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2-D Octonion Discrete Fourier Transform: Fast Algorithms

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Abstract

- The color image from one of the color models, for instance the RGB model, can be transformed into the quaternion algebra and be represented as one quaternion image which allows to process simultaneously of all color components of the image. The color image can be also considered in different models with transformation to the octonion space with following processing in the 8-D frequency domain.
- In this work, we describe the algorithm for the 2-D two-side octonion DFT (ODFT), by using two-side 2-D quaternion DFTs (QDFT).
- The calculation of the transform is reduced to calculation of two 2-D QDFT which has fast algorithms. The octonion algebra with the Fourier transform can be used in color imaging as the 2-D ODFT, which found effective applications in color imaging, medical imaging, in image filtration, image enhancement.
- The octonion 2-D DFT can be used not only in color imaging, but in gray-scale imaging as well, and for that there are many models of transferring one or a few gray-scale images into the octonion space.

Inroduction – Quanterions and Octonions

- **T**he quaternion q=(a,b,c,d) can be considered 4-dimensional generation of a complex number with one real part and three imaginary parts

$$q = a + bi + cj + dk = a + (bi + cj + dk) = a + (ib + jc + kd).$$

Here a, b, c, and d are real numbers and i, j, and k are three imaginary units with the following multiplication laws:

$$ij = -ji = k,\ \ jk = -kj = i,\ \ ki = -ik = -j,\ \ i^2 = j^2 = k^2 = ijk = -1.$$

The quaternion conjugate and modulus of q equal :

$$\bar{q} = a - (bi + cj + dk), \quad |q| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

A unit pure quaternion is $\mu = i\mu_i + j\mu_j + k\mu_k$ such that $|\mu| = 1$, $\mu^2 = -1$

Examples: The numbers $\mu=j$, $\mu=(i+j+k)/\sqrt{3}$, $\mu=(i+j)/\sqrt{2}$, and $\mu=(i-k)/\sqrt{2}$.

Given unit pure quaternion μ, the exponential number is calculated by

$$\exp(\mu x) = \cos(x) + \mu \sin(x) = \cos(x) + i\mu_i \sin(x) + j\mu_j \sin(x) + k\mu_k \sin(x)$$

Inroduction – Quanterions and Octonions

- The octonion o can be considered as the double-quaternion,

$$o = (q_1, q_2) = q_1 + q_2 E = (a_1 + q_1') + (a_2 + q_2')E$$

where quaternions $q_1 = a_1 + q_1'$ and $q_2 = a_2 + q_2'$ with imaginary parts

$$q_1' = ib_1 + jc_1 + kd_1 \text{ and } q_2' = ib_2 + jc_2 + kd_2$$

The octonion is a number with the eight component number,

$$o = (a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) = a_1 + ib_1 + jc_1 + kd_1 + a_2E + Ib_2 + Jc_2 + Kd_2.$$

$$o = a + ib + jc + kd + AE + IB + JC + KD,$$

Here, we denote the numbers iE, jE, kE by I, J, and K

The basic multiplications of imaginary units and the real number 1: →

	1	i	j	k	E	I	J	K
1	1	i	j	k	E	I	J	K
i	i	-1	$-k$	j	I	$-E$	$-K$	J
j	j	k	-1	$-i$	J	K	$-E$	$-I$
k	k	$-j$	i	-1	K	$-J$	I	$-E$
E	E	$-I$	$-J$	$-K$	-1	i	j	k
I	I	E	$-K$	J	$-i$	-1	$-k$	j
J	J	K	E	$-I$	$-j$	k	-1	$-i$
K	K	$-J$	I	E	$-k$	$-j$	i	-1

Inroduction – Octonion Exponent

- The multiplication of octonions is not associative, i.e., not for all octonions o_1, o_2, and o_3, the equality $(o_1o_2)o_3=o_1(o_2o_3)$ holds.
  A unit pure octonion: $|o|^2=1$ and $o^2 = -1$, (with its real part equals zero).
  Examples: Pure unit octonions λ are i, j, k, E, I, J, K, and

$$o_1 = \frac{i+j+k+E+I+J+2K}{\sqrt{10}}, \quad o_2 = \frac{i+j-2I+J}{7}.$$

The exponential function on octonions o is defined by the Taylor series as

$$e^o = \exp(o) = 1 + \sum_{n=1}^{\infty} \frac{o^n}{n!}.$$

Given pure octonion unit λ and real ϑ, we can define the exponent function as

$$\exp(\lambda\vartheta) = \cos(\vartheta) + \lambda\sin(\vartheta)$$

and its conjugate

$$\exp(-\lambda\vartheta) = \cos(\vartheta) - \lambda\sin(\vartheta).$$

Inroduction – *Octonion Exponent*

- *Example:* Let $o = 1 + i - 2j + k + 2E + I - J + 2K$ or

$$o = 1 + \lambda\vartheta = 1 + \frac{i - 2j + k + 2E + I - J + 2K}{4}4.$$

Then,

$$\vartheta = |o'| = \sqrt{1 + 2^2 + 1 + 2^2 + 1 + 1 + 2^2} = \sqrt{16} = 4,$$

and the pure unit octonion

$$\lambda = \frac{i - 2j + k + 2E + I - J + 2K}{4}.$$

The exponential number $\exp(o)$ is calculated by

$$e^o = e^1 \left(\cos(4) + \frac{i - 2j + k + 2E + I - J + 2K}{4} \sin(4) \right).$$

RGB and CMYK Models for Color Images

In quaternion imaging, each color triple is treated as a whole unit, and it thus is expected, that by using quaternion operations, a higher color information accuracy can be achieved.

A discrete color image $f_{n,m}$ in the RGB color space can be transformed into imaginary part of quaternion numbers form by encoding the red, green, and blue components of the RGB value as a pure quaternion (with zero real part):

$$f_{n,m} = 0 + (ir_{n,m} + jg_{n,m} + kb_{n,m}).$$

The zero real part of the quaternion image makes it pure quaternion, and we could use the gray-scale component in the real part

$$\begin{aligned} f_{n,m} &= a_{n,m} + (ir_{n,m} + jg_{n,m} + kb_{n,m}) \\ &= \frac{r_{n,m} + g_{n,m} + b_{n,m}}{3} + (ir_{n,m} + jg_{n,m} + kb_{n,m}). \end{aligned}$$

- The advantage of using quaternion based operations to manipulate color information in an image is that we do not have to process each color channel independently, but rather, treat each color triple as a whole unit.

RGB and CMYK Models for Color Images

In quaternion imaging, each color triple is treated as a whole unit, and it thus is expected, that by using quaternion operations, a higher color information accuracy can be achieved.

- RGB Model: A discrete color image $f_{n,m}$ can be transformed into imaginary part of quaternion numbers form by encoding the red (**R**), green (**G**), and blue (**B**) components of the RGB value as a pure quaternion (with zero real part):

$$f_{n,m} = 0 + (ir_{n,m} + jg_{n,m} + kb_{n,m}).$$

The zero real part of the quaternion image makes it pure quaternion, and we could use the gray-scale component in the real part

$$\begin{aligned} f_{n,m} &= a_{n,m} + (ir_{n,m} + jg_{n,m} + kb_{n,m}) \\ &= \frac{r_{n,m} + g_{n,m} + b_{n,m}}{3} + (ir_{n,m} + jg_{n,m} + kb_{n,m}). \end{aligned}$$

- CMYK Model: With the primary colors cyan (**C**), magenta (**M**), yellow (**Y**), the quaternion image is four-component image with the real part equal the black (**K**) color,

$$f_{n,m} = k_{n,m} + (ic_{n,m} + jm_{n,m} + ky_{n,m}).$$

Transfer of Color Images into the Octonion Space

There are different ways to compose or transfer color images into the octonion space and then introduce the concept of the octonion image. We consider one of such ways, by using the RGB color model.

1. We consider that the color image is the color image transformed to the quaternion space. This quaternion image with four components can be transformed to the octonion space, by composing the following "octonion image":

$$\begin{aligned} o_{n,m} &= f_{n,m} + f_{n+1,m}E \\ &= a_{n,m} + ir_{n,m} + jg_{n,m} + kb_{n,m} + Ea_{n+1,m} + Ir_{n+1,m} + Jg_{n+1,m} + Kb_{n+1,m} \\ & \quad n = 0:(N/2-1),\ m = 0:(M-1). \end{aligned}$$

Thus, the octonion image $o_{n,m} = (f_{n,m}, f_{n+1,m})$ is of size $(N/2) \times M$.

2. We also can consider the model of the "octonion image" with the following composition:

$$\begin{aligned} o_{n,m} &= f_{n,m} + f_{n,m+1}E \\ &= a_{n,m} + ir_{n,m} + jg_{n,m} + kb_{n,m} + Ea_{n,m+1} + Ir_{n,m+1} + Jg_{n,m+1} + Kb_{n,m+1}, \\ & \quad n = 0:(N-1),\ m = 0:(M/2-1). \end{aligned}$$

Thus, the octonion image $o_{n,m} = (f_{n,m}, f_{n,m+1})$ is of size $N \times (M/2)$.

RGB Color Images as "Octonion" Images: *Example 1*

- Fig. 1 shows the 256×256 color image $(r_{n,m}, g_{n,m}, b_{n,m})$ in part a. The octonion image $o_{n,m}$ with components of size 128×512 each was calculated.The i,j, and k-components of $o_{n,m}$ as a color image together with the color image composed by the I, J, and K-components of $o_{n,m}$ are shown in part b.

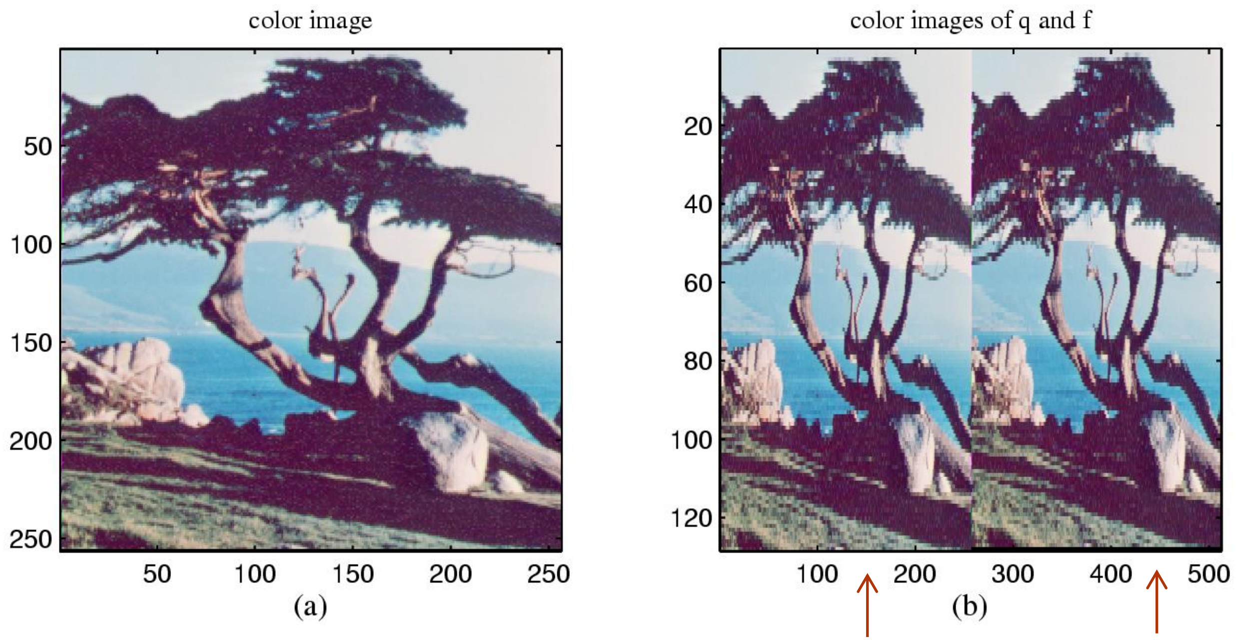

Fig. 1 (a) The color "tree" image and (b) the (i,j,k) and (I,J,K) color images.

RGB Color Images as "Octonion" Images: *Example 1*

- Figure 2 shows the 256×256 gray-scale image $f_{n,m}$ in part a. The real part $a_{n,m}$ and the E-component $a_{n+1,m}$ with components of size 128×512 each are shown in part b.

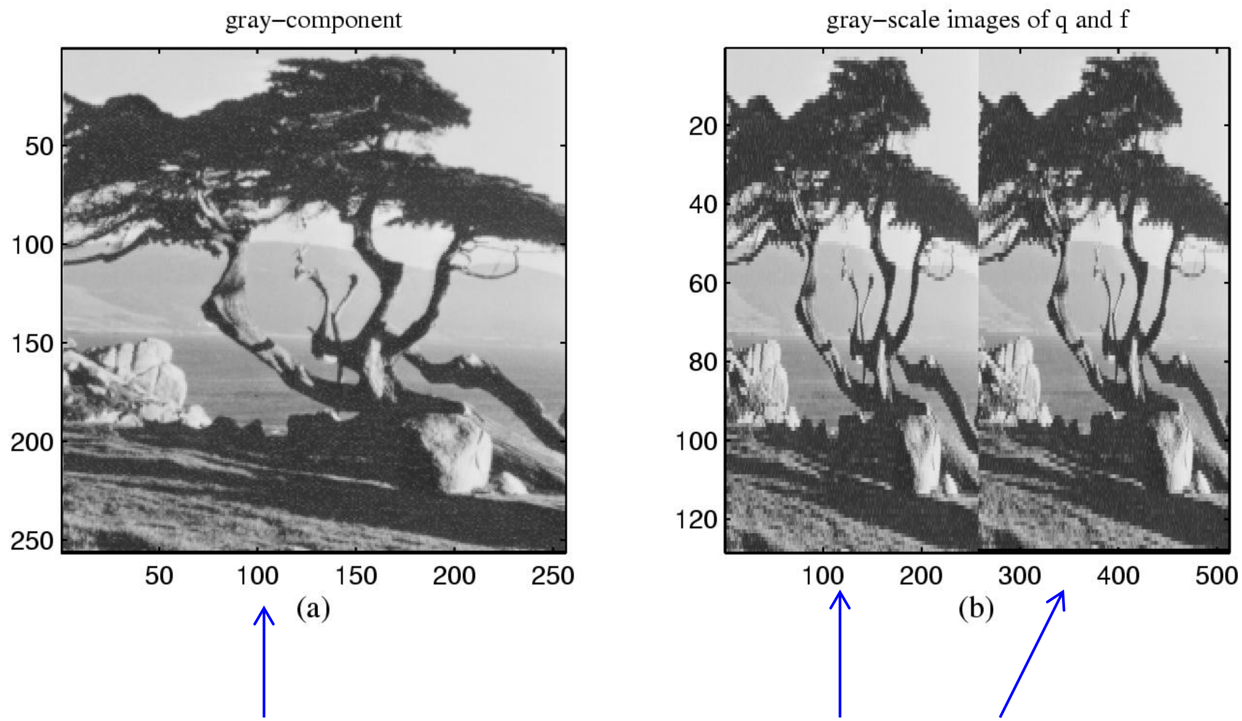

Fig. 2 (a) The gray-scale image $a_{n,m}$ and (b) the 1− and E−components of the "octonion" tree image.

RGB Color Images as "Octonion" Images: *Example 1*

- The 256×256 color "tree" image is presented as the "octonion" image with its eight components of size 128×256 each.

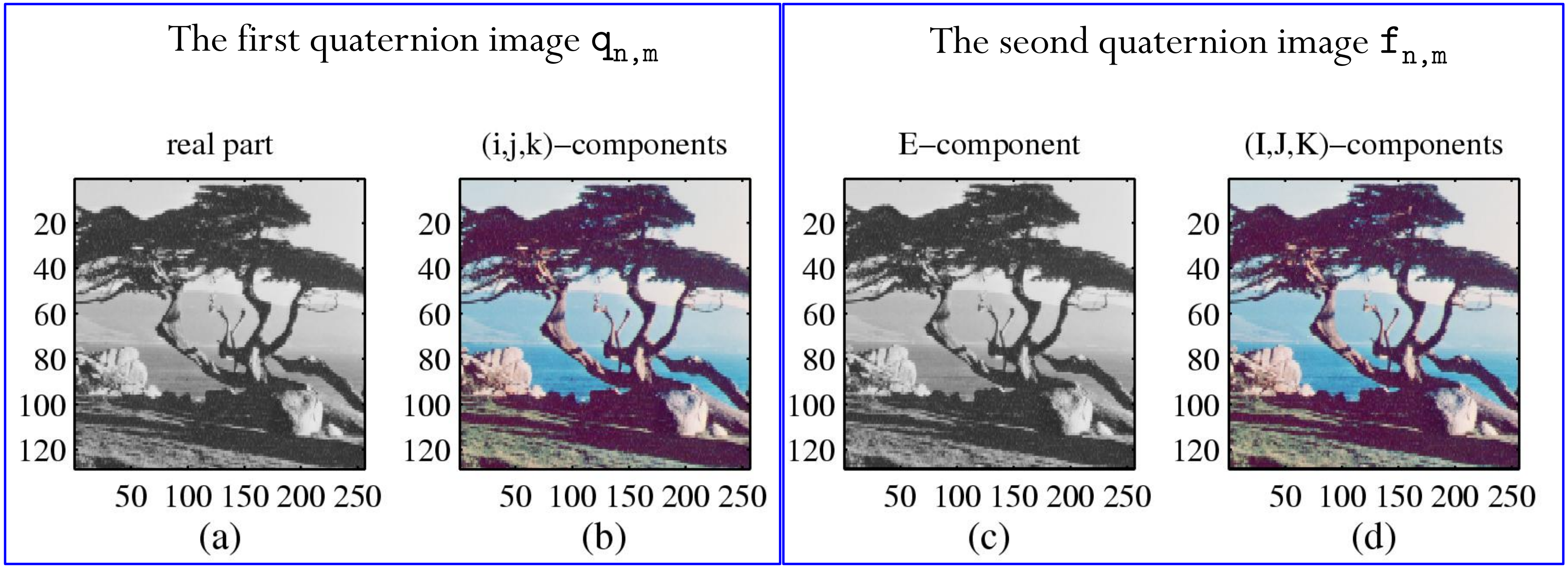

Fig. 3 (a) The gray-scale component, (b) the `(i,j,k)`-component color image, (c) the *E*-component, and (d) the `(I,J,K)`-component color image of the octonion "tree" image.

RGB Color Images as "Octonion" Images: *Example 2*

- The 362×500 color "flowers" image is presented as the "octonion" image with its eight components of size 181×500 each.

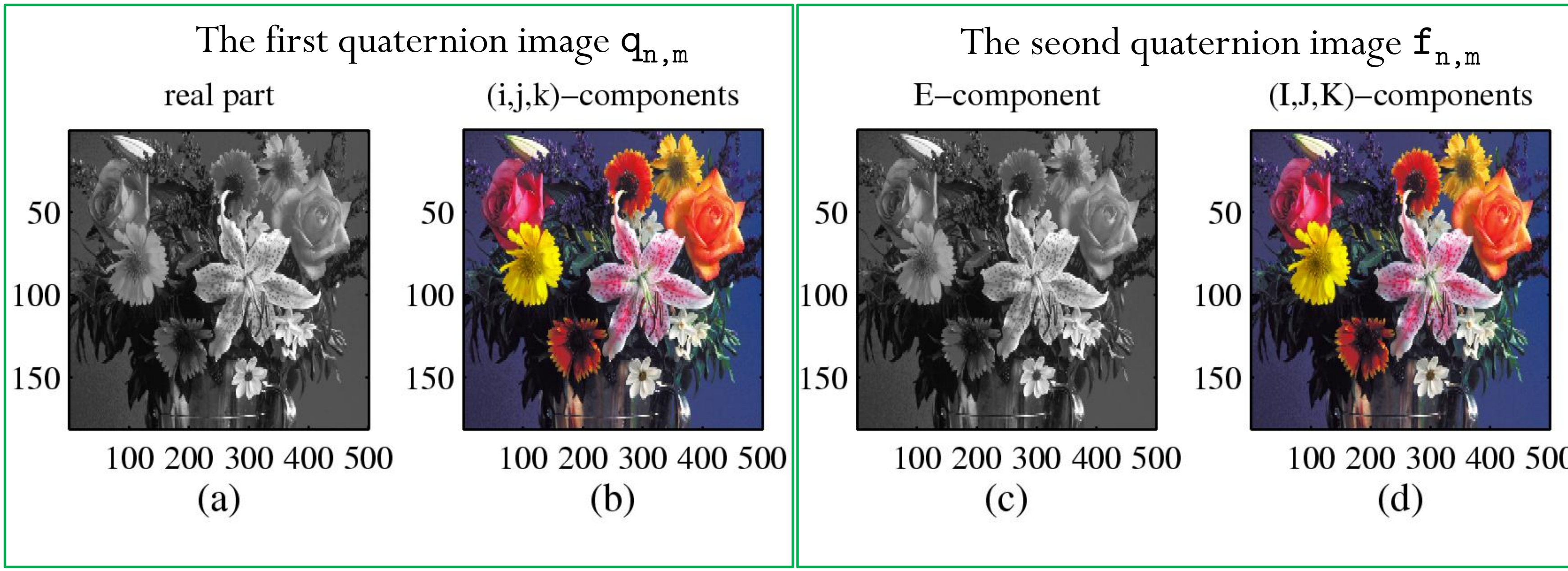

Fig. 4 (a) The gray-scale component, (b) the `(i,j,k)`-component color image, (c) the *E*-component, and (d) the `(I,J,K)`-component color image of the octonion "flowers" image.

Two-side Octonioin Discrete Fourier Transform (2-D ODFT)

- The 2-D ODFT is a new concept which allows for processing a few gray-scale or color images, as well as images with their local information as one octonion 8-D image in the spectral domain. This concept generalizes the traditional complex and quaternion 2-D DFT and can be used for parallel processing up to eight of gray-scale images or two color images.
- Given two pure unit octonions λ_1 and λ_2 which we consider equal to the pure unit quaternions μ_1 and μ_2, the two-side 2-D ODFT of the octonion image $o_{n,m}$ can be defined as

$$O_{p,s} = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} W_{\mu_1}^{np}\left(o_{n,m}W_{\mu_2}^{ms}\right) = \sum_{n=0}^{N-1} W_{\mu_1}^{np}\left(\sum_{m=0}^{M-1} o_{n,m}W_{\mu_2}^{ms}\right)$$

where p=0:(N-1) and s=0:(M-1).

The transform kernel is denoted by (W_{μ_1}, W_{μ_2}) .

The basis functions are defined by the exponential coefficients

$$\begin{aligned} W_{\mu_1} = W_{\mu_1;N} &= \cos(2\pi/N) - \mu_1 \sin(2\pi/N), \\ W_{\mu_2} = W_{\mu_2;M} &= \cos(2\pi/M) - \mu_2 \sin(2\pi/M). \end{aligned}$$

Two-side Octonioin Discrete Fourier Transform (2-D ODFT)

- One can apply the 2-D QDFTs to calculate the 2-D ODFT.

We consider the first and second quaternion components of the image

$$o_{n,m} = q_{n,m} + f_{n,m}E$$

and write

$$O_{p,s} \;=\; \sum_{n=0}^{N-1} W_{\mu_1}^{np}\left(\sum_{m=0}^{M-1} q_{n,m}W_{\mu_2}^{ms}\right) + \sum_{n=0}^{N-1} W_{\mu_1}^{np}\left(\sum_{m=0}^{M-1} (f_{n,m}E)W_{\mu_2}^{ms}\right)$$

- It not difficult to show the following property:

$$\underline{(q_{n,m}E)W_{\mu_2}^{ms} = (q_{n,m}W_{\mu_2}^{-ms})E.}$$

Therefore,

$$O_{p,s} = \sum_{n=0}^{N-1} W_{\mu_1}^{np}\left(\sum_{m=0}^{M-1} q_{n,m}W_{\mu_2}^{ms}\right) + \sum_{n=0}^{N-1} W_{\mu_1}^{np}\left(\sum_{m=0}^{M-1} f_{n,m}W_{\mu_2}^{-ms}\right)E.$$

and should be noted that $W_{\mu_2}^{-ms} = W_{\mu_2}^{m(M-s)} = W_{-\mu_2}^{ms}.$

Two-side Octonioin Discrete Fourier Transform (2-D ODFT)

- To calculate the two-side 2-D ODFT, we can perform the two-side 2-D QDFT of the quaternion image $q_{n,m}$ and the two-side 2-D QDFT of the quaternion image $f_{n,m}$.

- The first two-side QDFT is calculated with kernel (W_{μ_1}, W_{μ_2}), and the second one with the kernel (W_{μ_1}, W_{μ_2}) or $(W_{\mu_1}, W_{-\mu_2})$.

1. Let $Q_{p,s}$ be the two-side 2-D QDFT of the image $q_{n,m}$ and let $F_{p,s}$ be the two-side 2-D QDFT of $f_{n,m}$, with the kernel (W_{μ_1}, W_{μ_2}) both. Then,

$$O_{p,s} = Q_{p,s} + F_{p,M-s}E.$$

2. If $F_{p,s}$ is the two-side 2-D QDFT with the kernel $(W_{\mu_1}, W_{-\mu_2})$, then

$$O_{p,s} = Q_{p,s} + F_{p,s}E.$$

Example: 2-D ODFT of the "flowers" image

- The flower image of size 362×500 and its the 2-D ODFT.

The octonion "flowers" image is the pair of two quaternion images ($g_{n,m}$,$f_{n,m}$).

Color image of the imaginary part of $g_{n,m}$. Color image of the imaginary of $f_{n,m}$.

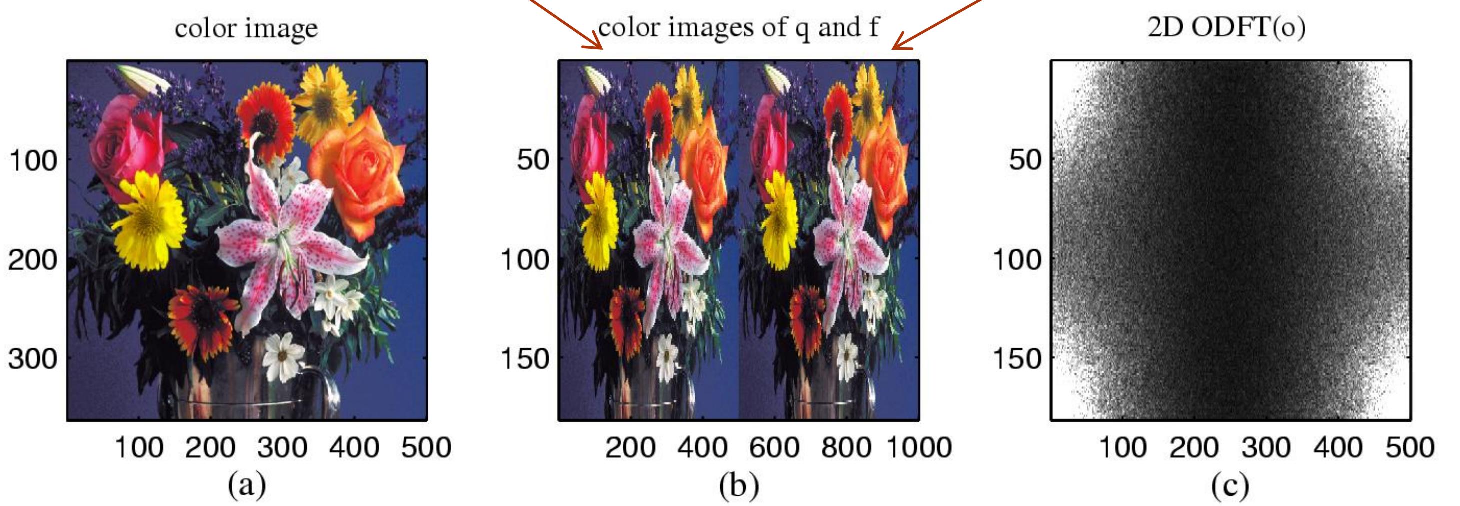

Fig. 5 (a) The color "flower" image, (b) the imaginary parts of two quaternions $g_{n,m}$ and $f_{n,m}$ of the quaternion image $o_{n,m}$, and (c) the 2-D ODFT of the octonion "flowers" image $o_{n,m}$. (The transform is shown in the absolute scale.)

The right-side 1-D QDFT is used in the 2-D OCDT

- Let $f_n = (a_n, b_n, c_n, d_n) = a_n + (ib_n + jc_n + kd_n)$ be the quaternion signal of length N.

  The right-side 1-D quaternion DFT (rs-QDFT) is defined as

$$Q_p = Q_1(p) + iQ_i(p) + jQ_j(p) + kQ_k(p) = \sum_{n=0}^{N-1} q_n W_\mu^{np}, \quad p = 0:(N-1),$$

where μ is a pure unit quaternion number; $\mu = m_1 i + m_2 j + m_3 k,\ \mu^2 = -1.$

Let φ_{np} be the angle $(2\pi/N)np$. For a real signal x_n of length N, the cosine and sine transforms being the real and imaginary parts of the 1-D DFT of this signal are

$$C_x(p) = \sum_{n=0}^{N-1} x_n \cos(\varphi_{np}), \quad S_x(p) = \sum_{n=0}^{N-1} x_n \sin(\varphi_{np}), \qquad p = 0:(N-1).$$

The calculation of the ls-QDFT at each point p can be written in the matrix form

$$\begin{bmatrix} Q_1(p) \\ Q_i(p) \\ Q_j(p) \\ Q_k(p) \end{bmatrix} = \begin{bmatrix} \Re(A_p) \\ \Re(B_p) \\ \Re(C_p) \\ \Re(D_p) \end{bmatrix} + \begin{bmatrix} 0 & m_1 & m_2 & m_3 \\ -m_1 & 0 & -m_3 & m_2 \\ -m_2 & m_3 & 0 & -m_1 \\ -m_3 & -m_2 & m_1 & 0 \end{bmatrix} \begin{bmatrix} \Im(A_p) \\ \Im(B_p) \\ \Im(C_p) \\ \Im(D_p) \end{bmatrix}$$

Here, A_p, B_p, C_p, and D_p, are the 1-D DFTs of a_n, b_n, c_n, and d_n, respectively.

The left-side 1-D QDFT is also used in the 2-D OCDT

- Let $f_n = (a_n, b_n, c_n, d_n) = a_n + (ib_n + jc_n + kd_n)$ be the quaternion signal of length N.
  The leftt-side 1-D quaternion DFT (ls-QDFT) is defined as

$$Q_p = Q_1(p) + iQ_i(p) + jQ_j(p) + kQ_k(p) = \sum_{n=0}^{N-1} W_\mu^{np} q_n, \quad p = 0 : (N-1).$$

where μ is a pure unit quaternion number; $\mu = m_1 i + m_2 j + m_3 k$, $\mu^2 = -1$.

The calculation of the ls-QDFT at each frequency-point p can be written in the matrix form

$$\begin{bmatrix} Q_1(p) \\ Q_i(p) \\ Q_j(p) \\ Q_k(p) \end{bmatrix} = \begin{bmatrix} \Re(A_p) \\ \Re(B_p) \\ \Re(C_p) \\ \Re(D_p) \end{bmatrix} + \begin{bmatrix} 0 & m_1 & m_2 & m_3 \\ -m_1 & 0 & m_3 & -m_2 \\ -m_2 & -m_3 & 0 & m_1 \\ -m_3 & m_2 & -m_1 & 0 \end{bmatrix} \begin{bmatrix} \Im(A_p) \\ \Im(B_p) \\ \Im(C_p) \\ \Im(D_p) \end{bmatrix}$$

By using the fast Fourier transforms, the calculation of the left-side and right-side 1-D QDFTs by the above equations is effective [1,2].

[1] A.M. Grigoryan and S.S. Agaian, Practical Quaternion Imaging with MATLAB, *SPIE Press*, 2017.
[2] A.M. Grigoryan, S.S. Agaian, “Tensor transform-based quaternion Fourier transform algorithm,” *Information Sciences*, vol. 320, pp. 62-74, November 2015, (doi: http://dx.doi.org/10.1016/j.ins.2015.05.018)

Multiplications and Additions for the left-side 1-D QDFT

- In the general case of the quaternion signal f_n, the number of operations of multiplication and addition for LS 1-D QDFT can be estimated as

$$m_{QFT}(N) = 4m_{FT}(N) + 12N \text{ and } a_{QFT}(N) = 4a_{FT}(N) + 12N$$

The number of operations for the left-side 1-D QDFT can be estimated as

$$\begin{aligned} m_{QFT}(N) &= 8\mu_{FT}(N) + 12N = 4Nr + 16, \\ a_{QFT}(N) &= 4\alpha_{FT}(N) + 12N = 2N(r+15) - 4(r^2 + 3r + 4). \end{aligned} \qquad (1)$$

Here, we consider that for the fast N-point discrete paired transform-based FFT[3,4], the estimation for multiplications and additions are

$$\mu_{FT}(N) = 2^{r-1}(r-3) + 2 \qquad \alpha_{FT}(N) = (2^r 6 - r^2 - 3r - 6)$$

and two 1-D DFTs with real inputs can be calculated by one DFT with complex input,

$$2m_{FT}(N) = 4 \times \mu_{FT}(N) \quad \text{and} \quad 2a_{FT}(N) = 2 \times \alpha_{FT}(N).$$

[3] A.M. Grigoryan, "2-D and 1-D multi-paired transforms: Frequency-time type wavelets," *IEEE Trans. on Signal Processing*, vol. 49, no. 2, pp. 344–353, Feb. 2001.

[4] A.M. Grigoryan, S.S. Agaian, "Split manageable efficient algorithm for Fourier and Hadamard transforms, *IEEE Trans. Signal Processing*, vol. 48, no. 1, pp. 172-183, 2000.

Conclusion

- The two-dimension two-side octonion discrete Fourier transform (2-D ODFT) is considered in the octonion algebra wherein the color image is transformed from such color models, as the RGB, CMY(K), or XYZ models.
- This transform can also be used in different models of gray-scale image representation in the octonion space.
- The calculation of the two-side 2-D ODFT can be accomplished was shown by using the fast 1-D left and right-side quaternion discrete Fourier transforms with any quaternion exponential kernel, when calculating two 2-D two-side QDFTs which define the two-side 2-D ODFT.

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THANK YOU VERY MUCH!

QUESTIONS, PLEASE?