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2-D Left-Side Quaternion Discrete Fourier Transform Fast Algorithm

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Abstract

We describe a fast algorithm for the 2-D left-side QDFT which is based on the concept of the tensor representation when the color or four-componnrnt quaternion image is described by a set of 1-D quaternion signals and the 1-D left-side QDFTs over these signals determine values of the 2-D left-side QDFT at corresponding subset of frequency-points. The efficiency of the tensor algorithm for calculating the fast left-side 2-D QDFT is described and compared with the existent methods.

- The proposed algorithm of the $2^r \times 2^r$-point 2-D QDFT uses $18N^2$ less multiplications than the well-known methods:
  - column-row method
  - method of symplectic decomposition.
- The proposed algorithm is simple to apply and design, which makes it very practical in color image processing in the frequency domain.
- The method of *quaternion image tensor representation* is uique in a sense that it can be used for both left-sida and right-side 2-D QDFTs.

Inroduction – Quanterions in Imaging

- The quaternion can be considered 4-dimensional generation of a complex number with one real part and three imaginary parts.

  Any quaternion may be represented in a hyper-complex form

$$Q = a + bi + cj + dk = a + (bi + cj + dk),$$

  where a, b, c, and d are real numbers and i, j, and k are three imaginary units with the following multiplication laws:

$$ij = -ji = k,\ \ jk = -kj = i,\ \ ki = -ik = -j,\ \ i^2 = j^2 = k^2 = ijk = -1.$$

- The commutativity does not hold in quaternion algebra, i.e., $Q_1Q_2 \neq Q_2Q_1$.

- A unit pure quaternion is $\mu = i\mu_i + j\mu_j + k\mu_k$ such that $|\mu| = 1$, $\mu^2 = -1$

  For instance, the number $\mu = (i+j+k)/\sqrt{3}$, $\mu = (i+j)/\sqrt{2}$, and $\mu = (i-k)/\sqrt{2}$

- The exponential number is defined as

$$\exp(\mu x) = \cos(x) + \mu \sin(x) = \cos(x) + i\mu_i \sin(x) + j\mu_j \sin(x) + k\mu_k \sin(x)$$

RGB Model for Color Images

- A discrete color image $f_{n,m}$ in the RGB color space can be transformed into imaginary part of quaternion numbers form by encoding the red, green, and blue components of the RGB value as a pure quaternion (with zero real part):

$$f_{n,m} = 0 + (r_{n,m}i + g_{n,m}j + b_{n,m}k)$$

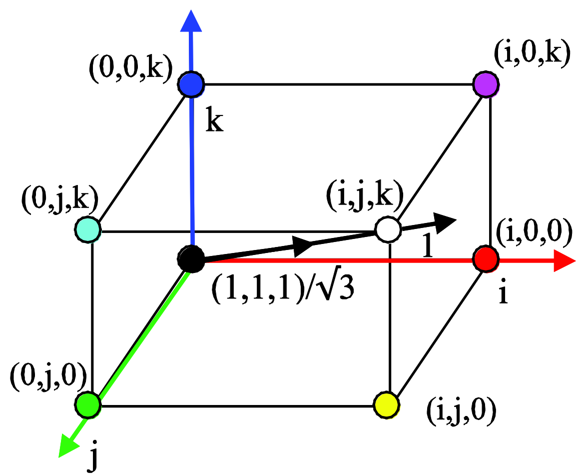


Figure 1: RBG color cube in quaternion space.

- The advantage of using quaternion based operations to manipulate color information in an image is that we do not have to process each color channel independently, but rather, treat each color triple as a whole unit.

Calculation of the left-side 1-D QDFT

- Let $f_n = (a_n, b_n, c_n, d_n) = a_n + ib_n + jc_n + kd_n$ be the quaternion signal of length N. The left-side 1-D quaternion DFT (LS QDFT) is defined as

$$F_p = Q_1(p) + iQ_i(p) + jQ_j(p) + kQ_k(p) = \sum_{n=0}^{N-1} W_\mu^{np} f_n, \quad p = 0 : (N-1),$$

where μ is a unit pure quaternion $\mu = im_1 + jm_2 + km_3$, $\mu^2 = -1$,

$$W_\mu = W_{N;\mu} = \exp(-\mu 2\pi/N) = \cos(2\pi/N) - \mu \sin(2\pi/N).$$

If we denote the N-point LS 1-D DFTs of the parts a_n, b_n, c_n, and d_n of the quaternion signal f_n by A_p, B_p, C_p, and D_p, respectively, we can calculate of the LS 1-D QDFT as

$$F_p = Q_1(p) + iQ_i(p) + jQ_j(p) + kQ_k(p), \quad p = 0 : (N-1).$$

$$\begin{aligned}
Q_1(p) &= \quad \mathrm{Real}(A_p) + m_1\mathrm{Imag}(B_p) + m_2\mathrm{Imag}(C_p) + m_3\mathrm{Imag}(D_p) \\
Q_i(p) &= -m_1\mathrm{Imag}(A_p) + \mathrm{Real}(B_p) + m_3\mathrm{Imag}(C_p) - m_2\mathrm{Imag}(D_p) \\
Q_j(p) &= -m_2\mathrm{Imag}(A_p) - m_3\mathrm{Imag}(B_p) + \mathrm{Real}(C_p) + m_1\mathrm{Imag}(D_p) \\
Q_k(p) &= -m_3\mathrm{Imag}(A_p) + m_2\mathrm{Imag}(B_p) - m_1\mathrm{Imag}(C_p) + \mathrm{Real}(D_p)
\end{aligned}$$

If the real part is zero, $a_n = 0$, and $f_n = (0, b_n, c_n, d_n) = a_n + ib_n + jc_n + kd_n$, the number of operations of multiplication and addition can be estimated as

$$m_{QFT}(N) = 3m_{FT}(N) + 9N \text{ and } a_{QFT}(N) = 3a_{FT}(N) + 9N$$

Multiplications and Additions for the left-side 1-D QDFT

- In the general case of the quaternion signal f_n, the number of operations of multiplication and addition for LS 1-D QDFT can be estimated as

$$m_{QFT}(N) = 4m_{FT}(N) + 12N \text{ and } a_{QFT}(N) = 4a_{FT}(N) + 12N$$

The number of operations for the left-side 1-D QDFT can be estimated as

$$\begin{aligned} m_{QFT}(N) &= 8\mu_{FT}(N) + 12N = 4Nr + 16, \\ a_{QFT}(N) &= 4\alpha_{FT}(N) + 12N = 2N(r+15) - 4(r^2 + 3r + 4). \end{aligned} \tag{1}$$

Here, we consider that for the fast N-point discrete paired transform-based FFT, the estimation for multiplications and additions are

$$\mu_{FT}(N) = 2^{r-1}(r-3) + 2 \qquad \alpha_{FT}(N) = (2^r 6 - r^2 - 3r - 6)$$

and two 1-D DFTs with real inputs can be calculated by one DFT with complex input,

$$2m_{FT}(N) = 4 \times \mu_{FT}(N) \quad \text{and} \quad 2a_{FT}(N) = 2 \times \alpha_{FT}(N).$$

Number of multiplications: Special case

In the special case when $\mu = (i + j + k)/\sqrt{3}$, equation (8) can be written as

$$F_p = Q_1(p) + iQ_i(p) + jQ_j(p) + kQ_k(p),$$

$$\begin{aligned}
Q_1(p) &= \mathrm{Real}(A_p) + [\mathrm{Imag}(B_p) + \mathrm{Imag}(C_p) + \mathrm{Imag}(D_p)]/\sqrt{3}, \\
Q_i(p) &= \mathrm{Real}(B_p) - [\mathrm{Imag}(A_p) - \mathrm{Imag}(C_p) + \mathrm{Imag}(D_p)]/\sqrt{3}, \\
Q_j(p) &= \mathrm{Real}(C_p) - [\mathrm{Imag}(A_p) + \mathrm{Imag}(B_p) - \mathrm{Imag}(D_p)]/\sqrt{3}, \\
Q_k(p) &= \mathrm{Real}(D_p) - [\mathrm{Imag}(A_p) - \mathrm{Imag}(B_p) + \mathrm{Imag}(C_p)]/\sqrt{3},
\end{aligned}$$

where $p = 0 : (N - 1)$.

The number of operations of multiplication and addition equal

$$m_{QFT}(N) = 4m_{FT}(N) + 4N,$$

or 8N operations of real multiplication less than in (1).

The direct and inverse left-side 2-D QDFTs

- Given color-in-quaternion image $f_{n,m} = a_{n,m} + ib_{n,m} + jc_{n,m} + kd_{n,m}$, we consider the concept of the left-side 2-D QDFT in the following form:

$$F_{p,s} = \sum_{n=0}^{N-1} W_{N;\mu}^{np} \left[\sum_{m=0}^{M-1} W_{M;\mu}^{ms} f_{n,m} \right] \qquad \text{or} \qquad F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} W_{K;\mu}^{M_1 np + N_1 ms} f_{n,m},$$

where $p = 0 : (N-1)$ and $s = 0 : (M-1)$ and $K = NM/\text{g.c.d.}(N, M)$.

The inverse left-side 2-D QDFT is:

$$f_{n,m} = \frac{1}{NM} \sum_{p=0}^{N-1} \sum_{s=0}^{M-1} W_{K;\mu}^{-(M_1 np + N_1 ms)} F_{p,s}$$

1. Column-row algorithm: The calculation of the separable 2-D N×N-point QDFT by formula requires 2N N-point LS 1-D QDFTs. Each of the 1-D QDFT requires two N-point LS 1-D DFTs. Therefore, the column-row method uses 4N N-point LS 1-D DFTs and multiplications

$$m_{QFT}(N, N) = 8N^2 \log_2 N + 32N$$

2. The calculation the LS 2-D QDFT by the symplectic decomposition of the color image

$$m_{QFT}(N, N) = 8N^2 \log_2 N - 6N^2 + 32N.$$

Example: N×N-point left-side 2-D QDFT

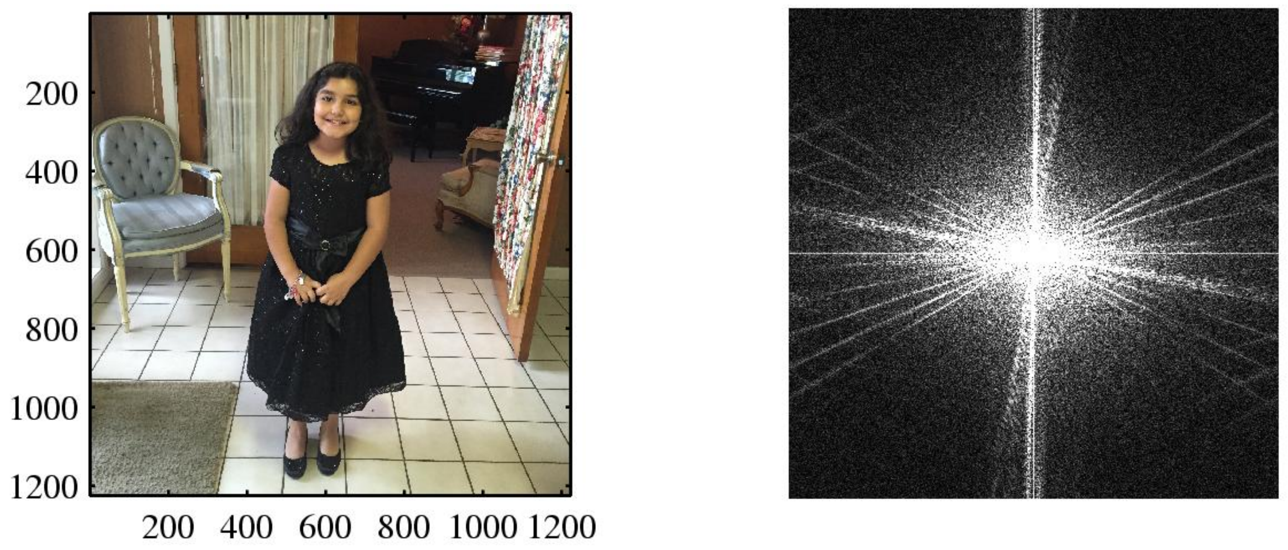


Figure 2. (a) The color image of size 1223×1223 and (b) the 2-D left-side quaternion discrete Fourier transform of the color-in-qiuaternion image (in absolute scale and cyclically shifted to the middle).

Tensor Representation of the regular 2-D DFT

Let $f_{n,m}$ be the gray-scale image of size $N{\times}N$.

- The tensor representation of the image $f_{n,m}$ is the 2D-frequency-and-1D-time representation when the image is described by a set of 1-D splitting-signals each of length N

$$\boxed{\chi : \{f_{n,m}\} \to \{f_{T_{p,s}} = \{f_{p,s,t}; t = 0 : (N-1)\}\}_{(p,s)\in J_{N,N}}\,.}$$

The components of the signals are the ray-sums of the image along the parallel lines

$$f_{p,s,t} = \sum_{(n,m)\in X} \{f_{n,m};\ np + ms = t \bmod N\}.$$

Each splitting-signals defines 2-D DFT at N frequency-points of the set

$$T_{p,s} = \{(kp \bmod N, ks \bmod N);\ k = 0 : (N-1)\}$$

on the cartesian lattice $\quad X = X_{N,N} = \{(n,m);\ n,m = 0,1,\ldots,(N-1)\}$

$$\boxed{F_{kp \bmod N, ks \bmod N} = \sum_{t=0}^{N-1} f_{p,s,t} W_N^{kt}, \quad k = 0 : (N-1).}$$

Example: Tensor Representation of the 2-D DFT

1-D splitting-signal of the tensor represntation of the image 512×512

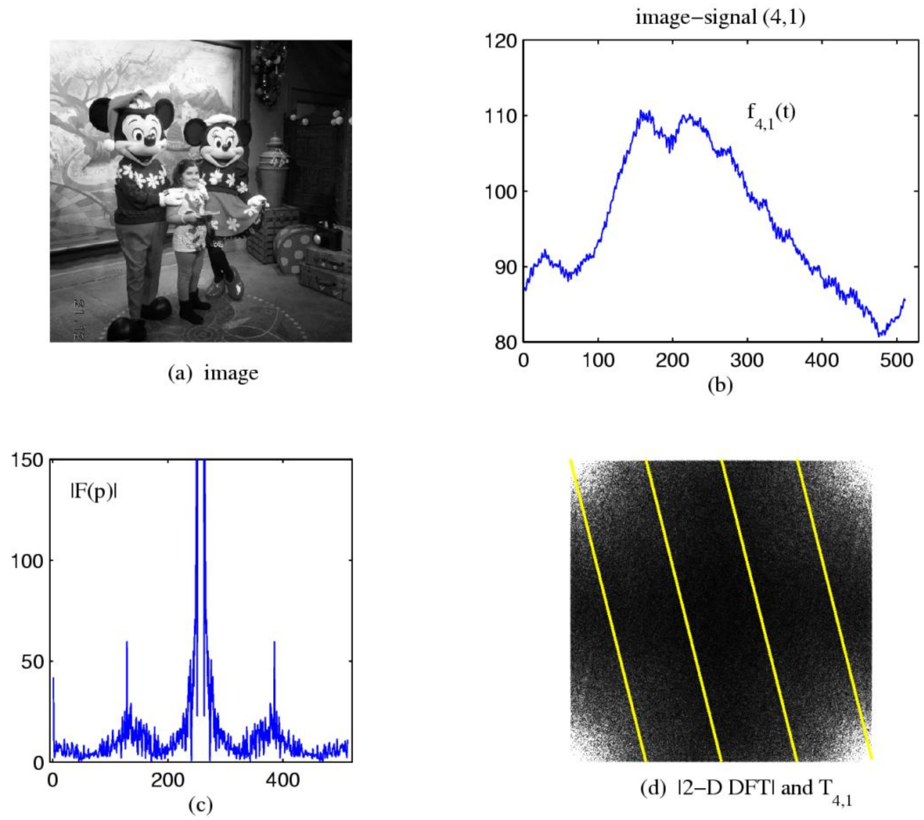


Figure 3. (a) The Miki-Anoush-Mini image, (b) splitting-signal for the frequency-point (4,1), (c) magnitude of the shifted to the middle 1-D DFT of the signal, and (d) the 2D DFT of the image with the frequency-points of the set $T_{4,1}$.

Tensor Representation of the left-side 2-D QDFT

Let $f_{n,m} = a_{n,m} + ib_{n,m} + jc_{n,m} + kd_{n,m}$ be the quaternion image of size $N \times N$, ($a_{n,m} = 0$).

In the tensor representation, the quaternion image is represented by a set of 1-D quaternion splitting-signals each of length N and generated by a set of frequencies (p,s),

$$\boxed{\chi : \{f_{n,m}\} \to \{f_{T_{p,s}} = \{f_{p,s,t}; t = 0 : (N-1)\}\}_{(p,s) \in J_{N,N}}.}$$

$$J_{N,N} = \big\{(1,s); s = 0 : (N-1)\big\} \cup \big\{(2p,1); p = 0 : (N/2-1)\big\}.$$

The components of the signals are defined as

$$\begin{aligned} f_{p,s,t} &= (r_{p,s,t})i + (g_{p,s,t})j + (b_{p,s,t})k = \sum_{(n,m)\in V_{p,s,t}} f_{n,m} \\ &= \left(\sum_{(n,m)\in V_{p,s,t}} r_{n,m}\right) i + \left(\sum_{(n,m)\in V_{p,s,t}} g_{n,m}\right) j + \left(\sum_{(n,m)\in V_{p,s,t}} b_{n,m}\right) k. \end{aligned}$$

Here, the subsets $V_{p,s,t} = \{(n,m) \in X;\ np + ms = t \bmod N\}$

Property of the TT:

$$F_{kp \bmod N, ks \bmod N} = \sum_{t=0}^{N-1} W_{\mu}^{kt} f_{p,s,t}, \quad k = 0 : (N-1).$$

Example: Tensor Representation of the 2-D LS QDFT

The splitting-signal of the tensor represntation of the color image 1223×1223:

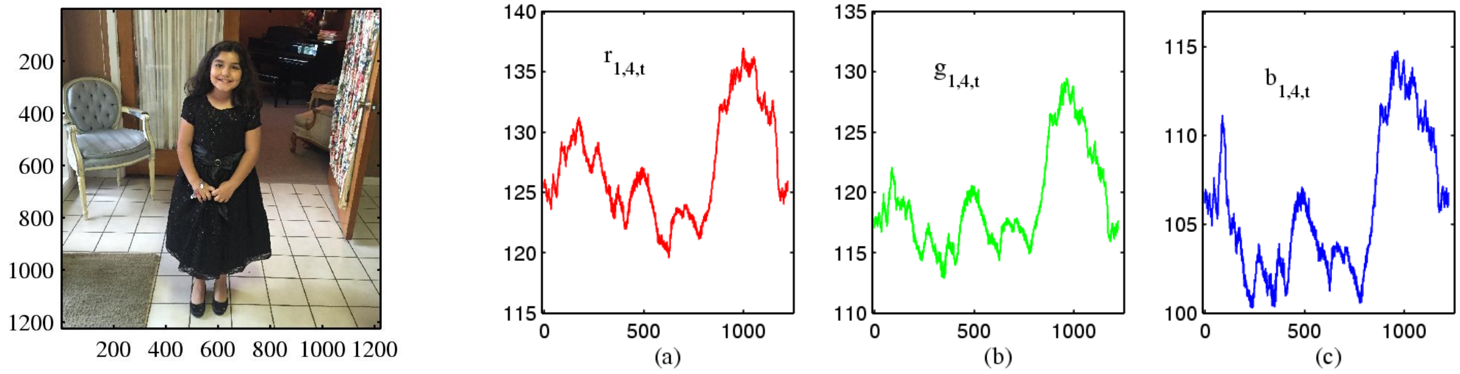


Figure 4. Color image and (a,b,c) components of the splitting-signal generated by (1,4).

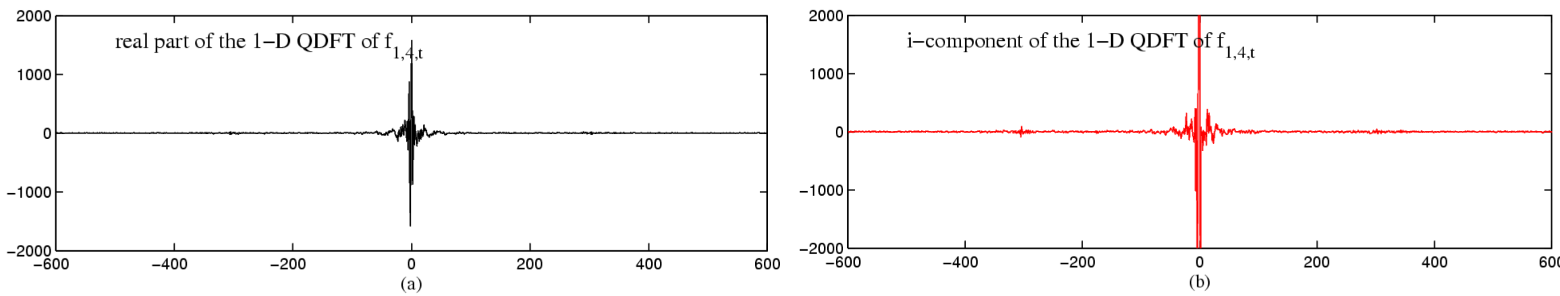
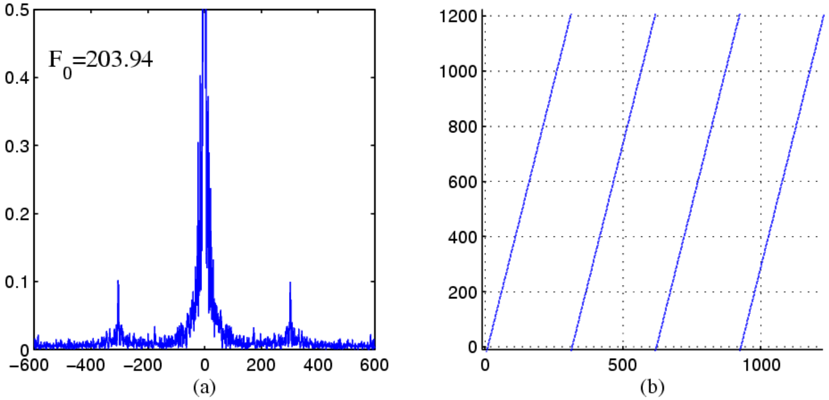


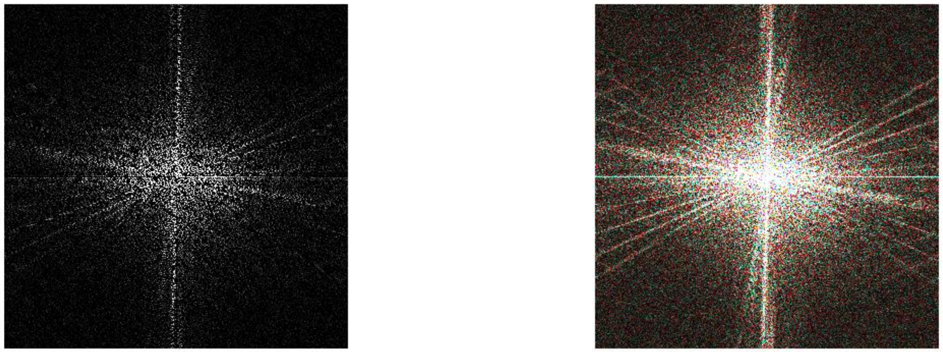
Figure 5. The 123-point left-side DFT of the (1,4) quaternion splitting-signal; (a) the real part and (b) the i-component of the signal.

Example: Tensor Representation of the 2-D LS QDFT



$\mu=(i+2j+k)/\sqrt{6}$

Figure 6. (a) The 1-D left-side QDFT the quaternion splitting-signal $f_{1,4,t}$ (in absolute scale), and (b) the location of 1223 frequency-points of the set $T_{1,4}$ on the Cartesian grid, wherein this 1-D LS QDFT equals the 2-D LS QDFT of the quaternion image.



(a) (b)

Figure 7. (a) The real part and (b) the imaginary part of the left-side 2-D QDFT of the 2-D color-in-quaternion `girl Anoush" image.

Tensor Transform: Direction Quaternion Image Components

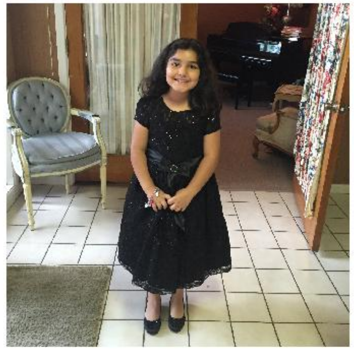
Color image can be reconstructed by its 1-D quaternion splitting-signals or direction color image components defined by

$$d_{n,m} = d_{n,m;p,s} = \frac{1}{N} f_{p,s,(np+ms) \bmod N} \qquad n, m = 0 : (N-1).$$

<u>Statement 1:</u> The discrete quaternion image of size $N \times N$, where N *is* prime, can be composed from its *(N+1)* quaternion direction images or splitting-signals as

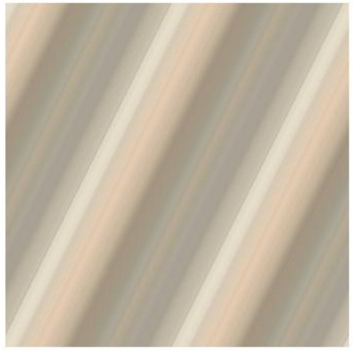
$$\begin{aligned} f_{n,m} &= \sum_{(p,s) \in J_{N,N}} d_{n,m;p,s} \\ &= \frac{1}{N} \left[\sum_{p=0}^{N-1} f_{p,1,(np+m) \bmod N} + f_{1,0,n} \right], \\ & \quad n, m = 0 : (N-1). \end{aligned}$$

Color-or-Quaternion Image is The Sum of Direction Image Components



(a)

(b) (1,1)–DI



(c) (1,2)–DI

(d) (1,4)–DI

Figure 8: (a) The color image and direction images generated by (p,s) equal (b) (1,1), (c) (1,2), and (d) (1,4).

The Paired Image Representation: Splitting-Signals and Direction Quaternion Image Components

The tensor transform, or representation is redndant for the case $N \times N$, where N is a power of 2. Therefore the tensor transform is modified and new1-D quaternion splitting-signals or direction color image components are calculated by

$$d'_{n,m;p,s} = f'_{p,s,(np+ms) \bmod N} = f_{p,s,(np+ms) \bmod N} - f_{p,s,(np+ms+N/2) \bmod N}$$

Such representation of the quaternion image is called the paired transform; it is unitary and therefore not redundant.

Statement 2: The discrete quaternion image of size $N \times N$, where $N=2^r$, $r>1$, can be composed from its $(3N-2)$ quaternion direction images as

$$f_{n,m} = \sum_{(p,s)\in J'_{N,N}} d'_{n,m;p,s} = \frac{1}{2N}\sum_{k=0}^{r-1}\frac{1}{2^k}\sum_{(p,s)\in 2^k J_{N/2^k,N/2^k}} f'_{p,s,(np+ms) \bmod N} + \frac{1}{N^2} f'_{0,0,0}\,.$$

Here $J'_{N,N}$ is a set of generators (p,s).

Conclusion

- We presented a new concept of the tensor representation for color images in the quaternion algebra. By means of this representation, the processing of the color image in the frequency domain is reduced to calculation of the 1-D signals and the 2-D left-side QDFT is calculated through the 1-D QDFTs.
- The proposed algorithm leads to a more efficient performance of the 2-D left-side quaternion DFT than the existent fast algorithms.

  The 2-D QDFT with symplectic decomposition requires $18N^2$ more multiplications, than the method of tensor representation.
- The discrete color image and its left-side quaternion Fourier transform was considered on the Cartesian lattice, and the presented concept of tensor representation can also be used for other lattices, such as hexagonal lattices, as in the case of gray-scale images.

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