Method of Flow Graph Simplification for the 16-point Discrete Fourier Transform

Artyom M. Grigoryan and Veerabhadra S. Bhamidipati

Abstract

Efficient method of realization of the paired algorithm for calculation of the 1-D discrete Fourier transform (DFT), by simplifying the signal-flow graph of the transform, is described. The signal-flow graph is modified by separating the calculation for real and imaginary parts of all inputs and outputs in the signal-flow graph and using properties of the transform. The examples for calculation of the 8- and 16-point DFTs are considered in detail. The calculation of the 16-point DFT of real data requires 12 real multiplications and 58 additions. 2 multiplications and 20 additions are used for the 8-point DFT.

Index Terms – The fast Fourier transform, paired function and transform.

I. Introduction

FAST Fourier transform is one of the most frequently used tools in signal and image processing, communication systems, and many other areas of science and engineering [1]-[6]. In this paper we present an effective calculation of the fast Fourier transform that is based on the simplification of the signal-flow graph of calculation of the transform by the paired transform developed by Grigoryan [11], [13]. It is known that the paired transform splits the DFT into a minimum set of short transforms, and the algorithm of calculation of the DFT by the paired transform uses a minimum number of multiplications by twiddle factors. The question arises how to define the exact minimum number of real multiplications by maximum simplifying the flow graph of the algorithm. This question addresses also to many other algorithms of the DFT. Instead of finding new effective formulas for calculation of transform coefficients, we will work directly on the flow graph of the transform. In many cases of order N of transform, the simplification of the signal-flow graph can be done easily. We will consider in detail the $N = 8$ and 16 cases.

We refer to the paired algorithm of the DFT, but we belive that signal-flow graphs of other fast algorithms, such as the fractional DFT [7], split-radix, vector split-radix, mixed radix [8], [9], can be considered and modified in a similar way. The advantage of using the paired transform is in the fact that this transform reveals completely the mathematical structure of many other unitary transforms, such as cosine, Hartley, and Hadamard transforms, and requires

The authors are with the Department of Electrical Engineering, The University of Texas at San Antonio. 6900 North Loop 1604 West, San Antonio, TX 78249-0669 USA (email: amgrigoryan@utsa.edu; vbhamidi@lonestar.utsa.edu)

The author version: Artyom Grigoryan, December 12, 2018.

a minimum number of operations [12], [15]. Therefore the method described here for the DFT can be applied for fast calculation of these transforms, too.

As an example, Figure 1 shows the diagram of splitting an N-point discrete unitary transform (DUT) by the paired transform χ'_N, in the $N = 16$ case. The calculation of the N-point DUT is reduced to calculation of the 8-, 4-, 2-, and 1-point DUTs. When weighted coefficients w_k for the output of the paired transform are equal to $\exp(-j2\pi k/N)$, $k = 1 : (N/2-1)$, and DUT is the discrete Fourier transform (DFT), then the diagram describes the algorithm of calculation of the N-point DFT. When all coefficients $\mathrm{w}_k \equiv 1$ and DUT is considered to be the discrete Hadamard transform (DHdT), then we obtain the diagram of calculation of the N-point DHdT by transforms of smaller orders.

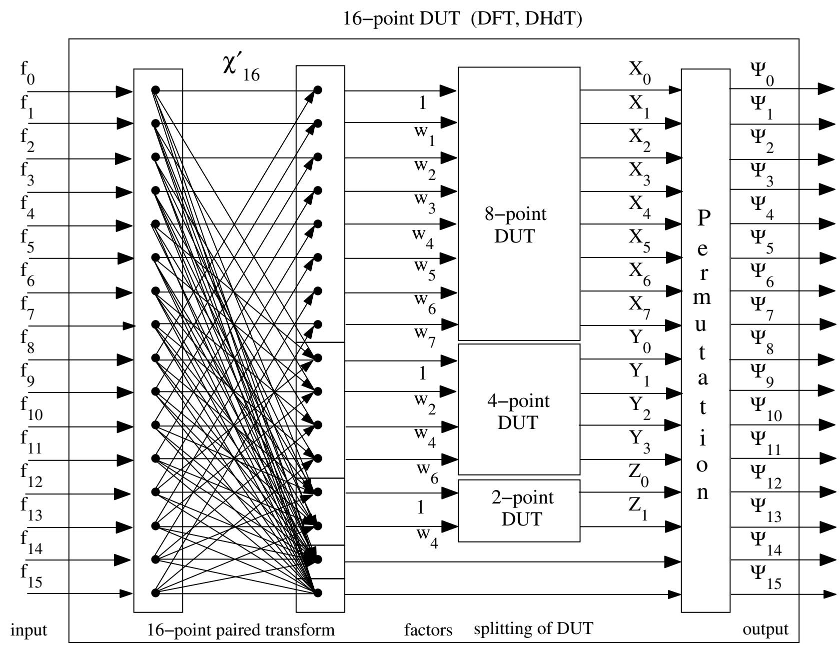


Fig. 1. Diagram of calculation of the 16-point discrete unitary transform (DUT) by the paired transforms and 8-, 4-, and 2-point DUTs. The permutation is calculated by $(2k+1) \bmod 16 \leftarrow k$, $k = 0 : 15$.

The paired transform is fast, requires $2N-2$ operations of addition/subtraction. The arithmetical complexity of the paired algorithm for the N-point DFT, when N is a power of 2, is calculated by $\alpha(N) = N/2(r+9) - r^2 - 3r - 6$ and $m(N) = N/2(r-3) + 2$, where functions $\alpha(N)$ and $m(N)$ stand respectively for number of additions and multiplications by non-trivial twiddle factors. The detail description of the paired transform-based algorithms for calculation of the discrete Fourier and Hadamard transforms, as well as examples of MATLAB-based programs for calculation of these and paired transforms can be found in [14].

II. Fast Fourier transform by paired transforms

Let f_n be a finite sequence or discrete-time signal of length N, where N is a power of two, $N = 2^r$, $r \geq 1$. The N-point Fourier transform of the sequence f_n is defined by

$$F_p = (\mathcal{F}_N \circ f)_p = \sum_{n=0}^{N-1} f_n W^{np}, \qquad p = 0 : (N-1), \tag{1}$$

where $W = \exp(-j2\pi/N)$. We use the notation $n = 0 : (N-1)$ to denote the numbers that run from 0 to $(N-1)$.

The splitting of the signal is performed by the N-point discrete paired transform χ'_N whose basis functions $\chi'_{p,t}$ are defined by

$$\chi'_{p,t}(n) = \begin{cases} 1, & \text{if } np = t \bmod N, \\ -1, & \text{if } np = t + N/2 \bmod N, \\ 0, & \text{otherwise,} \end{cases} \quad n = 0 : (N-1), \tag{2}$$

where $p = 2^k$, $k = 0 : (r-1)$, $t = 0 : (2^{r-k-1} - 1)$, and $\chi'_{0,0}(n) \equiv 1$.

In the paired representation, the sequence f_n is considered as a set of $(r+1)$ splitting-signals

$$f \xrightarrow{\chi'_N} \begin{cases} f'_{T'_1} = \{f'_{1,0}, f'_{1,1}, f'_{1,2}, ..., f'_{1,N/2-1}\} \\ f'_{T'_2} = \{f'_{2,0}, f'_{2,2}, f'_{2,4}, ..., f'_{2,N/2-2}\} \\ f'_{T'_4} = \{f'_{4,0}, f'_{4,4}, f'_{4,8}, ..., f'_{4,N/2-4}\} \\ \cdots \quad ... \quad ... \quad ... \\ f'_{T'_{N/2}} = \{f'_{N/2,0}\} \\ f'_{T'_0} = \{f'_{0,0}\} \end{cases} \tag{3}$$

where the components of the splitting-signals are defined by

$$f'_{p,t} = \chi'_{p,t} \circ f = \left[\sum_{np = t \bmod N} f_n \right] - \left[\sum_{np = (t+N/2) \bmod N} f_n \right]. \tag{4}$$

The N-point DFT is split into a set of short transformations $\{\mathcal{F}_{N/2}, \mathcal{F}_{N/4}, ..., \mathcal{F}_2, 1, 1\}$ by

$$F_{\overline{(2m+1)p}} = \sum_{t=0}^{L_p - 1} (f'_{p,t} W^t_{2L_p}) W^{mt}_{L_p}, \qquad m = 0 : (L_p - 1), \tag{5}$$

where $\bar{l} = l \bmod N$ for an integer l, and $L_p = N/(2p)$, when $p = 2^k$, $k = 0 : (r-1)$. Values of p are taken in a such a way that subsets

$$T'_p = \{\overline{(2m+1)p};\ m = 0 : (N/2 - 1)\}, \quad T'_0 = \{0\},$$

compose a partition of the set of samples $X_N = \{0, 1, 2, ..., N-1\}$.

Example 1: We consider the $N = 4$ case. The set $X = \{0, 1, 2, 3\}$ is covered by the partition $\sigma' = (T'_1, T'_2, T'_0)$ with subsets $T'_1 = \{1, 3\}$, $T'_2 = \{2\}$, and $T'_0 = \{0\}$. For the generators $p = 1, 2$, and 0 of the subsets T'_p, the following (4×3) matrix with values of $t = (np) \bmod 4$, when $n = 0 : 3$, is composed

$$||t||_{n=0:3,p=1,2,0} = \left\| \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right\|.$$

Therefore, a length-4 sequence f_n can be represented as three short sequences

$$\begin{aligned} f_{T'_1} &= \{f'_{1,0},\ f'_{1,1}\} = \{f_0 - f_2, f_1 - f_3\}, \\ f_{T'_2} &= \{f'_{2,0}\} = \{f_0 - f_1 + f_2 - f_3\}, \\ f_{T'_0} &= \{f'_{0,0}\} = \{f_0 + f_1 + f_2 + f_3\}. \end{aligned} \tag{6}$$

This representation is performed by the paired transformation χ'_4 with the following matrix

$$[\chi'_4] = \begin{bmatrix} [\chi'_{1,0}] \\ [\chi'_{1,1}] \\ [\chi'_{2,0}] \\ [\chi'_{0,0}] \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{7}$$

The decomposition of the 4-point DFT by the paired transform into 2-, 1-point DFTs can be written in the matrix form as

$$[\mathcal{F}_4] = ([\mathcal{F}_2] \oplus 1 \oplus 1)\, D_4[\chi'_4], \tag{8}$$

where D_4 is the diagonal matrix with coefficients $\{1, -j, 1, 1\}$. Operation $\oplus$ denotes the Kronecker sum of matrices.

We now will consider the method of fast calculation of the 8-point DFT by the paired transform and we will analyze the signal-flow graph of calculation, in order to obtain an effective calculation of the DFT.

III. The paired algorithm for the 8-point DFT

Let f_n be a sequence (signal) of length 8. The set $X = \{0, 1, ..., 7\}$ is covered by the following partition of subsets $\sigma' = (T'_1, T'_2, T'_4, T'_0)$, where subsets $T'_1 = \{1, 3, 5, 7\}$, $T'_2 = \{2, 6\}$, $T'_4 = \{4\}$, and $T'_0 = \{0\}$. The partition of X by subsets T'_p is unique. For the generators of these subsets T'_p, the following (8×4) matrix is composed

$$||t = (np) \bmod 8||_{n=0:7,p=1,2,4,0} = \left\| \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & 8 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\|. \tag{9}$$

Elements of this matrix show a way of calculating the matrix of the 8-point discrete paired transform. Indeed, it follows directly from (9) that the signal f_n is represented in paired form as the following four splitting-signals

$$\begin{aligned} f_{T'_1} &= \{f'_{1,0},\ f'_{1,1},\ f'_{1,2},\ f'_{1,3}\} = \{f_0 - f_4, f_1 - f_5, f_2 - f_6, f_3 - f_7\} \\ f_{T'_2} &= \{f'_{2,0}, f'_{2,2}\} = \{f_0 - f_2 + f_4 - f_6, f_1 - f_3 + f_5 - f_7\} \\ f_{T'_4} &= \{f'_{4,0}\} = \{f_0 - f_1 + f_2 - f_3 + f_4 - f_5 + f_6 - f_7\} \\ f_{T'_0} &= \{f'_{0,0}\} = \{f_0 + f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7\}. \end{aligned}$$

All components of these four signals are calculated by the paired transformation χ'_8 with the following matrix

$$[\chi'_8] = \begin{bmatrix} [\chi'_{1,0}] \\ [\chi'_{1,1}] \\ [\chi'_{1,2}] \\ [\chi'_{1,3}] \\ [\chi'_{2,0}] \\ [\chi'_{2,2}] \\ [\chi'_{4,0}] \\ [\chi'_{0,0}] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{10}$$

Each splitting-signal carries the spectral information at the samples of the corresponding subset T'_p. Thus, for $p = 1$

$$\sum_{n=0}^{3} \left(f'_{1,t} W_8^t\right) W_4^{kt} = F_{2k+1}, \qquad k = 0, 1, 2, 3,$$

for $p = 2$

$$f'_{2,0} + \left(f'_{2,2} W_4\right) W_2^k = F_{4k+2}, \qquad k = 0, 1,$$

and for $p = 4$ and 0, we obtain respectively $f'_{4,0} = F_4$ and $f'_{0,0} = F_0$.

The decomposition of the 8-point DFT by the paired transform can be thus written in the matrix form as

$$[\mathcal{F}_8] = ([\mathcal{F}_4] \oplus [\mathcal{F}_2] \oplus 1 \oplus 1)\, D_8 [\chi'_8] \tag{11}$$

where D_8 is the diagonal matrix with coefficients $\{1, W^1, -j, W^3, 1, -j, 1, 1\}$ and $W = \exp(-j\pi/4)$.

The signal-flow graph for calculation of the 8-point DFT by paired transforms is shown in Fig. 2. The matrix of the 2-point paired transformation is defined by

$$[\chi'_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We now consider the signal-flow graph of Fig. 2 for the case when the input sequence f_n is real. There are two operations of multiplication by non-trivial complex factors $W = (1-j)a$ and

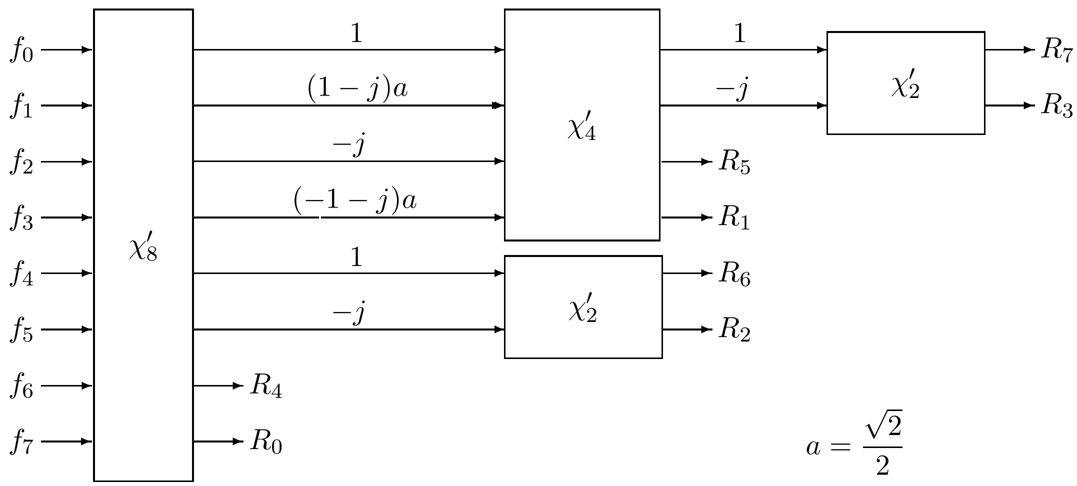


Fig. 2. Block-scheme of calculation of the 8-point DFT by paired transforms.

$W^3 = (-1-j)a$, where $a = \sqrt{2}/2$, to be used over the output of the 8-point paired transform. Three trivial multiplications by $-j$ are also used in calculation. All inputs and outputs of paired transforms χ'_8, χ'_4, and χ'_2 as well as multiplications by complex factors can be split into two parts as shown in Fig. 3. The new signal-flow graph can be then simplified. One can see that calculations for real R_p and imaginary I_p parts of transform coefficients F_p have been separated in a symmetric way. As an example, Figure 3 shows also values of all inputs and outputs for each transform in the block-scheme, when the 8-point sequence $\{f_n\}$ equals $\{1,2,4,4,3,7,5,8\}$.

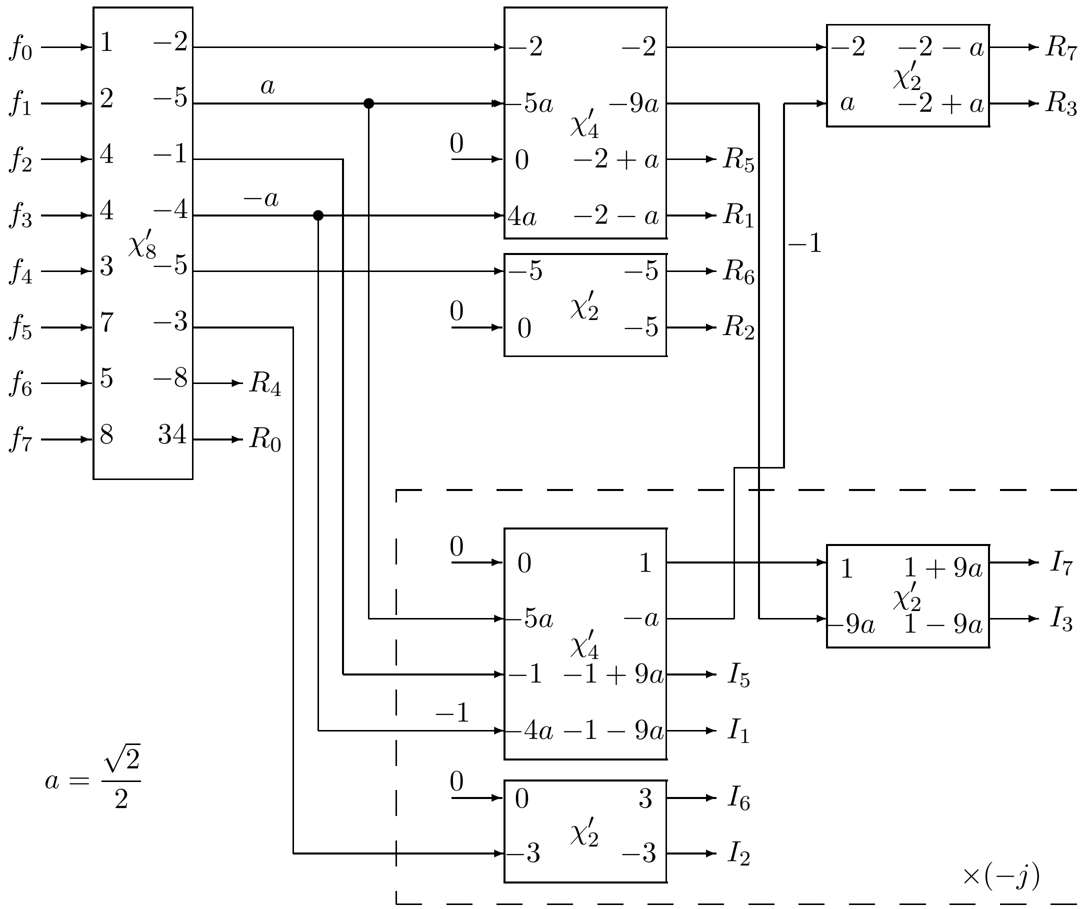
Using the property of complex conjugacy of Fourier coefficients for a real input, $F_{N-p} = \bar{F}_p$ for $p = 1:(N/2-1)$, we obtain

$$R_{N-p} = R_p, \quad I_{N-p} = -I_p, \quad p = 1:(N/2-1), \tag{12}$$

and $I_N = I_{N/2} = 0$. Therefore, the signal-flow graph of Fig. 3 can be reduced to the signal-flow graph shown in Fig. 4. We denote by $\chi'_{4;in}$ two incomplete 4-point paired transforms for which only the last two outputs are calculated. Since one of inputs for both the incomplete 4-point paired transforms is zero, they can be considered as 3-to-2-point transforms with the following matrices

$$[\chi'_{4;in}] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad [\chi'_{4;in}] = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{13}$$

The calculation of the 8-point DFT by the signal-flow graph of Fig. 4 requires two real multiplications by factor $a = \sqrt{2}/2$ and $14 + 2 \times 3 = 20$ additions. Indeed, the fast 2^n-point paired transform uses $2^{n+1} - 2$ additions [13]. The 8-point paired transform requires 14 additions, and each of the 4-point incomplete paired transform uses 3 additions.



$$a = \frac{\sqrt{2}}{2}$$

Fig. 3. Block-scheme of calculation of real and imaginary parts of the 8-point DFT.

IV. The paired algorithm for the 16-point DFT

The paired transform in the $N = 16$ case is defined by the partition $\sigma' = (T'_1, T'_2, T'_4, T'_8, T'_0)$ of the set of samples $X = \{0, 1, 2, 3, \ldots, 15\}$. The subsets $T'_1 = \{1, 3, 5, 7, 9, 11, 13, 15\}$, $T'_2 = \{2, 6, 10, 14\}$, $T'_4 = \{4, 12\}$, $T'_8 = \{8\}$, and $T'_0 = \{0\}$. Let f_n be a sequence of length 16, which is split in the paired representation by five signals

$$f \xrightarrow{\chi'_{16}} \left\{ f'_{T'_1}, f'_{T'_2}, f'_{T'_4}, f'_{T'_8}, f'_{T'_0} \right\}.$$

The signal-flow graph for calculation of the 16-point DFT by paired transforms is given in Fig. 5. Ten multiplications by non-trivial twiddle factors W^k_{16}, $k = 1, 2, 3, 5, 6, 7$, plus seven trivial multiplications by $-j$ are used in the calculation.

Similar to the $N = 8$ case described above, for a real input f_n we can redraw the signal-flow graph of the 16-point DFT by separating the calculation for the real and imaginary parts of

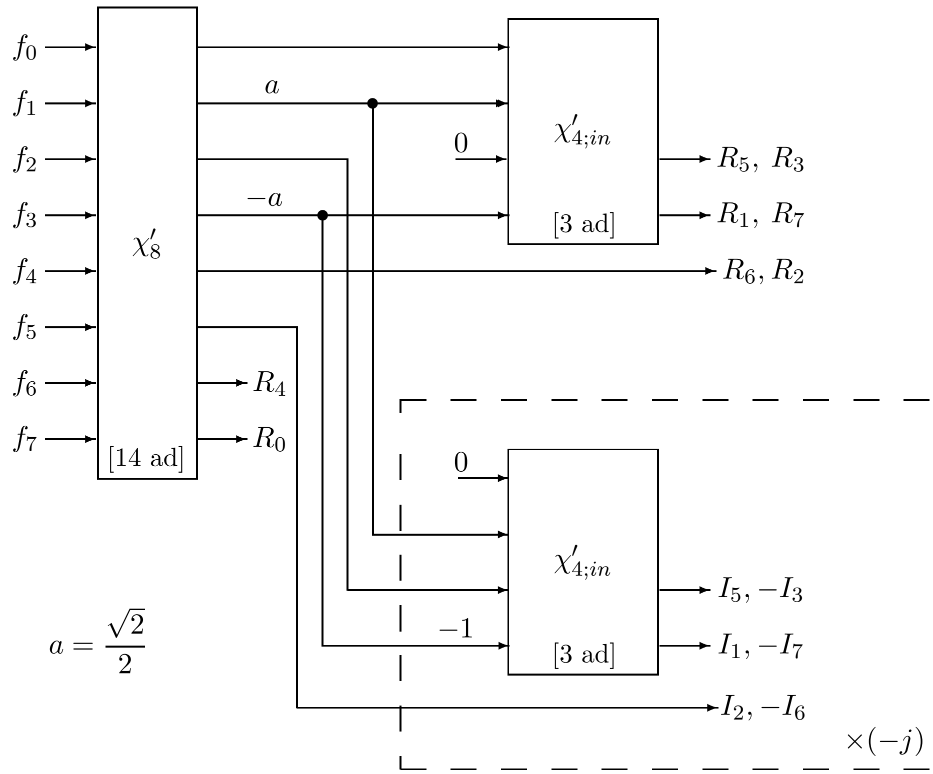


Fig. 4. Simplified block-scheme of calculation of real and imaginary parts of the 8-point DFT.

Fourier coefficients $F_p, p = 0 : 15$. After such a separation the signal-flow graph can be simplified, because of the following relations between Fourier coefficients for a real input, $R_{16-p} = R_p$, $I_{16-p} = -I_p$, when $p = 1 : 7$. The simplified signal-flow graph for calculation of the 16-point DFT is shown in Fig. 6.

The calculation of the 16-point DFT by the simplified signal-flow graph requires $m'(16) = 12$ real multiplications by factors $a = \cos(\pi/4)$, $b = \cos(\pi/8)$, and $c = \cos(3\pi/8)$. We denote by $\chi'_{8;in}$ two incomplete 8-point paired transforms for which only the last four outputs are calculated. Two incomplete 8-point paired transforms requires 9 additions each. Two incomplete 4-point paired transforms $\chi'_{4;in}$ are also used for calculation of the last two outputs. The incomplete 4-point paired transforms requires 3 additions each. The total number of the required additions is thus calculated as

$$\alpha'(16) = \alpha(\chi'_{16}) + 2\alpha(\chi'_{8;in}) + 2\alpha(\chi'_{4;in}) = 30 + 2 \times 9 + 2 \times 3 + 2 \times 2 = 58.$$

In conclusion, Table I shows the estimates for numbers of multiplication and addition that have been received in radix-2 algorithms with 1, 2, 3, and 5 butterflies [3], [8]-[10]. The data are given for a complex input f_n. It is assumed for these estimates that the complex multiplication

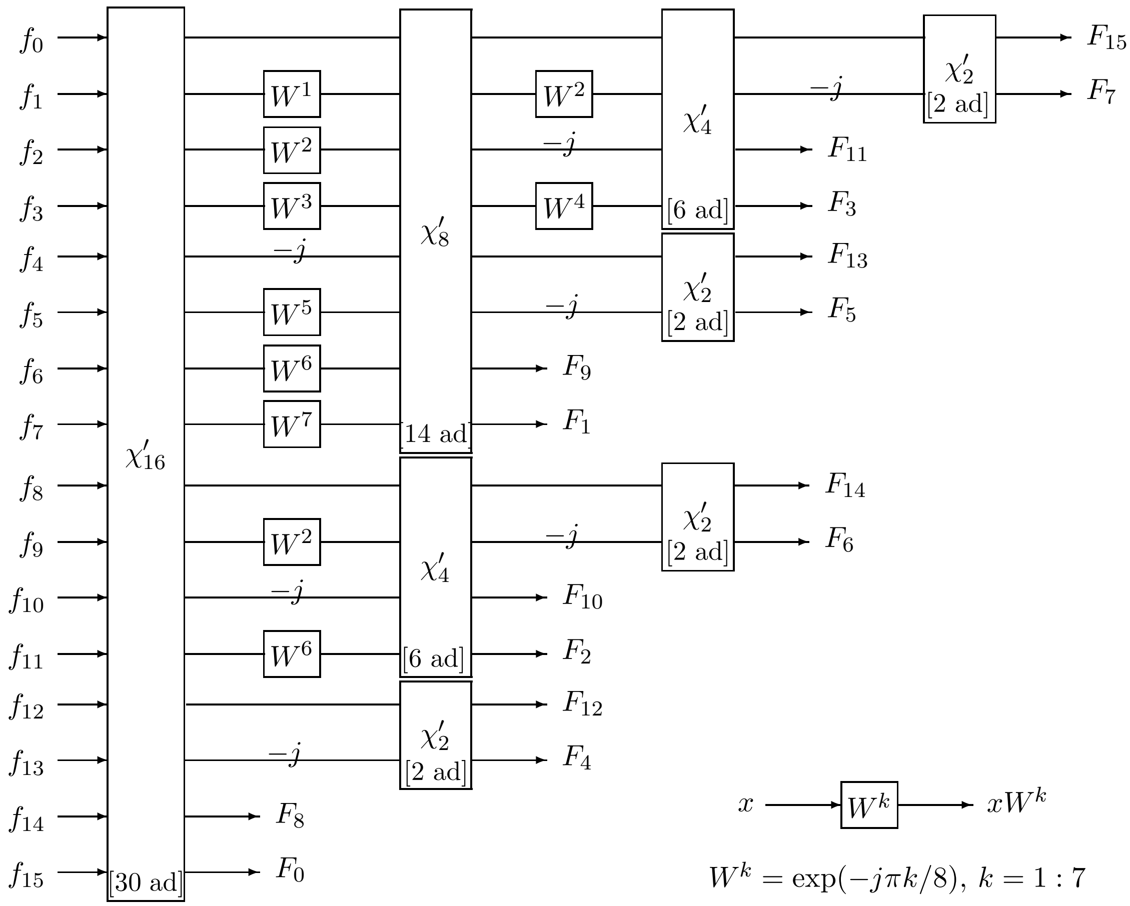


Fig. 5. Block-scheme of calculation of the 16-point DFT by paired transforms.

by a non-trivial twiddle factor in radix-2 algorithms is performed with two additions and four multiplications.

The proposed calculation of the N-point DFT by the simplified flow graph can be used for real and imaginary inputs separately. Therefore, the number of operations of multiplication is counted as twice those estimates derived for real inputs. The number of additions is counted as twice those estimates derived for real inputs, plus extra additions are needed to combine the first $(N-2)$ DFT outputs produced from real and imaginary inputs. For instance, for the 16-point DFT of complex data, the number of additions equals $2(58) + 2(14) = 144$. For the 8-point DFT of complex data, the number of additions equals $2(20) + 2(6) = 52$. We see that the paired algorithm is the best, by operations of multiplication and addition.

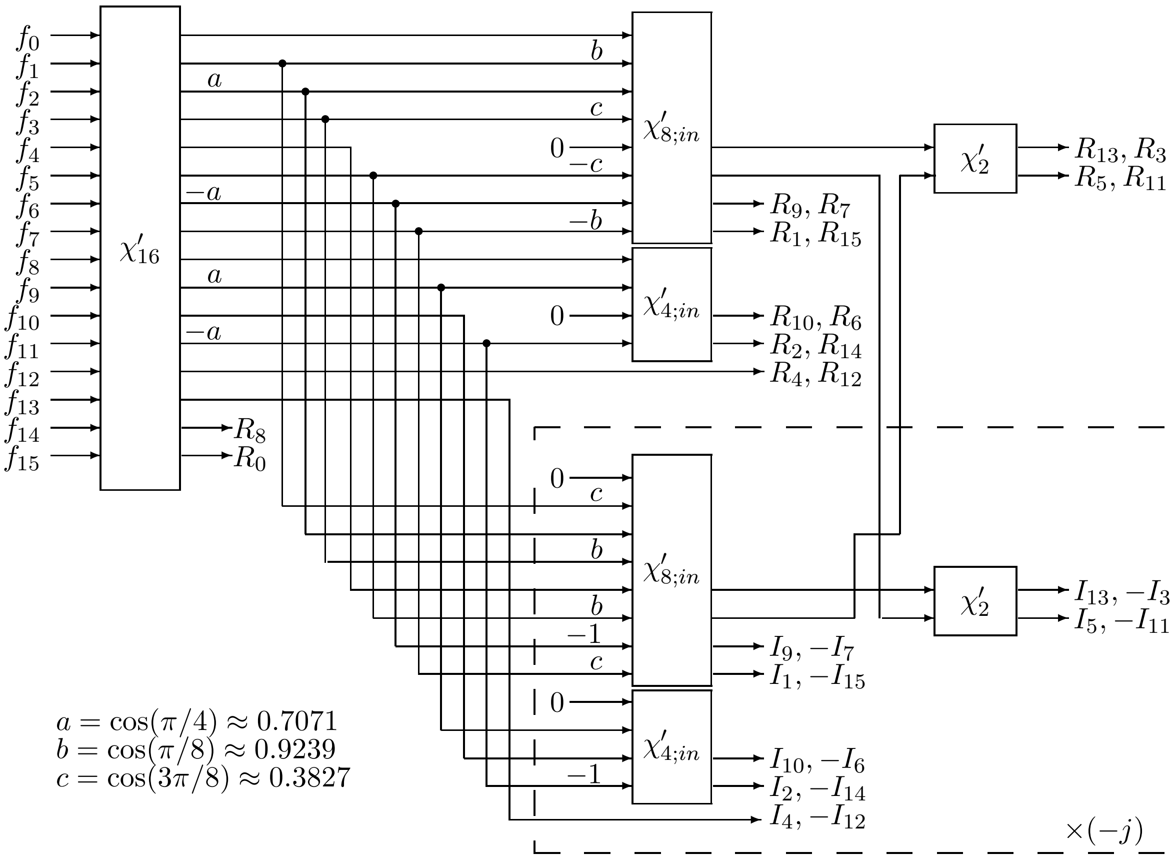


Fig. 6. Block-scheme of calculation of the 16-point DFT of a real input f_n, $n = 0 : 15$.

N	$m_{2\|1}$	$\alpha_{2\|1}$	$m_{2\|2}$	$\alpha_{2\|2}$	$m_{2\|3}$	$\alpha_{2\|3}$	$m_{2\|5}$	$\alpha_{2\|5}$	m'	α'
8	48	72	20	58	8	52	4	52	4	52
16	128	192	68	162	40	148	28	148	24	144

TABLE I

Number of multiplications/additions for calculation of the N-point FFT by algorithms radix-2 by $1, 2, 3, 5$ butterflies [9] and paired algorithm.

References

[1] J.W. Cooley and J.W. Tukey, "An algorithm for the machine computation of complex Fourier series," *Math. Comput.* 1965. vol. 9, no. 2, pp. 297-301.

[2] A.V. Oppenheim and R.W. Schafer, *Digital Signal Processing,* Prentice-Hall, Englewood Cliffs, N.J., 1975.

[3] L.R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing,* Prentice-Hall, Englewood Cliffs, N.J., 1975.

[4] R.E. Blahut, *Fast Algorithm for Digital Signal Processing,* Reading, MA: Addison-Wesley, 1984.

[5] O. Ersoy, *Fourier-Related Transform, Fast Algorithms and Applications,* Prentice Hall, 1997.

[6] D.F. Elliot and K.R. Rao, *Fast transforms - Algorithms, Analyzes, and Applications,* New York, Academic, 1982.

[7] L.B. Almeida, "The fractional Fourier transform and time frequency representations," *IEEE Trans. Signal Processing,* vol. 42, no. 11, pp. 3084-3090, Nov. 1994.

[8] I.C. Chan and K.L. Ho, "Split vector-radix fast Fourier transform," *IEEE Trans. Signal Processing,* vol. 40, no 8, pp. 2029-2040, Aug. 1992.

[9] P. Duhamel and M. Vetterli, "Fast Fourier transforms: A tutorial review and state of the art," *Signal Processing,* vol. 19, pp. 259-299, 1990.

[10] *Handbook for Digital Signal Processing,* Edited by J.F. Kaiser and S.K. Mitra, Wiley-Interscience, New-York, 1993.

[11] A.M. Grigoryan, "Algorithm for computing the discrete Fourier transform with arbitrary orders," *Jour. Vich. Mat. i Mat. Fiziki*, vol. 30, no. 10, pp. 1576-1581, 1991.

[12] A.M. Grigoryan, "An algorithm of computation of the one-dimensional discrete Hadamard transform," *Izvestiya VUZov SSSR Radioelectronica,* vol. 34, no. 8, pp. 100-103, 1991.

[13] A.M. Grigoryan, "2-D and 1-D multipaired transforms: Frequency-time type wavelets," *IEEE Trans. Signal Processing,* vol. 49, no. 2, pp. 344-353, Feb. 2001.

[14] A.M. Grigoryan and S.S. Agaian, "Split manageable efficient algorithm for Fourier and Hadamard transforms," *IEEE Trans. Signal Processing*, vol. 48, no. 1, pp. 172-183, Jan. 2000.

[15] A.M. Grigoryan and S.S. Agaian, *Multidimensional Discrete Unitary Transforms: Representation, Partitioning, and Algorithms,* New York, Marcel Dekker, Inc., 2003.

The additional information about the paired FFT and MATLAB-based codes can be fond in the book

[16] A.M. Grigoryan and M.M. Grigoryan, *Brief Notes in Advanced DSP: Fourier Analysis with MATLAB,* CRC Press Taylor and Francis Group, 2009.