Two General Models for Gradient Operators in Imaging

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Abstract

- In this paper we describe two general parametric, non symmetric 3×3 gradient models.
- Equations for calculating the coefficients of matrices of gradients are presented.
- These models for generating gradients in x-direction include the known gradient operators and new operators that can be used in graphics, computer vision, robotics, imaging systems and visual surveillance applications, object enhancement, edge detection and classification.
- The presented approach can be easier extended for large windows.

General Model of Gradients of the 2nd order

The case of 3×3 window is considered.

Type I: Let $\boldsymbol{A}$ be the following matrix:

$$\boldsymbol{A} = \frac{1}{K}\begin{bmatrix} \lambda a & d & -a \\ \lambda b & \underline{0} & -b \\ \lambda c & e & -c \end{bmatrix}, \qquad (1)$$

Here, the triplet of numbers (a, b, c) and $\lambda > 0$ are given.

The coefficients d and e will be found or selected from the condition that the sum of all coefficients equals zero.
The factor $1/K$ is calculated after the coefficients d and e.
The matrix $\boldsymbol{A}$ is called the $(\lambda, a, b, c|d)$-matrix.

Equations for coefficients after zeroing the sum of all coefficients:

$$(a+b+c)(\lambda-1)+(d+e)=0, \qquad (2)$$

The symmetric case, $e=d$,

$$d=-\frac{1}{2}(a+b+c)(\lambda-1). \qquad (3)$$

For simplicity, it is assumed that $a, b, c > 0$.

The values of d and e will be negative if $\lambda > 1$ and positive when $\lambda < 1$.

The non symmetric case, $e \neq d$.

Example 1: $a = b = c = 1, \lambda = 1$

The $d = e$ case corresponds to the Prewitt gradient operator,

$$d = -\frac{1}{2}(3)(0) = 0, \quad \boldsymbol{A} = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ 1 & \underline{0} & -1 \\ 1 & 0 & -1 \end{bmatrix}. \qquad (4)$$

Non-symmetric case: $d + e = 0$, the matrix has the form

$$\boldsymbol{A} = \frac{1}{3 + |d|}\begin{bmatrix} 1 & d & -1 \\ 1 & \underline{0} & -1 \\ 1 & -d & -1 \end{bmatrix};$$

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & -1 \\ 1 & \underline{0} & -1 \\ 1 & -1 & -1 \end{bmatrix}, \qquad \frac{1}{5}\begin{bmatrix} 1 & 2 & -1 \\ 1 & \underline{0} & -1 \\ 1 & -2 & -1 \end{bmatrix}, \qquad \frac{1}{7}\begin{bmatrix} 2 & 1 & -2 \\ 2 & \underline{0} & -2 \\ 2 & -1 & -2 \end{bmatrix},$$

when $d = 1, 2$, and 0.5.

Example 2: $a = c = 1,\ b = 2,\ \lambda = 1$

The $d = e$ case corresponds to the Sobel gradient operator,

$$d = -\frac{1}{2}(4)(0) = 0, \quad \boldsymbol{A} = \frac{1}{4}\begin{bmatrix} 1 & 0 & -1 \\ 2 & \underline{0} & -2 \\ 1 & 0 & -1 \end{bmatrix}. \tag{5}$$

In the non symmetric case, $d + e = 0$, the matrix is

$$\boldsymbol{A} = \frac{1}{4 + |d|}\begin{bmatrix} 1 & d & -1 \\ 2 & \underline{0} & -2 \\ 1 & -d & -1 \end{bmatrix}.$$

For instance, when $d = 1, 2,$ and 0.5, such matrices are

$$\frac{1}{5}\begin{bmatrix} 1 & 1 & -1 \\ 2 & \underline{0} & -2 \\ 1 & -1 & -1 \end{bmatrix}, \quad \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & \underline{0} & -2 \\ 1 & -2 & -1 \end{bmatrix}, \quad \frac{1}{9}\begin{bmatrix} 2 & 1 & -2 \\ 4 & \underline{0} & -4 \\ 2 & -1 & -2 \end{bmatrix}.$$

Example 3: $a = c = 1$, $b = 2$, $\lambda = 2$

In the symmetric case,

$$d = -\frac{1}{2}(4)(1) = -2, \quad \boldsymbol{A} = \frac{1}{8}\begin{bmatrix} 2 & -2 & -1 \\ 4 & \underline{0} & -2 \\ 2 & -2 & -1 \end{bmatrix}.$$

In the non-symmetric case, $d + e = -4$ and the matrix is

$$\boldsymbol{A} = \frac{1}{8+d}\begin{bmatrix} 2 & d & -1 \\ 4 & \underline{0} & -2 \\ 2 & -4-d & -1 \end{bmatrix} \qquad (d > 0).$$

When $d = 1$ and 2, we obtain the matrices

$$\frac{1}{9}\begin{bmatrix} 2 & 1 & -1 \\ 4 & \underline{0} & -2 \\ 2 & -5 & -1 \end{bmatrix} \quad \text{and} \quad \frac{1}{10}\begin{bmatrix} 2 & 2 & -1 \\ 4 & \underline{0} & -2 \\ 2 & -6 & -1 \end{bmatrix}.$$

Example 4: $a = c = 1,\ b = \sqrt{2},\ \lambda = 1$

In the symmetric case, $d = 0$, we obtain Frei-Chen gradient

$$\boldsymbol{A} = \frac{1}{2+\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ \sqrt{2} & \underline{0} & -\sqrt{2} \\ 1 & 0 & -1 \end{bmatrix}. \tag{7}$$

In the non-symmetric case, $d + e = 0$ and

$$\boldsymbol{A} = \frac{1}{2+d+\sqrt{2}}\begin{bmatrix} 1 & d & -1 \\ \sqrt{2} & \underline{0} & -\sqrt{2} \\ 1 & -d & -1 \end{bmatrix} \qquad (d > 0).$$

for d=1, $\boldsymbol{A} = \frac{1}{3+\sqrt{2}}\begin{bmatrix} 1 & 1 & -1 \\ \sqrt{2} & \underline{0} & -\sqrt{2} \\ 1 & -1 & -1 \end{bmatrix}.$

Example 5: $a = c = 1$, $b = (1+\sqrt{5})/2$, $\lambda = 1$

In the symmetric case, we obtain the Gold-Ratio matrix

$$\boldsymbol{A} = \frac{1}{5+\sqrt{5}}\begin{bmatrix} 2 & 0 & -2 \\ 1+\sqrt{5} & \underline{0} & -1-\sqrt{5} \\ 2 & 0 & -2 \end{bmatrix}, \qquad (d = 0).$$

In the non-symmetric case, $d + e = 0$ and

$$\boldsymbol{A} = \frac{1}{5+2d+\sqrt{5}}\begin{bmatrix} 2 & 2d & -2 \\ 1+\sqrt{5} & \underline{0} & -1-\sqrt{5} \\ 2 & -2d & -2 \end{bmatrix} \quad (d > 0).$$

$$\boldsymbol{A} = \frac{1}{7+\sqrt{5}}\begin{bmatrix} 2 & 2 & -2 \\ 1+\sqrt{5} & \underline{0} & -1-\sqrt{5} \\ 2 & -2 & -2 \end{bmatrix} \quad \text{If } d = 1. \quad (8)$$

In the $d = (1+\sqrt{5})/2$ case,

$$\boldsymbol{A} = \frac{1}{2(3+\sqrt{5})}\begin{bmatrix} 2 & 1+\sqrt{5} & -2 \\ 1+\sqrt{5} & \underline{0} & -1-\sqrt{5} \\ 2 & -1-\sqrt{5} & -2 \end{bmatrix}.$$

Example 6: $a = b = c = 1, \lambda = 5/3$

In the symmetric case, we obtain

$$d = -1,\ \ \boldsymbol{A} = \frac{1}{15}\begin{bmatrix} 5 & -3 & -3 \\ 5 & \underline{0} & -3 \\ 5 & -3 & -3 \end{bmatrix}.$$

This matrix corresponds to the Kirsch gradient operator.

Model of Matrices of Type II

Type II: Let **A** be the following matrix:

$$\boldsymbol{A} = \frac{1}{K}\begin{bmatrix} \lambda a & d & -a \\ \lambda b & \underline{f} & -b \\ \lambda c & e & -c \end{bmatrix}, \tag{10}$$

where the triplet $(a, b, c) > 0$ and number $\lambda > 0$ are given. The coefficients $d, e,$ and f will be found from the condition that the sum of all coefficients equals zero.

The scale factor $1/K$ will be found after the coefficients $d, e,$ and f.

The matrix is called **the ($\boldsymbol{\lambda, a, b, c|d, e}$)-matrix**.

Equations for the matrix coefficients:

$$(a+b+c)(\lambda-1)+(d+e+f)=0. \qquad (11)$$

In the symmetric case when $e=d$,

$$(2d+f)=-(a+b+c)(\lambda-1).$$

Example 9: $a=b=c=1,\ \lambda=1$

$$2d+f=-\frac{1}{2}(1)(0)=0,\ \boldsymbol{A}=\frac{1}{K}\begin{bmatrix}1 & d & -1\\ 1 & f & -1\\ 1 & d & -1\end{bmatrix}.$$

If $d=1$, then $f=-2$, $\boldsymbol{A}$ is the Prewitt gradient matrix

$$\boldsymbol{A}=\frac{1}{5}\begin{bmatrix}1 & 1 & -1\\ 1 & -2 & -1\\ 1 & 1 & -1\end{bmatrix}. \qquad (12)$$

Example 10: $a = c = 0, b = 1, \lambda = 1$

$$2d + f = 0, \quad \boldsymbol{A} = \frac{1}{K}\begin{bmatrix} 0 & d & 0 \\ 1 & f & -1 \\ 0 & d & 0 \end{bmatrix}.$$

If $d = 1$, then $f = -2$ and we obtain the matrix

$$\boldsymbol{A} = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If $d = 1/2$, then $f = -1$ and we obtain the matrix

$$\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1 & -1 \\ 0 & 1/2 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & -2 \\ 0 & 1 & 0 \end{bmatrix}.$$

If $d = 1/4$, then $f = -1/2$ and the matrix is

$$\boldsymbol{A} = \frac{2}{3}\begin{bmatrix} 0 & 1/4 & 0 \\ 1 & -1/2 & -1 \\ 0 & 1/4 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 1 & 0 \\ 4 & -2 & -4 \\ 0 & 1 & 0 \end{bmatrix}.$$

Case $d = 0$: $f = 0$ and (the separate 1st order gradient matrix)

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (13)$$

Example 11: $a = c = 1, b = 0, \lambda = 1$

$$2d + f = 0, \;\; \boldsymbol{A} = \frac{1}{K}\begin{bmatrix} 1 & d & -1 \\ 0 & -2d & 0 \\ 1 & d & -1 \end{bmatrix}.$$

$$\boldsymbol{A} = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \qquad (d = 1).$$

Example 12: $a = c = 1, b = -2, \lambda = 1$

$$2d + f = 0, \quad \boldsymbol{A} = \frac{1}{4 + |2d|}\begin{bmatrix} 1 & d & -1 \\ -2 & -2d & 2 \\ 1 & d & -1 \end{bmatrix}.$$

If $d = 1$, then $f = -2$, $K = 6$, and we obtain the matrix

$$\boldsymbol{A} = \frac{1}{6}\begin{bmatrix} 1 & 1 & -1 \\ -2 & -2 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

If $d = 2$, then $f = -4$, $K = 8$, and we obtain the matrix

$$\boldsymbol{A} = \frac{1}{8}\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}.$$

Examples of a few gradients on images

Given $(\lambda, a, b, c|d)$-matrix $\boldsymbol{A}$ of type I, the gradient operator is defined with this matrix in x-direction, i.e., $[G_x^2] = \boldsymbol{A}$.

The matrix of this gradient in y-direction is considered to be calculated as

$$\left[G_y^2\right] = -[G_x^2]' = -\boldsymbol{A}'.$$

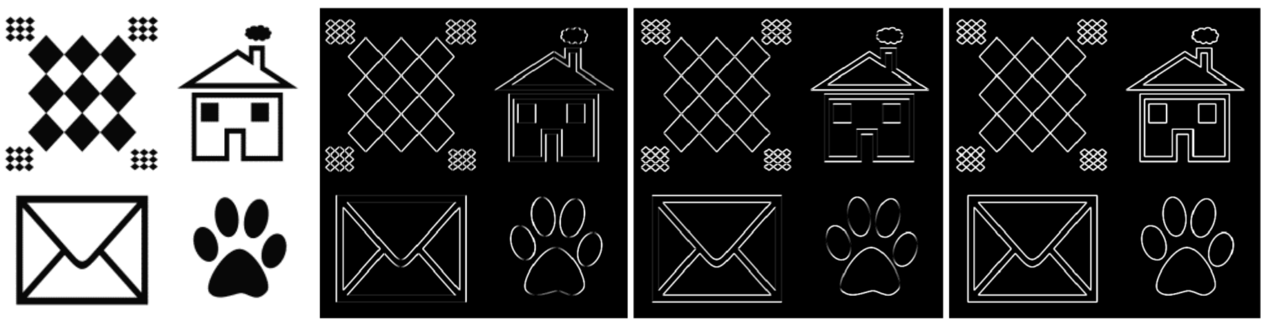
Here, $\boldsymbol{A}'$ is the transpose matrix $\boldsymbol{A}$.

The gradient operator with these matrices can be named the $(\lambda, a, b, c|d)$-gradient operator.

The $(\mathbf{1,1,1,1|1/4})$-gradient is defined with matrices

$$[G_x^2] = \frac{4}{13}\begin{bmatrix} 1 & 1/4 & -1 \\ 1 & \underline{0} & -1 \\ 1 & -1/4 & -1 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 4 & 1 & -4 \\ 4 & \underline{0} & -4 \\ 4 & -1 & -4 \end{bmatrix}$$

and $$[G_y^2] = \frac{1}{13}\begin{bmatrix} -4 & -4 & -4 \\ -1 & \underline{0} & 1 \\ 4 & 4 & 4 \end{bmatrix}.$$



(a) Image (b) x-gradient (c) y-gradient (d) maximum

***Fig. 1**. (a) The image, (b) the horizontal, (c) the vertical, and (d) the maximum* $G_m(f) = \max\{G_x^2(f), G_y^2(f)\}$ *gradient images.*

One can notice the slightly bright horizontal and vertical lines in the gradient images $G_x^2(f)$ and $G_y^2(f)$, respectively, which are due to the non zero coefficient $d = -e = 1/4$. Indeed,

$$[G_x^2] = \frac{12}{13}\cdot\underbrace{\frac{1}{3}\begin{bmatrix}1 & 0 & -1\\ 1 & \underline{0} & -1\\ 1 & 0 & -1\end{bmatrix}} + \frac{1}{13}\cdot\underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & \underline{0} & 0\\ 0 & -1 & 0\end{bmatrix}},$$

It is the arithmetic mean of two matrices, one matrix is the matrix of the Prewitt gradient in the x-direction and another one is matrix of the separated gradient in the y-direction

$$[G_x^2] = \frac{12}{13}[P_x^2] + \frac{1}{13}\left[G_y\right]. \qquad (14)$$

The weight of the separated gradient G_y is 1/13 which is a small number, when comparing with the weight 12/13 of the gradient P_x^2. Therefore, in the gradient image shown in Fig. 1(a), the extracted horizontal lines do not have high intensities. For the gradient G_y^2, similar calculations hold

$$[G_y^2] = -[G_x^2]' = -\frac{12}{13}[P_x^2]' - \frac{1}{13}[G_x]' = \frac{12}{13}[P_y^2] + \frac{1}{13}\left[G_y\right],$$

and the intensity of the vertical lines in the gradient image in Fig. 2(b) have low intensity because of the weighted coefficient 1/13. To see better the horizontal and vertical lines in the gradient images $G_x^2(f)$ and $G_y^2(f)$, the value of the coefficient in the $(1,1,1,1|d)$-matrix should be increased.

The $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}|\mathbf{1/2})$-gradient is defined with matrices

$$[G_x^2] = \frac{2}{7}\begin{bmatrix} 1 & 1/2 & -1 \\ 1 & \underline{0} & -1 \\ 1 & -1/2 & -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 & -2 \\ 2 & \underline{0} & -2 \\ 2 & -1 & -2 \end{bmatrix}$$

$$[G_y^2] = \frac{1}{7}\begin{bmatrix} -2 & -2 & -2 \\ -1 & \underline{0} & 1 \\ 2 & 2 & 2 \end{bmatrix},$$

Note: $$[G_x^2] = \frac{6}{7}\cdot\underbrace{\frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ 1 & \underline{0} & -1 \\ 1 & 0 & -1 \end{bmatrix}} + \frac{1}{7}\cdot\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & \underline{0} & 0 \\ 0 & -1 & 0 \end{bmatrix}}$$

$$[G_x^2] = \frac{6}{7}[P_x^2] + \frac{1}{7}[G_y]. \qquad (15)$$

The weight 6/7 of the Prewitt gradient in x-direction is smaller than 12/13, and the weight 1/7 of the separated gradient G_y is larger than the weight 1/13 in the (1,1,1,1|1/4)-gradient. The additional horizontal and vertical lines in the gradient images $G_x^2(f)$ and $G_y^2(f)$ can be better observed.

(a) x-gradient (b) y-gradient (c) maximum

Figure 2. *(a) The horizontal, (b) vertical, and (c) maximum gradient images*.

The $(\mathbf{1,1,1,1|1})$-gradient is defined with matrices

$$[G_x^2] = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 \\ 1 & \underline{0} & -1 \\ 1 & -1 & -1 \end{bmatrix} \text{ and } [G_y^2] = \frac{1}{4}\begin{bmatrix} -1 & -1 & -1 \\ -1 & \underline{0} & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) x-gradient (b) y-gradient (c) maximum

Figure 3. *(a) The horizontal, (b) vertical, and (c) maximum gradient images.*

The matrix $[G_x^2]$ can be written as

$$[G_x^2] = \frac{3}{4}\cdot\underbrace{\frac{1}{3}\begin{bmatrix}1 & 0 & -1\\ 1 & \underline{0} & -1\\ 1 & 0 & -1\end{bmatrix}} + \frac{1}{4}\cdot\underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & \underline{0} & 0\\ 0 & -1 & 0\end{bmatrix}}.$$

It is the arithmetic mean of two matrices,

$$[G_x^2] = (3/4)[P_x^2] + (1/4)\left[G_y\right].$$

Thus, when increasing the value of d in the $(1,1,1,1|d)$-gradient matrix, the weight of the Prewitt gradient is decreasing and the weight of the gradient G_y is increasing. Therefore, the additional horizontal in the gradient image $G_x^2(f)$ becomes more visible.

The $(1, 1, 1.5, 1|d)$-gradient

The gradient is defined by the matrices

$$[G_x^2] = \frac{2}{7+2|d|}\begin{bmatrix} 1 & d & -1 \\ 1.5 & \underline{0} & -1.5 \\ 1 & -d & -1 \end{bmatrix}$$

$$= \frac{1}{7+2|d|}\begin{bmatrix} 2 & 2d & -2 \\ 3 & \underline{0} & -3 \\ 2 & -2d & -2 \end{bmatrix}$$

and

$$[G_y^2] = \frac{1}{7+2|d|}\begin{bmatrix} -2 & -3 & -2 \\ -2d & \underline{0} & 2d \\ 2 & 3 & 2 \end{bmatrix}.$$

The cases when $d = 0$ and $1/2$:

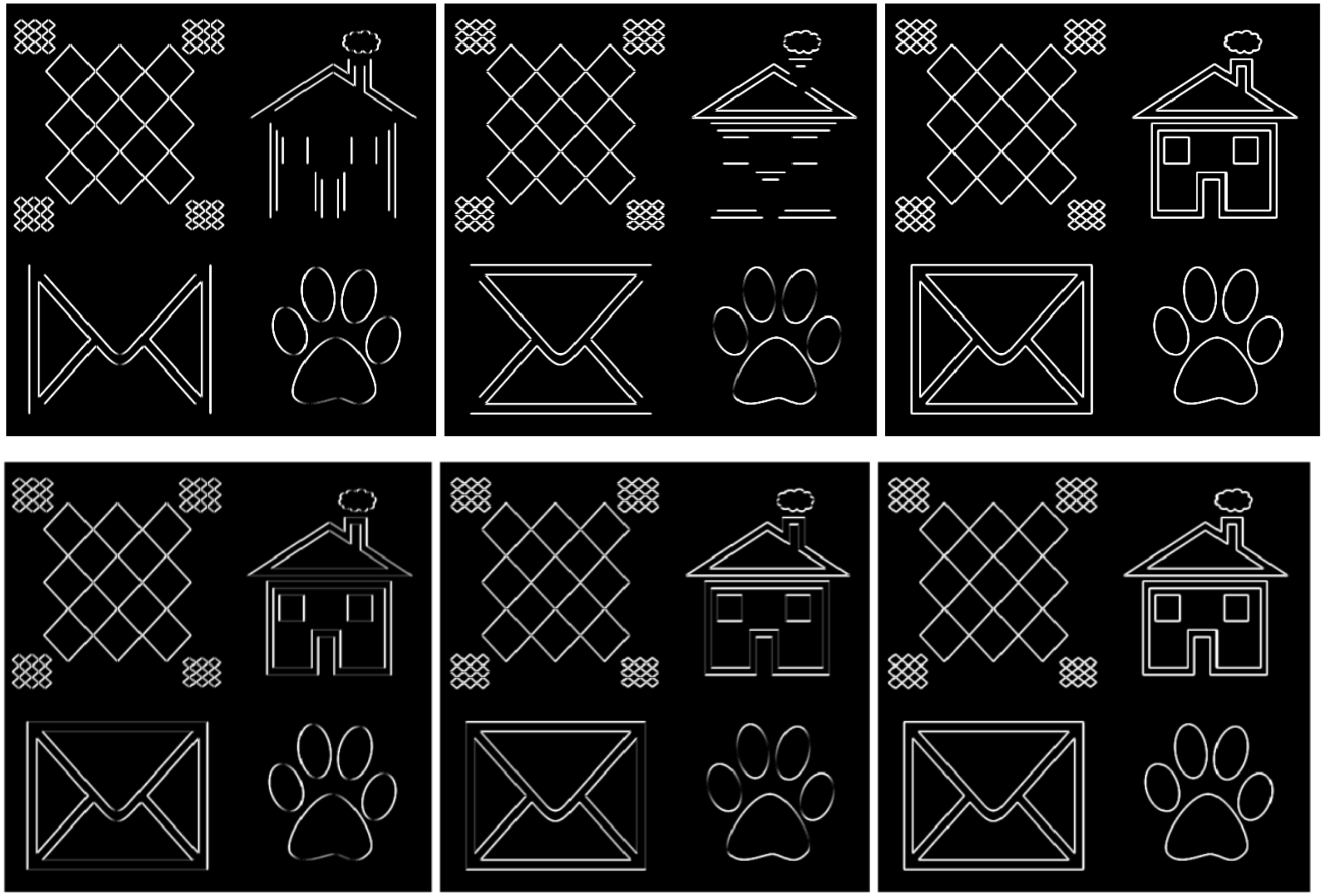
$$[G_x^2] = \frac{1}{7}\begin{bmatrix} 2 & 0 & -2 \\ 3 & \underline{0} & -3 \\ 2 & 0 & -2 \end{bmatrix} \quad \text{and} \quad \frac{1}{8}\begin{bmatrix} 2 & 1 & -2 \\ 3 & \underline{0} & -3 \\ 2 & -1 & -2 \end{bmatrix}.$$

The second matrix is the arithmetic mean of the first matrix and the matrix of the separated gradient in the y-direction,

$$[G_x^2] = \frac{7}{8} \cdot \underbrace{\frac{1}{7}\begin{bmatrix} 2 & 0 & -2 \\ 3 & \underline{0} & -3 \\ 2 & 0 & -2 \end{bmatrix}} + \frac{1}{8} \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & \underline{0} & 0 \\ 0 & -1 & 0 \end{bmatrix}}.$$

In image $G_x^2(f)$ with $d = 1/2$, the additional horizontal lines are extracted, when comparing with the $d = 0$ case. In $G_y^2(f)$ with $d = 1/2$, the vertical lines are extracted.

The cases when $d = 0$ and $d = 1/2$:

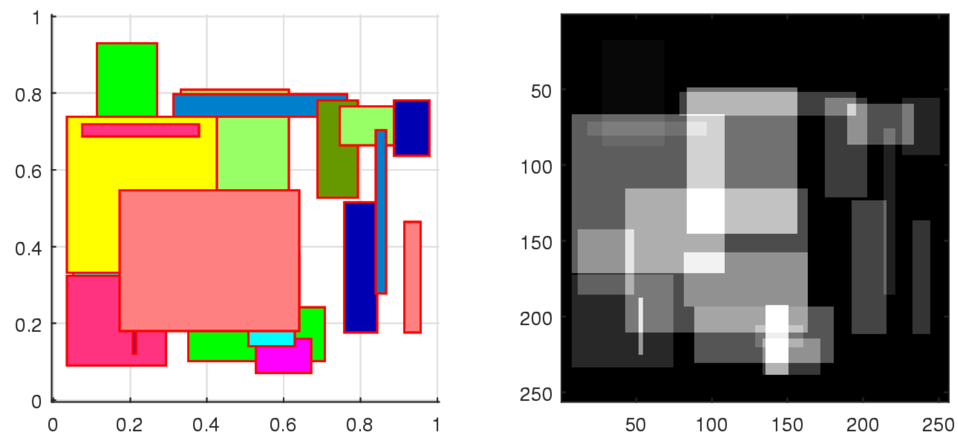


(d=0)

(d=0.5)

(b) x-gradient (b) y-gradient (c) maximum

Fig 3. *(a) The horizontal, (b) vertical, and (c) maximum gradient images***.**



(a) Model (b) image

Fig 5. *The grayscale image modeled by 20 random rectangles*

(a) x-gradient (b) y-gradient (c) maximum

Fig 6. *(a) The horizontal, (b) vertical, and (c) maximum gradient images*

Together with the maximum gradient, the square-root gradient operation is also used for edge detection

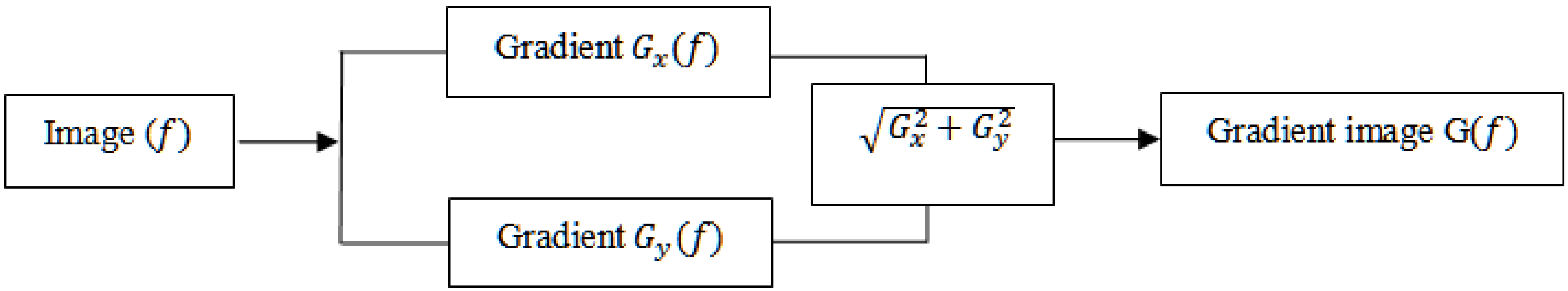


Fig. 7. *The diagram of calculation for the square-root gradient image*.

The magnitude and square-root gradient images are defined as

$$G^2(f) = |G_x^2(f)| + \left|G_y^2(f)\right|$$

$$G(f) = \sqrt{[G_x^2(f)]^2 + [G_y^2(f)]^2}.$$

The 3×3 Frei-Chen Gradients (the (1,2,3,2|0)-matrices)

The Frei-Chen gradient operators can be simplified, by using the coefficients 1.5 instead of coefficients $\sqrt{2} = 1.4142$,

$$[G_x^2] = \frac{1}{3.5}\begin{bmatrix} 1 & 0 & -1 \\ 1.5 & \underline{0} & -1.5 \\ 1 & 0 & -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 0 & -2 \\ 3 & \underline{0} & -3 \\ 2 & 0 & -2 \end{bmatrix},$$

$$[G_y^2] = \frac{1}{7}\begin{bmatrix} -2 & -3 & -2 \\ 0 & \underline{0} & 0 \\ 2 & 3 & 2 \end{bmatrix}. \qquad (16)$$

These matrices are the (1,1,1.5,1|0)-gradient matrices, when $a = 1$, $b = 1.5$, $c = 1$, and $\lambda = 1$. The matrix of the gradient G_y^2 is defined as $\left[G_y^2\right] = -[G_x^2]'$.

The 3×3 Gold-Ratio Gradients

The golden ratio number $a = \varphi$ is considered instead of $\sqrt{2}$,

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180. \qquad (17)$$

For the GR, $\varphi^2 = 1 + \varphi$ and $\varphi^3 = 1 + 2\varphi$. We call the differencing operators in x- and y-directions with matrices

$$[G_x^2] = \frac{1}{2+\varphi}\begin{bmatrix} 1 & 0 & -1 \\ \varphi & 0 & -\varphi \\ 1 & 0 & -1 \end{bmatrix},$$

$$[G_y^2] = \frac{1}{2+\varphi}\begin{bmatrix} -1 & -\varphi & -1 \\ 0 & 0 & 0 \\ 1 & \varphi & 1 \end{bmatrix}.$$

Gold-Ratio gradient generalization by $(1,1,\alpha,1|0)$-matrices

$$[G_x^2] = \frac{1}{2+|\alpha|}\begin{bmatrix} 1 & 0 & -1 \\ \alpha & 0 & -\alpha \\ 1 & 0 & -1 \end{bmatrix},$$

$$[G_y^2] = \frac{1}{2+|\alpha|}\begin{bmatrix} -1 & -\alpha & -1 \\ 0 & 0 & 0 \\ 1 & \alpha & 1 \end{bmatrix}.$$

When $\alpha = 1$, these operators are the Prewitt operators and for $\alpha = 2$, these operators are the Sobel operators. In many cases, the gradient images of these operators look similar; it is visually difficult to distinguish which operator results in the best gradient image after thresholding.

For example, we consider the square-root gradient image

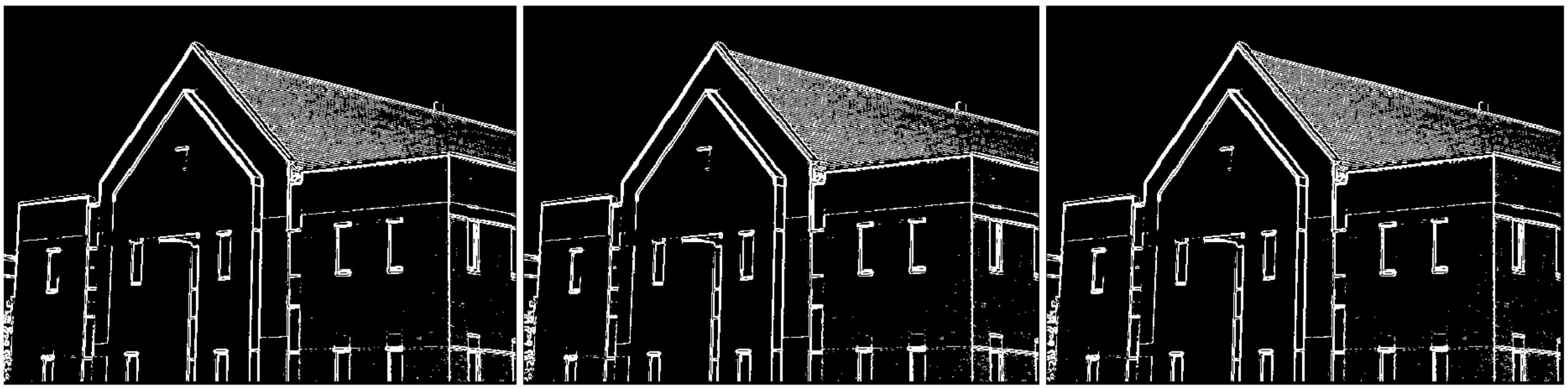
$$G(f) = \sqrt{[G_x^2(f)]^2 + [G_y^2(f)]^2}$$

for the grayscale image

***Fig. 8**. The grayscale "building" image* **.**

The cases when $\alpha = 1, \varphi$, and 2: The images are shown after the thresholding by $T = 20$, i.e., the binary threshold images are calculated at each pixel (n, m) by

$$G_T(f)_{n,m} = \begin{cases} 1, & if\ \ G(f)_{n,m} \geq T; \\ 0, & if\ G(f)_{n,m} < T. \end{cases} \tag{18}$$



(a) x-gradient (b) y-gradient (c) maximum

Fig 9. *The square-root gradient images after thresholding, when using (a) the Prewitt operator, (b) the Gold-Ratio operator, and (c) the Sobel operator***.**

Summary

Two models of 3×3 gradient operators have been introduced, which include many known gradients and new gradients that can be used in imaging.

Simple equations for calculating the coefficients of the gradient matrices are presented.

Such models can also similarly be described for the gradient operators with masks 5×5 and 7×7.

References

1. A.M. Grigoryan, S.S. Agaian, *Practical Quaternion Imaging With MATLAB*, SPIE PRESS, 2017
2. ...