, the QR-decomposition is an important task for many MIMO signal detection schemes.

QR-decomposition is the problem of the solution of the linear system of equations, written in matrix form as

$$Ax = y, \qquad (x = A^{-1}y).$$

The solution x can be found after the factorization of the matrix $A = QR$, where Q is an orthogonal matrix and R is a right triangular matrix, in the case when the dimensions of the known vector y and unknown x are equal.

There are several methods for computing the QR-decomposition:

1. The Gramm-Schmidt process,

2. The method of Cholesky factorization,

3. The Householder transformations, and

4. The Givens rotations.

• In the Given rotations, each rotation zeros one element in the sub-diagonal of the matrix. Therefore, a sequence of $N(N-1)/2$ plane rotations are required for reduction of a square $(N \times N)$ matrix to triangular form.

The Givens rotations require a large number of arithmetical operations, including multiplications and $N(N-1)/2$ square roots.

• The method of Householder transforms is the most applied method for QR-decomposition, which reduces the number of square roots to at most $2(N-1)$ and uses approximately $4N^3/3$ multiplications.

Since many signal processing systems are required to process complex matrices and solve linear equations containing complex values, algorithms which are able to handle such equations are in demand.

• The heap transform which is a fast and efficient method for signal processing can be used for QR-decomposition of the complex matrices. The heap transform is a fast transform, can be used for different length of the signal, and easy to implement. In heap transform different paths for calculation can be used and selecting a suitable and "optimal" paths may lead to reduction of the number of operations used for the QR-decomposition of matrices.

General Approach: Discrete Heap Transforms

Let $f_1(\mathbf{z},\varphi_1,...,\varphi_m),...,f_n(\mathbf{z},\varphi_1,...,\varphi_m)$ be parameterized functions of n-dimensional vector-variable $\mathbf{z}$. $\varphi_1,...,\varphi_m$ are parameters that will be used for tuning the transformation

$$T:\ \mathbf{z} \rightarrow (f_1(\mathbf{z}),\ldots,f_n(\mathbf{z})) \tag{1}$$

to the desired one.

The choice of such parameters φ_k, $k=1:m$, based on specified vector generators $\mathbf{x}$ and the so-called decision equations, to achieve a uniqueness of parameters.

A separable transform T is considered, which means that there exist such transforms T_{φ_1}, T_{φ_2}, ..., T_{φ_m} that

$$T = T_{\varphi_1,\ldots,\varphi_m} = T_{\varphi_{i(m)}}\ldots T_{\varphi_{i(2)}}T_{\varphi_{i(1)}} \tag{2}$$

where $i(k)$ is a permutation of numbers $k=1,2,...,m$.

We consider the case when each T_{φ_k} changes only two components of the input vector $\mathbf{z}=(z_1,...,z_{N-1})'$,

$$\begin{aligned} T_{\varphi_k}:\ \mathbf{z} \rightarrow (z_1,...,z_{k_1-1},\underline{f_{k_1}(\mathbf{z},\varphi_k)},z_{k_1+1},...,\\ z_{k_2-1},\underline{f_{k_2}(\mathbf{z},\varphi_k)},z_{k_2+1},...,z_m), \end{aligned} \tag{3}$$

where the pair of numbers (k_1,k_2) is uniquely defined by k, and $1 \le k_1 < k_2 \le m$.

The operation $k \rightarrow (k_1,k_2)$ defines a path of processing the signal, or *the path of the transform T.*

We assume that all first functions $f_{k_1}(\mathbf{z},\varphi)$ in (3) are equal to a function $f(\mathbf{z},\varphi)$, and all functions $f_{k_2}(\mathbf{z},\varphi)$ equal a function $g(\mathbf{z},\varphi)$. These two functions are considered to be given.

The n-dimensional transformation $T = T(\varphi_1,...,\varphi_m)$ is defined by the transformations

$$T_{k_1,k_2}(\varphi_k):\ (z_{k_1},z_{k_2}) \to (f(z_{k_1},z_{k_2},\varphi_k), g(z_{k_1},z_{k_2},\varphi_k)).$$

The selection of parameters φ_k, $k = 1:m$, is based on the given decision equations, to achieve a uniqueness of parameters and desired properties of the transform T.

Given a system of equations

$$\begin{cases} f(x,y,\varphi) = y_0 \\ g(x,y,\varphi) = a \end{cases} \tag{4}$$

which is called *the system of decision equations*, it is assumed that the two-point transformation

$$T_\varphi : (z_0,z_1) \to (z_0', z_1') = (f(z_0,z_1,\varphi), g(z_0,z_1,\varphi)) \tag{5}$$

is unitary. T_φ is *the basis transformation* of T.

It is also assumed that for a specified subset of numbers a, the equation

$$g(x,y,\varphi) = a$$

has a unique solution with respect to φ for each point (x,y) on the plane or its chosen subset.

Example 1 *Given a real number a, we consider the elementary rotation:*

$$\begin{aligned} f(x,y,\varphi) &= x\cos\varphi - y\sin\varphi, \\ g(x,y,\varphi) &= x\sin\varphi + y\cos\varphi. \end{aligned} \tag{6}$$

The basis transformation is defined as the rotation of the point (x,y) to the horizontal $Y=a$,

$$T_\varphi : (x,y) \to (x\cos\varphi - y\sin\varphi, a), \tag{7}$$

where the rotation angle φ is calculated by

$$\varphi = r(y,x,a) = \arccos\left(\frac{a}{\sqrt{x^2+y^2}}\right) + \arctan\left(\frac{x}{y}\right),$$
$$(\varphi = \arcsin(a/x)\,, \;\; \textit{if } y=0).$$

The signal can be processed as shown in Figure 1 for the five-point signal $\mathbf{x}$.

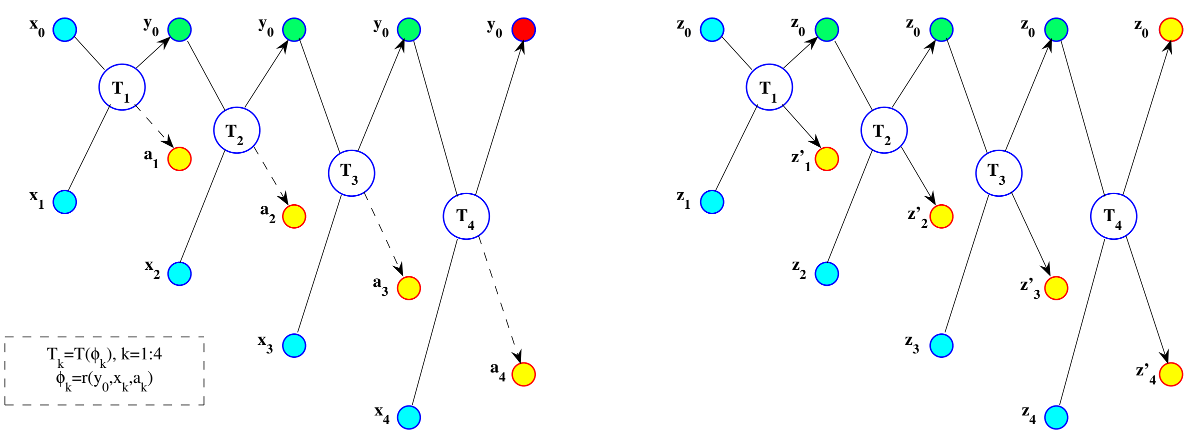

Figure 1. Signal-flow graphs of determination of the five-point transformation by a vector $\mathbf{x}=(x_0,x_1,x_2,x_3,x_4)'$ and processing the input vector $\mathbf{z}=(z_0,z_1,z_2,z_3,z_4)'$.

The special selection of a set of parameters is initiated by the vector-generator through the decision equations with a given set of constants $a \in A = \{a_1, a_2, \ldots, a_{N-1}\}$. We define the following "heap" of elements

$$\begin{aligned} y_0^{(1)} &= f(x_0, x_1, \varphi_1), \\ y_0^{(2)} &= f(y_0^{(1)}, x_2, \varphi_2), \\ y_0^{(3)} &= f(y_0^{(2)}, x_3, \varphi_3), \ \ldots, \\ y_0^{(N-1)} &= f(y_0^{(N-2)}, x_{N-1}, \varphi_{N-1}), \end{aligned} \tag{8}$$

where angles φ_k are calculated by

$$\varphi_k = r(y^{(k-1)}, x_k, a_k),\ k = 1 : (N-1),\ (y^{(0)} = x_0),$$

The heap transform of the vector-generator $\mathbf{x}$ itself equals

$$T(\mathbf{x}) = (y_0, a_1, a_2, \ldots, a_{N-1}), \quad (y_0 = x_0^{(N-1)}).$$

When all parameters $a_k = 0$, i.e., when the whole energy

$$E[\mathbf{x}] = \sqrt{x_0^2 + x_1^2 + \cdots + x_{N-1}^2}$$

of the vector $\mathbf{x}$ is collected in one heap, and then transfered to the first component. This is the case of the Givens rotations of vectors, or points (y_0, x_k) on the horizontal $Y = 0$.

The input signal $\mathbf{z}$ is processed in the same order, or the path P, as the vector-generator $\mathbf{x}$, when composed the heap transform.

Example 2:

$N = 8$ and the generator is $\mathbf{x} = (1, -1, 2, -1, 1, -1, 1, -1)'$. The matrix $\mathbf{T}$ of the heap transformation generated by this vector can be written as $\mathbf{T} = \mathbf{DM}$, where $\mathbf{D}$ is the diagonal matrix and $\mathbf{M}$ is the integer matrix,

$$\mathbf{T} = \mathbf{DM} = \mathbf{D}\begin{bmatrix} 1 & -1 & 2 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 6 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 1 & 7 & 0 & 0 & 0 \\ 1 & -1 & 2 & -1 & 1 & 8 & 0 & 0 \\ -1 & 1 & -2 & 1 & -1 & 1 & 9 & 0 \\ 1 & -1 & 2 & -1 & 1 & -1 & 1 & 10 \end{bmatrix}.$$

The rows of the matrix are the basis functions of the heap transformation and are shown in Figure 2.

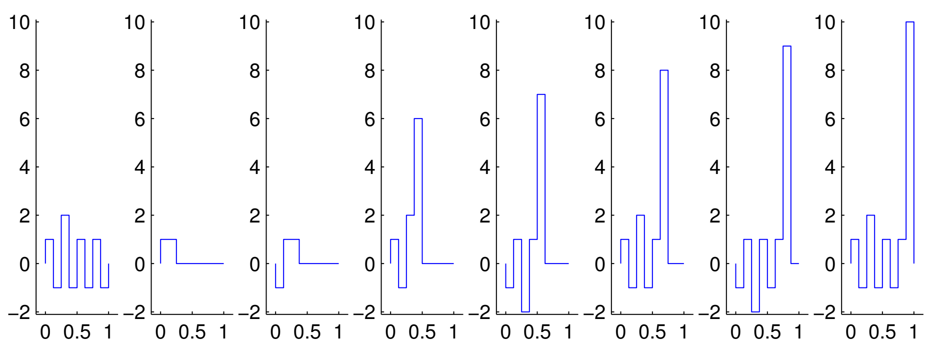

Figure 2. Basis functions of the heap transformation.

The transformation T is called the *N-point discrete* $\mathbf{x}$-*signal-induced heap transformation* (DsiHT), and the vector $\mathbf{x}$ is the generator of this transformation.

Analytical expressions

Analytical formulas can be derived for calculation of components of the heap transform

$$T:(z_0,z_1,...,z_{N-1})\rightarrow(z_0^{(N-1)},z_1^{(1)},...,z_{N-1}^{(1)}). \qquad (9)$$

We consider the partial cross-correlation of $\mathbf{z}$ with the vector-generator $\mathbf{x}$ and energy of $\mathbf{x}$:

$$\begin{array}{lcl} E_k(\mathbf{z},\mathbf{x}) & = & z_0x_0+z_1x_1+...+z_{k-1}x_{k-1}, \\ E_k^2(\mathbf{x}) & = & E_k(\mathbf{x},\mathbf{x})=x_0^2+x_1^2+...+x_{k-1}^2, \end{array}$$

where $k=1:(N-1)$.

The components of the heap transform on the kth iteration ($k=1:(N-1)$) are expressed by correlation data as

$$z_0^{(k)}=\frac{E_{k+1}(\mathbf{z},\mathbf{x})}{E_{k+1}(\mathbf{x})},\qquad z_k^{(1)}=-\frac{E_k(\mathbf{z},\mathbf{x})x_k-z_kE_k^2(\mathbf{x})}{E_{k+1}(\mathbf{x})E_k(\mathbf{x})}. \quad (10)$$

On the final step, the value of the first component is

$$z_0^{(N-1)}=\frac{E_N(\mathbf{z},\mathbf{x})}{E_N(\mathbf{x})} \qquad (11)$$

which is the correlation coefficient of the input signal, $\mathbf{z}$, with the normalized signal-generator $\mathbf{x}$.

The coefficients $t_{n,m}$ of the matrix T of the N-point discrete heap transform can also be obtained from equations in (10), *Complex matrix QR-decomposition methods*

There are several methods for qr-decomposition of the real and complex matrices including, Householder, Givens rotations and Modified Gramm-Schmidt.

We introduce a novel method for complex matrix QR-decomposition based on the heap transformation. For that we consider only the heap transformations defined by decision equations with zero angular equation.

For a real vector $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{N-1})'$, the Householder transformation has a symmetric $(N \times N)$ matrix and is defined as $\mathbf{H} = \mathbf{I} - 2\mathbf{w}\mathbf{w}'$, where

$$\mathbf{w} = \frac{\mathbf{x} - (||\mathbf{x}||, 0, 0, \ldots, 0)'}{\sqrt{2||\mathbf{x}||(||\mathbf{x}|| - x_0)}}. \tag{12}$$

For comparison with the heap transform, Figure 4 shows the complete set of basis functions of the Householder transformation for the signal in Example 2.

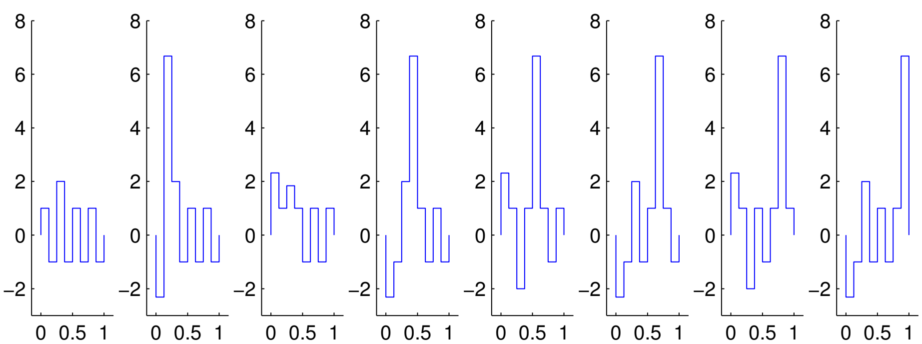

Figure 4. Basis functions of the Householder transform.

In the complex case, the Householder transform matrix $\mathbf{H}_k$, $k = 0 : N-1$, is defined as $\mathbf{H}_k = \mathbf{I} - 2\mathbf{w}_k\mathbf{w}_k^*$ where $\mathbf{w}_k^*$ denotes the complex conjugate of $\mathbf{w}_k$.

Complex DsiHT

For the heap transforms of complex vectors, we first calculate the correlation data as follows:

$$\begin{aligned} E_k^2(\mathbf{x}) &= ||\mathbf{x}|| = |x_0|^2 + |x_1|^2 + \cdots + |x_{k-1}|^2 \\ E_k(\mathbf{z},\mathbf{x}) &= z_0\bar{x}_0 + z_1\bar{x}_1 + \cdots + z_{k-1}\bar{x}_{k-1} \end{aligned}$$

$k = 0 : (N-1)$. The transform defined by similar equations (10) and (11) is unitary and is called *the complex heap transformation.*

The basic 2-D T, which is defined by a complex vector $(x_0, x_1)'$ and then applied to a complex input $(z_0, z_1)'$, is calculated in matrix form as follows:

$$\mathbf{T} : \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \to \frac{\text{sign}(\text{Real}(x_0))}{\sqrt{|x_0|^2 + |x_1|^2}} \begin{bmatrix} \bar{x}_0 & \bar{x}_1 \\ -x_1 & x_0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}. \quad (13)$$

When $(z_0, z_1)' = (x_0, x_1)'$, we obtain the real transform $\mathbf{T}\cdot(x_0, x_1)' = \left(A\sqrt{|x_0|^2 + |x_1|^2}, 0\right)'$ where $A = \text{sign}(\text{Real}(x_0))$.

This transform differs from the known definition of the complex Givens rotation, whose matrix is calculated as

$$\mathbf{R}(c,s) = \begin{bmatrix} c & s \\ -\bar{s} & c \end{bmatrix} = \frac{1}{\sqrt{|x_0|^2 + |x_1|^2}} \begin{bmatrix} |x_0| & \text{sign}(x_0)\bar{x}_1 \\ -\overline{\text{sign}}(x_0)x_1 & |x_0| \end{bmatrix}$$

Let $B = \text{sign}(x_0)$ be a complex number with norm one. The Givens rotation results in the complex transform

$$\mathbf{R}(c,s) \cdot (x_0, x_1)' = \left(B\sqrt{|x_0|^2 + |x_1|^2}, 0\right)'.$$

Example 5:

$N = 5$ and $\mathbf{x} = (1+3i, -2+4i, 3+2i, 4-2i, 2+1i)'$.

The complex heap transformation generated by this vector has the following matrix $\mathbf{T}_\mathbf{x}$:

$$\begin{bmatrix} 0.1213-0.3638i & -0.2425-0.4851i & 0.3638-0.2425i & \cdots \\ -0.5774-0.5774i & 0.5774 & 0 & \cdots \\ -0.2506+0.1949i & -0.0557+0.4455i & 0.8353 & \cdots \\ 0.0384+0.2690i & 0.3074+0.2306i & -0.1537+0.2690i & \cdots \\ -0.0764+0.0764i & 0+0.1528i & -0.1222+0.0153i & \cdots \end{bmatrix}$$

$$\begin{matrix} \cdots & 0.4851+0.2425i & 0.2425-0.1213i \\ \cdots & 0 & 0 \\ \cdots & 0 & 0 \\ \cdots & 0.8262 & 0 \\ \cdots & -0.0917-0.1222i & 0.9625 \end{matrix} \Bigg] .$$

The determinant of this matrix

$$\det(T) = 0.3162 - 0.9487i \quad \text{and} \quad |\det(T)| = 1.$$

The transform of the vector-generator equals

$$\mathbf{T}^*_\mathbf{x}[\mathbf{x}] = (8.2462, 0, 0, 0, 0)'$$

where 8.2462 is $||\mathbf{x}||$. The matrix $\mathbf{T}^*_\mathbf{x}$ is the complex conjugate to $\mathbf{T}_\mathbf{x}$; its first row is the normalized generator,

$$(0.1213{-}0.3638i, \cdots, 0.2425{-}0.1213i) = \frac{(1+3i, \cdots, 4-2i, 2+1i)}{8.2462}.$$

Triangularization of the square complex matrix

The complex heap transformations can be used for triangularization of the square complex matrix. As an example, we consider the following matrix:

$$\mathbf{X} = \begin{bmatrix} 1+1i & -2+1i & 2-3i & 3+4i \\ 3+1i & 1+1i & 1+2i & 3-2i \\ 2-3i & 3+1i & 4+1i & 2-2i \\ 1+1i & 3-1i & 4+3i & 4-2i \end{bmatrix},$$

where $\det(X) = 74 + 23i$. The method of heap transforms results in the decomposition of

$$\mathbf{X} = \mathbf{R}'\mathbf{X}_2$$

where $\mathbf{R}$ is the unitary matrix

$$\mathbf{R} = \left[\begin{array}{rrc} 0.1925-0.1925i & 0.5774-0.1925i & \cdot \\ -0.4216-0.0590i & 0.1265+0.1433i & \cdot \\ 0.6424+0.4976i & -0.2617+0.0839i & \cdot \\ 0.1193-0.2650i & -0.5875-0.4152i & \cdot \end{array} \right.$$

$$\left. \begin{array}{crr} \cdot & 0.3849+0.5774i & 0.1925-0.1925i \\ \cdot & 0.2445-0.2276i & 0.7167+0.3963i \\ \cdot & 0.4471-0.0668i & 0.1876+0.1572i \\ \cdot & -0.2745+0.3567i & 0.4420 \end{array} \right].$$

$\mathbf{X}_2$ is the upper triangular matrix

$$\mathbf{X}_2 = \begin{bmatrix} 5.1962 & 1.5396+2.3094i & 3.0792+2.5019i & 5.0037-2.3094i \\ 0 & 4.3928 & 1.7031+4.6120i & 3.3304-2.4788i \\ 0 & 0 & 4.4820 & 1.1447+4.0632i \\ 0 & 0 & 0 & 0.7574-0.0102i \end{bmatrix}$$

QR-decomposition: Complex matrices

$$\mathbf{x} = (3+12i, 2+1i, 2-4i, 9+2i).$$

The matrix $\mathbf{H}_1$ of the Householder transformation is

$$\mathbf{H}_1 = \begin{bmatrix} -0.76 & -0.08-0.10i & 0.20-0.17i & -0.25-0.50i \\ -0.08+0.10i & 0.98 & -0.02i & -0.04-0.01i \\ 0.20+0.17i & 0.02i & 0.95 & -0.02+0.08i \\ -0.25+0.50i & -0.04+0.01i & -0.02-0.08i & 0.81 \end{bmatrix}$$

The transform of the vector $\mathbf{x}$ is

$$\mathbf{H}_1\mathbf{x} = (-3.93-15.73i, 0, 0, 0)^{'}.$$

The matrix $\mathbf{T}_1$ of the the heap transformation generated with the same vector $\mathbf{x}$ equals

$$\mathbf{T}_1 = \begin{bmatrix} 0.18-0.74i & 0.12-0.06i & 0.12+0.24i & 0.55-0.12i \\ -0.11+0.13i & 0.98 & 0 & 0 \\ 0.25+0.21i & 0.05i & 0.94 & 0 \\ -0.23+0.47i & -0.09+0.02i & -0.04-0.18i & 0.82 \end{bmatrix}$$

The transform of the vector $\mathbf{x}$ is

$$\mathbf{T}_1\mathbf{x} = (16.21, 0, 0, 0)^{'}.$$

The matrix $\mathbf{T}_1$ has three zero elements and it for larger matrices the number of zeros of matrix $\mathbf{T}$ of the heap transformation considerably increases.

The number of zeros in the $(N \times N)$ matrix of the heap transformation equals $n_0 = N(N-3)/2+1$.

Matrix decomposition

Let $\mathbf{X}$ be a real square matrix $(N \times N)$ with $\det(\mathbf{X}) \neq 0$. We denote coefficients of this matrix as

$$\mathbf{X} = \begin{bmatrix} a_0 & b_0 & c_0 & \dots & d_0 \\ a_1 & b_1 & c_1 & \dots & d_1 \\ a_2 & b_2 & c_2 & \dots & d_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{N-1} & b_{N-1} & c_{N-1} & \dots & d_{N-1} \end{bmatrix}. \tag{14}$$

Let vector $\mathbf{a}$ be the first column of this matrix $\mathbf{a} = (a_0, a_1, a_2, \dots, a_{N-1})'$. We denote by $\mathbf{T_a}$ the matrix of the heap transformation $T_\mathbf{a}$ generated by this vector. This matrix contains an upper triangular submatrix with $N(N-3)/2+1$ zeros. The product of two matrices

$$\mathbf{X}_1 = \mathbf{T_a X}$$

results in a matrix $(N \times N)$ with the first column

$$\bar{\mathbf{a}} = \mathbf{T_a}(\mathbf{a}) = (||\mathbf{a}||, 0, 0, \dots, 0)'.$$

The matrix $\mathbf{X}_1$ has the following form:

$$\mathbf{X}_1 = \begin{bmatrix} ||\mathbf{a}|| & \bar{b}_0 & \bar{c}_0 & \dots & \bar{d}_0 \\ 0 & \bar{b}_1 & \bar{c}_1 & \dots & \bar{d}_1 \\ 0 & \bar{b}_2 & \bar{c}_2 & \dots & \bar{d}_2 \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \bar{b}_{N-1} & \bar{c}_{N-1} & \dots & \bar{d}_{N-1} \end{bmatrix} = \begin{bmatrix} ||\mathbf{a}|| & \bar{b}_0 & \bar{c}_0 & \dots & \bar{d}_0 \\ 0 & & & & \\ 0 & & \mathbf{Y}_1 & & \\ \dots & & & & \\ 0 & & & & \end{bmatrix}$$

$\mathbf{Y}_1$ is the $(1,1)$ minor of this product.

We can repeat the process described above for the submatrix $\mathbf{Y}_1$, by considering its first vector-column $\mathbf{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_{N-1})'$ as a generator for the $(N-1)$-point heap transformation $T_\mathbf{b}$, whose matrix we denote by $\mathbf{T_b}$.

The product of two matrices

$$\mathbf{X}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{T_b} \end{bmatrix} \mathbf{X}_1 = (1 \oplus \mathbf{T_b})\mathbf{T_a}\mathbf{X}$$

is a matrix $(N \times N)$ with the second column equal

$$\bar{\mathbf{b}} = \begin{pmatrix} \bar{b}_0 \\ \mathbf{T_b}(\mathbf{b}) \end{pmatrix} = (\bar{b}_0, ||\mathbf{b}||, 0, 0, \ldots, 0)'.$$

On this step, the matrix $\mathbf{X}$ is transformed to the matrix $\mathbf{X}_2$ which has the following form:

$$\mathbf{X}_2 = \begin{bmatrix} ||\mathbf{a}|| & \bar{b}_0 & \bar{c}_0 & \ldots & \bar{d}_0 \\ 0 & ||\mathbf{b}|| & \ldots & \ldots & \ldots \\ 0 & 0 & & & \\ \ldots & \ldots & & \mathbf{Y}_2 & \\ 0 & 0 & & & \end{bmatrix} \tag{15}$$

where $\mathbf{Y}_2$ is the $(2, 2)$ minor of this product and has the size $(N-2) \times (N-2)$.

On the next step, we consider the first vector $\mathbf{c}$ of the submatrix $\mathbf{Y}_2$ and construct the matrix $\mathbf{T_c}$ of heap transformation generated by $\mathbf{c}$. Then, the $(3, 3)$ minor $\mathbf{Y}_3$ of the product of the matrices

$$\mathbf{X}_3 = (1 \oplus 1 \oplus \mathbf{T_c})\mathbf{X}_2 = (1 \oplus 1 \oplus \mathbf{T_c})(1 \oplus \mathbf{T_b})\mathbf{T_a}\mathbf{X}$$

is transformed into a matrix where the first column is zero, except in the first row.

The process of such transformations is repeated until we obtain the upper triangular matrix:

$$\begin{aligned}\mathbf{X}_{N-1} &= (1 \oplus \cdots \oplus 1 \oplus \mathbf{T_d}) \cdots (1 \oplus 1 \oplus \mathbf{T_c})(1 \oplus \mathbf{T_b})\mathbf{T_a}\mathbf{X} \\ &= \left[\begin{array}{ccccc} \|\mathbf{a}\| & \bar{b}_0 & \bar{c}_0 & \dots & \bar{d}_0 \\ 0 & \|\mathbf{b}\| & \dots & \dots & \dots \\ 0 & 0 & \|\mathbf{c}\| & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \|\mathbf{d}\| \end{array}\right].\end{aligned}$$

The determinant of the matrix equals

$$\det \mathbf{X}_{N-1} = \det \mathbf{X} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \|\mathbf{c}\| \cdots \|\mathbf{d}\|,$$

since all heap transformations have determinant one.

We denote by $\mathbf{R}$ the product of the following matrices

$$\mathbf{R} = (1 \oplus \cdots \oplus 1 \oplus \mathbf{T_d}) \cdots (1 \oplus 1 \oplus \mathbf{T_c})(1 \oplus \mathbf{T_b})\mathbf{T_a}. \quad (16)$$

Each heap transformation is unitary, and $\mathbf{R}$ is unitary.

The inverse matrix $\mathbf{R}^{-1}$ can thus be represented by

$$\mathbf{R}^{-1} = \mathbf{R}' = \mathbf{T}'_{\mathbf{a}}(1 \oplus \mathbf{T}'_{\mathbf{b}})(1 \oplus 1 \oplus \mathbf{T}'_{\mathbf{c}}) \cdots (1 \oplus \cdots \oplus 1 \oplus \mathbf{T}'_{\mathbf{d}}).$$

We obtain the following decompositions of the matrix:

$$\mathbf{X} = \mathbf{R}^{-1}\mathbf{X}_{N-1}, \qquad \mathbf{X}^{-1} = \mathbf{X}_{N-1}^{-1}\mathbf{R}, \qquad (17)$$

where $\mathbf{X}_{N-1}$ and $\mathbf{X}_{N-1}^{-1}$ are the upper triangular matrices and $\mathbf{R}$ is unitary.

Example (3 × 3):

Consider the following (3 × 3) matrix:

$$\mathbf{X} = \begin{bmatrix} 1+1i & 2+2i & 3+3i \\ 2+2i & 3+3i & 2+2i \\ 3+3i & 4+4i & 4+4i \end{bmatrix}.$$

We obtain the following decomposition of the matrix $\mathbf{X}$:

$$\mathbf{X} = \mathbf{R}^{-1}\mathbf{X}_2 = \begin{bmatrix} 0.18+0.18i & 0.61+0.61i & 0.40 \\ 0.37+0.37i & 0.15+0.15i & -0.81 \\ 0.56+0.56i & -0.30-0.30i & 0.40 \end{bmatrix}$$

$$\times \begin{bmatrix} 5.29 & 7.55 & 7.18 \\ 0 & 0.92 & 1.85 \\ 0 & 0 & 1.2247+1.22i \end{bmatrix}.$$

The MATLAB function *"[Q1,R1]=qr(X)"* calculates the matrix decomposition with the following matrices:

$$\mathbf{Q}_1 = \begin{bmatrix} 0.18-0.18i & -0.61-0.61i & -0.28-0.28i \\ -0.37-0.37i & -0.15-0.15i & 0.57+0.57i \\ -0.56-0.56i & 0.30+0.30i & -0.28-0.28i \end{bmatrix}$$

$$\mathbf{R}_1 = \begin{bmatrix} -5.29 & -7.55 & -7.18 \\ 0 & -0.92 & -1.85 \\ 0 & 0 & -1.73 \end{bmatrix}.$$

Example (4×4):

We consider the following (4×4) complex matrix:

$$\mathbf{X} = \begin{bmatrix} 1+1j & -2+1j & 2-3j & 3+4j \\ 3+1j & 1+1j & 1+2j & 3-2j \\ 2-3j & 3+1j & 4+1j & 2-2j \\ 1+1j & 3-1j & 4+3j & 4-2j \end{bmatrix}.$$

The calculation of the matrix decomposition by our function *"[U,A]=m_lu_encomp(X)"* results in the following matrices $\mathbf{U}$ and $\mathbf{A}$:

$$\begin{bmatrix} 0.1925-0.1925i & 0.5774-0.1925i & 0.3849+0.5774i & 0.1925-0.192 \\ -0.4216-0.0590i & 0.1265+0.1433i & 0.2445-0.2276i & 0.7167+0.396 \\ 0.6424+0.4976i & -0.2617+0.0839i & 0.4471-0.0668i & 0.1876+0.157 \\ 0.1193-0.2650i & -0.5875-0.4152i & -0.2745+0.3567i & 0.4420+0.000 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 5.1962 & 1.5396+2.3094i & 3.0792+2.5019i & 5.0037-2.3094i \\ 0 & 4.3928 & 1.7031+4.6120i & 3.3304-2.4788i \\ 0 & 0 & 4.4820 & 1.1447+4.0632i \\ 0 & 0 & 0 & 0.7574-0.0102i \end{bmatrix}$$

The number of operations of multiplication in the QR-decomposition by the heap transforms is estimated as $(4/3)N^3 + 2N^2 - 2$. In QR-decomposition of the complex matrices by the heap transform, the path is a very important characteristic.

The heap transforms with the strong path or optimal path may result in effective QR-decompositions of real and complex matrices (Grigoryan, *ALAM,* vol. 4, no. 2, June 2014).

Conclusion

The signal induced transformation, DsiHT, and its particular case, the heap transform can be used in calculation of the QR-decomposition of the real and complex matrices.

For both real and complex matrices, the N-point DsiHT has a fast and unique algorithm for any length N of the signal and can be applied for different paths of the transforms.

The selection of "an optimal" path is very important in the QR-decomposition.

The proposed method was described for square matrices, and it also can be used for non square real and complex matrices.

References

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