Discrete Integer Fourier Transform in Real Space: Elliptic Fourier Transform

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Introduction

- The traditional N-point DFT is defined as the decomposition of the signal by N roots of the unit, which are located on the unit circle. Multiplication of a complex number by a twiddle factor can be considered in matrix form; as the Givens transformation.
- The definition of the DFT in the real space can be generalized by using a two-point transform T different from the rotations. It is assumed that the matrix of T defines a one-parametric group with period N.
- We introduce a concept of the T-generated N-block discrete transformation, or N-block T-GDT. For the N-block T-GDT the inner product is defined with respect to which rows/columns of the matrix are orthogonal.

  T is parameterized; selection of parameters can be done among the integer numbers, which leads to integer-valued metric.

DFT in Real Space

- The N-point DFT: decomposition of the signal by N roots of the unit

$$W^k = W_N^k = e^{-j\frac{2\pi k}{N}} = c_k - js_k = \cos(2\pi k / N) - j\sin(2\pi k / N)$$

Multiplication in matrix form:

$$x = (x_1, x_2) \rightarrow W^k x = (c_k - js_k)(x_1 + jx_2)$$

$$T^k x = \begin{pmatrix} c_k & s_k \\ -s_k & c_k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

N-point DFT of vector $f = (f_0, f_1, f_2, \ldots, f_{N-1})'$

$$F_p = \sum_{n=0}^{N-1} W^{np} f_n, \quad p = 0 : (N-1).$$

Matrix of the N-point DFT

$$[F_N] = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & W^1 & W^2 & \ldots & W^{N-1} \\ 1 & W^2 & W^4 & \ldots & W^{N-2} \\ 1 & \ldots & \ldots & \ldots & \ldots \\ 1 & W^{N-1} & W^{N-2} & \ldots & W^1 \end{bmatrix}$$

Transform: C^N to R^{2N}

- Vector representation

$$f \rightarrow \bar{f} = (r_0, i_0, r_1, i_1, r_2, i_2, \ldots, r_{N-1}, i_{N-1})'$$

with the vector component

$$\bar{f}_n = (\bar{f}_{2n}, \bar{f}_{2n+1})' = (r_n, i_n)' = (\operatorname{Re} f_n, \operatorname{Im} f_n)'.$$

N-point DFT as $2N$-point in R^{2N}

$$\bar{F}_p = \begin{bmatrix} R_p \\ I_p \end{bmatrix} = \sum_{n=0}^{N-1} T^{np} \bar{f}_n = \sum_{n=0}^{N-1} T^{np} \begin{bmatrix} r_n \\ i_n \end{bmatrix}, \quad p = 0 : (N-1),$$

has the following matrix:

$$[F_{N-b}] = \begin{bmatrix} I & I & I & \ldots & I \\ I & T^1 & T^2 & \ldots & T^{N-1} \\ I & T^2 & T^4 & \ldots & T^{N-2} \\ I & \ldots & \ldots & \ldots & \ldots \\ I & T^{N-1} & T^{N-2} & \ldots & T^1 \end{bmatrix},$$

$$T^{k_1+k_2} = T^{k_1} T^{k_2}, \ \forall k_1, k_2, \quad (T^0 = T^N = I).$$

Exm: 6-point DFT in R^{12}

- Six-point DFT with the matrix

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 1 & 0 & 1 \\ 1 & 0 & c_1 & s_1 & \dots & c_4 & s_4 & c_5 & s_5 \\ 0 & 1 & -s_1 & c_1 & \dots & -s_4 & c_4 & -s_5 & c_5 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & c_4 & s_4 & \dots & c_4 & s_4 & c_2 & s_2 \\ 0 & 1 & -s_4 & c_4 & \dots & -s_4 & c_4 & -s_2 & c_2 \\ 1 & 0 & c_5 & s_5 & \dots & c_2 & s_2 & c_1 & s_1 \\ 0 & 1 & -s_5 & c_5 & \dots & -s_2 & c_2 & -s_1 & c_1 \end{bmatrix}, \det(X) = 6^6.$$

Since $c_{6-k} = c_k$ and $s_{6-k} = -s_k$ for $k=1,2$ and $c_3 = -1, s_3 = 0$,

$$\hat{X}f = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & c_1 & -c_1 & -1 & -c_1 & c_1 \\ 0 & -s_1 & -s_1 & 0 & s_1 & s_1 \\ 1 & -c_1 & -c_1 & 1 & -c_1 & -c_1 \\ 0 & -s_1 & s_1 & 0 & -s_1 & s_1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} R_0 \\ R_1 \\ I_1 \\ R_2 \\ I_2 \\ R_3 \end{bmatrix}$$

$R_4 = R_2, I_4 = -I_2$ and $R_5 = R_1, I_5 = -I_1$.

[Two multiplications by $s_1, (c_1 = -0.5)$]

The inner product

- The DFT is linear in the R^{2N} and the inner product is preserved for extended DFT:

$$(\bar{f},\bar{g}) = \sum_{n=0}^{2N-1} \bar{f}_n \bar{g}_n = (X\bar{f}, X\bar{g}) = \sum_{k=0}^{2N-1} [F]_k [G]_k$$

where $[F]_k$ and $[G]_k$ denote components of the extended DFTs of the vectors f and g, respectively.

Integer representation

- The definition of the DFT in the real space R^{2N} can be generalized by 2-D transforms different from the rotations.

  The $2N$-point transform of the vector

$$\bar{f} = (f_0, f_1, f_2, f_3, \ldots, f_{2N-2}, f_{2N-1})'$$

  is defined by

$$\bar{F}_p = \begin{bmatrix} F_{2p} \\ F_{2p+1} \end{bmatrix} = \sum_{n=0}^{N-1} T^{np} \bar{f}_n = \sum_{n=0}^{N-1} T^{np} \begin{bmatrix} f_{2n} \\ f_{2n+1} \end{bmatrix},$$

$$p = 0 : (N-1).$$

  T is a matrix 2×2, *det $T=1$*, and it defines a one-parametric group with period N.

  We call the transformation $X : f \to F$

  the T-generated N-block discrete transformation, or the N-block T-GFT.

  Case *T=W*: *N-block W-GFT (N-block DFT)*

  when $(f_{2n}, f_{2n+1}) = (r_n, i_n)$.

$N=6$ case (space R^{12})

Consider the 6-block T-GDT:

$$T=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix},\qquad \det T=1,\quad T^6=I,$$

$$X=X(T)=\begin{bmatrix} I & I & I & I & I & I\\ I & T^1 & T^2 & T^3 & T^4 & T^5\\ I & T^2 & T^4 & I & T^2 & T^4\\ I & T^3 & I & T^3 & I & T^3\\ I & T^4 & T^2 & I & T^4 & T^2\\ I & T^5 & T^4 & T^3 & T^2 & T^1\end{bmatrix},\quad \det(X)=6^6.$$

The inverse matrix:

$$X^{-1}=\frac{1}{6}\begin{bmatrix} I & I & I & I & I & I\\ I & T^{-1} & T^{-2} & T^{-3} & T^{-4} & T^{-5}\\ I & T^{-2} & T^{-4} & I & T^{-2} & T^{-4}\\ I & T^{-3} & I & T^{-3} & I & T^{-3}\\ I & T^{-4} & T^{-2} & I & T^{-4} & T^{-2}\\ I & T^{-5} & T^{-4} & T^{-3} & T^{-2} & T^{-1}\end{bmatrix}=\begin{bmatrix} I & I & I & I & I & I\\ I & T^5 & T^4 & T^3 & T^2 & T^1\\ I & T^4 & T^2 & I & T^4 & T^2\\ I & T^3 & I & T^3 & I & T^3\\ I & T^2 & T^4 & I & T^2 & T^4\\ I & T^1 & T^2 & T^3 & T^4 & T^5\end{bmatrix}$$

$$\underline{X(T)^4=X(T^{-1})^4=6^2 I.}$$

Cyclic shift:

$$\bar{f}=(f_0,f_1,f_2,f_3,f_4,f_5)'\to\vec{\bar{f}}=(f_5,f_0,f_1,f_2,f_3,f_4)'$$

$$(X\bar{f})_p\to(X\vec{\bar{f}})_p=T^p(X\bar{f})_p,\quad p=0:5,$$

$$\bar{f},\vec{\bar{f}}\in R^{12}.$$

N=6: Inner product

$$(\bar{f},\bar{g}) \neq (X\bar{f}, X\bar{g})$$

We define the inner product as

$$(\bar{f},\bar{g})_A = \bar{f}'A\bar{g}$$

where $A \neq I$ is a matrix $12 \times 12.$

For that we first define a matrix $R = T'RT$ and, then, $A = I_6 \otimes R.$

- We obtain solutions

$$R = R(a,b) = \begin{bmatrix} a & b \\ -a-b & a \end{bmatrix}.$$

Example: $$R = R(1,0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

$$|x|^2 = (x,x)_R = x_0^2 + x_1^2 - x_0 x_1,$$

$$|\bar{f}|^2 = (\bar{f},\bar{f})_A = \sum_{n=0}^{11} f_n^2 - \sum_{n=0}^{11} f_{2n} f_{2n+1}.$$

"Inner product" in R^{12}

- The metric can be defined as

$$d(\bar f,\bar g)=|\bar f-\bar g|^2=(\bar f-\bar g,\bar f-\bar g)_A\geq 0$$

however $(\bar f,\bar g)_A\neq(\bar g,\bar f)_A,\ \bar f\neq\bar g\in R^{12}$.

To obtain the property

$$(\bar f,\bar g)_A=(\bar g,\bar f)_A \Leftarrow \underline{\underline{b=-a/2}}.$$

Example:

$$R=\frac{1}{\sqrt{3}}R(2,-1)=\frac{1}{\sqrt{3}}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix},\quad \det R=1.$$

$$(\bar f,\bar g)_A=\frac{1}{\sqrt{3}}\sum_{n=0}^{5}[f_{2n}\quad f_{2n+1}]\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}g_{2n}\\ g_{2n+1}\end{bmatrix}$$

$$(X\bar f,X\bar g)_A=\frac{1}{6\sqrt{3}}\sum_{p=0}^{5}[F_{2p}\quad F_{2p+1}]\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}G_{2p}\\ G_{2p+1}\end{bmatrix}$$

$$\forall\bar f,\bar g\in R^{12}.$$

Roots of the unit matrix

? Integer matrix such that $\underline{T^5 = I}$.

For $\varphi = 2\pi/5, c = \cos(\varphi) = 0.3090,$

$$C = C(\varphi) = \begin{bmatrix} c & c-1 \\ c+1 & c \end{bmatrix} = \begin{bmatrix} 0.3090 & -0.6910 \\ 1.3090 & 0.3090 \end{bmatrix}$$

$$= cU + V = c \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\det(C) = 1$ and $C^5 = I.$

- *5*-block C-GDT is defined by

$$X(C) = \begin{bmatrix} I & I & I & I & I \\ I & C^1 & C^2 & C^3 & C^4 \\ I & C^2 & C^4 & C^1 & C^3 \\ I & C^3 & C^1 & C^4 & C^2 \\ I & C^4 & C^3 & C^2 & C^1 \end{bmatrix}, \quad X^4(C) = 25I, \det X(C) = 5^5.$$

$$X^{-1}(C) = \frac{1}{5} X(C^{-1}) = \frac{1}{5} \begin{bmatrix} I & I & I & I & I \\ I & C^4 & C^3 & C^2 & C^1 \\ I & C^3 & C^1 & C^4 & C^2 \\ I & C^2 & C^4 & C^1 & C^3 \\ I & C^1 & C^2 & C^3 & C^4 \end{bmatrix}.$$

Elliptic DFT

- Consider the orbit of a point x with respect to the group of motion $\{C^k; k = 0:4\}$,

$$x \to y_1 = Cx \to y_2 = Cy_1 \to y_3 = Cy_2 \to y_4 = Cy_3 = x$$

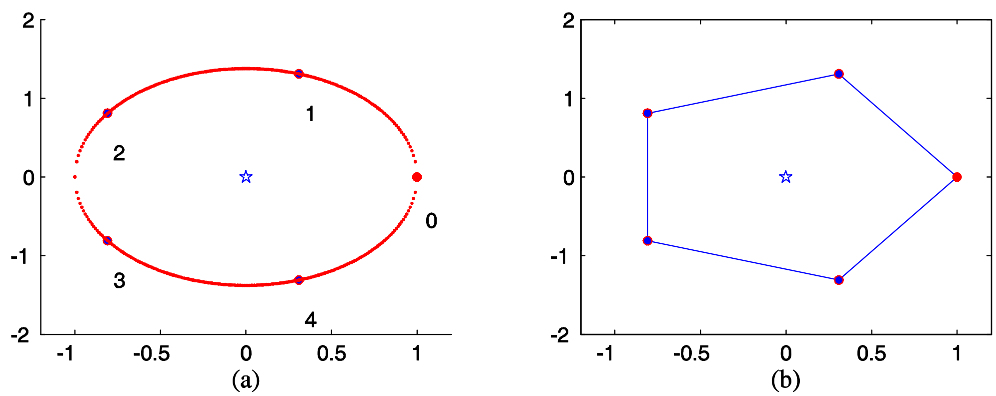


Fig 1: (a) Location of all points y_k and (b) scheme of the movement of the point (1,0).

The points move along the perimeter of the ellipse,

$$x^2 + \frac{y^2}{b^2} = 1, \qquad \left(b = \sqrt{\frac{1+c}{1-c}} = 1.3764 \right)$$

General Elliptic DFT

- Given $N > 1$ and angle $\varphi = 2\pi / N,$

the following matrix is defined

$$C = C(\varphi) = \begin{bmatrix} -\cos\varphi & \cos\varphi - 1 \\ \cos\varphi + 1 & -\cos\varphi \end{bmatrix} = \cos\varphi \cdot U + V.$$

$$C = C(\varphi) = -\cos\varphi \cdot I + \sin\varphi \cdot R,$$

$$R = R(\varphi) = \begin{bmatrix} 0 & -\tan\varphi / 2 \\ \cot\varphi / 2 & 0 \end{bmatrix}, \quad \det R = 1.$$

Such a matrix can be also defined as

$$C = C(\varphi) = \cos\varphi \cdot I + \sin\varphi \cdot R.$$

$$C^N(\varphi) = I, \quad \forall N > 1.$$

- *Definition*: Given $N > 1$ and angle ϕ

  the matrix

  $$C = C_\phi(\varphi) = \cos\varphi \cdot I + \sin\varphi \cdot R(\phi)$$

  is called *the generalized elliptic matrix (GEM)*.

Example: N=32

Consider the sinusoidal signal

$$x_r(t) = \cos(2\omega_0 t), \omega_0 = 2\pi / N, t = t_n \in [0,2\pi].$$

$$\bar{x} = (x_r(0),0,x_r(1),0,\ldots,0,x_r(N-1),0)'$$

$$\bar{y} = X(C)\bar{x}$$

$$\bar{y}_r = (\bar{y}(0),\bar{y}(2),\bar{y}(4),\bar{y}(6),\ldots,\bar{y}(2N-4),\bar{y}(2N-2))'$$

$$\bar{y}_i = (\bar{y}(1),\bar{y}(3),\bar{y}(5),\bar{y}(7),\ldots,\bar{y}(2N-3),\bar{y}(2N-1))'$$

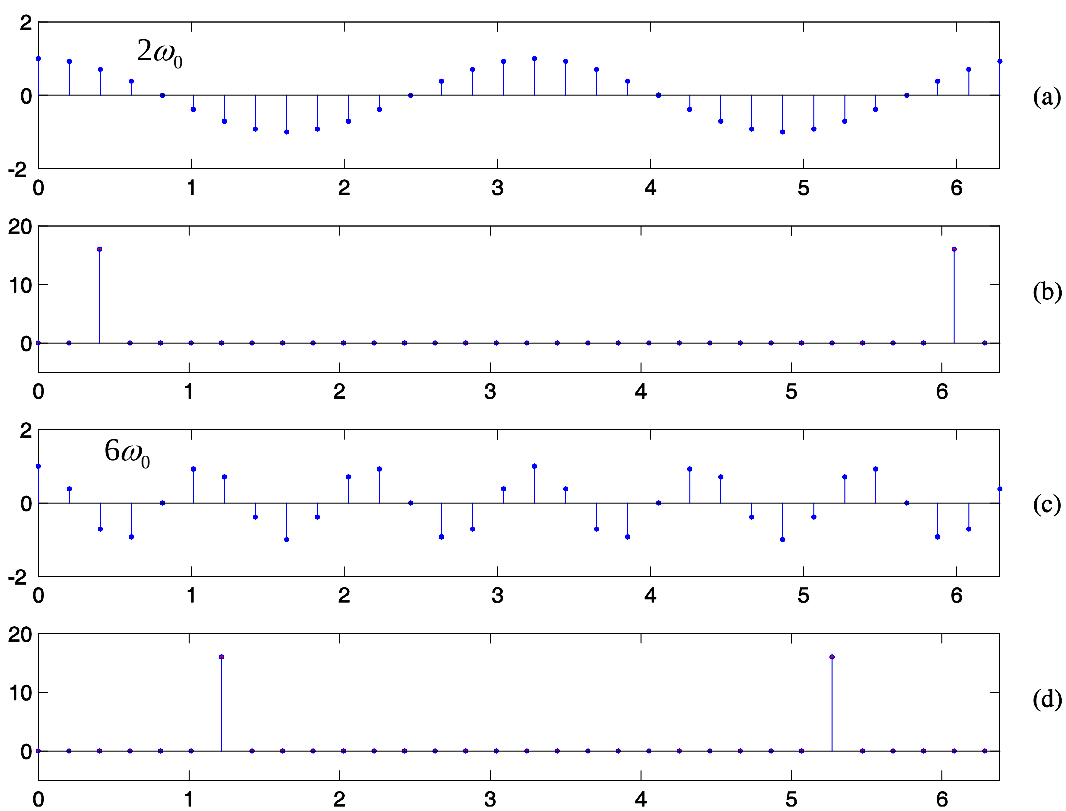


Fig 2: (a,c) Sinusoidal signals and (b,d) the 32-block DEFTs of the signals.

Imaginary part of EDFT

The N-block EDFT recognizes the carrying frequency at the frequency-points $p=6$ and $28=32-p$.

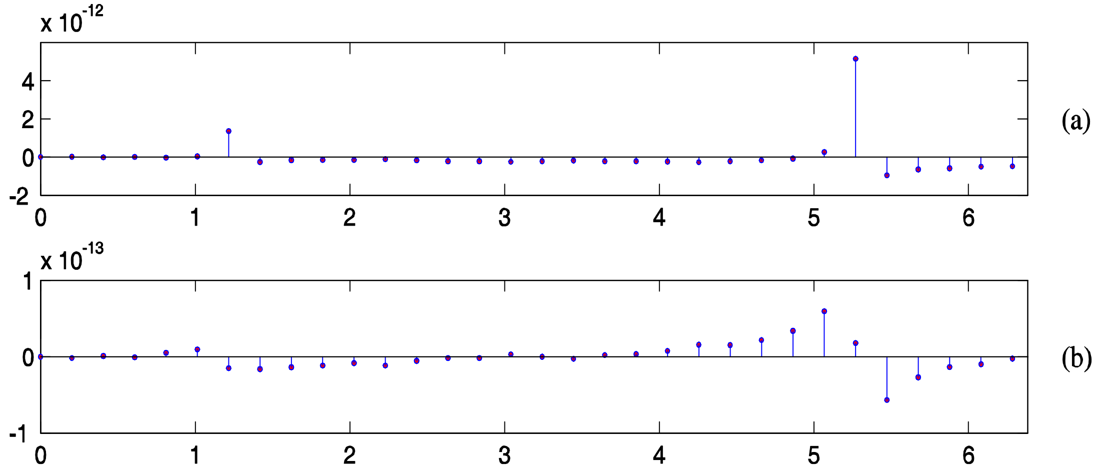


Fig 2: Imaginary parts of the (a) 32-block EDFT and (b) 32-block DFT of the signal $\cos(6\omega_0 t)$.

DFT does not have maximums on the carrying frequency-points *6,26*.

The imaginary part of the DFT by amplitude is smaller than the EDFT.

Properties of GEM

- GEM defines a one-parametric group with period *N*

$$C_\phi(\varphi_1)C_\phi(\varphi_2) = C_\phi(\varphi_1+\varphi_2),\ \forall \varphi_1,\varphi_2.$$

$$C_\phi^N(\varphi) = C_\phi(N\varphi) = C_\phi(2\pi) = I.$$

$$x \to y_1 = Cx \to y_2 = Cy_1 \to y_3 = Cy_2 \to \ldots \to y_{N-1} = Cy_{N-2} \to x = Cy_{N-1}$$

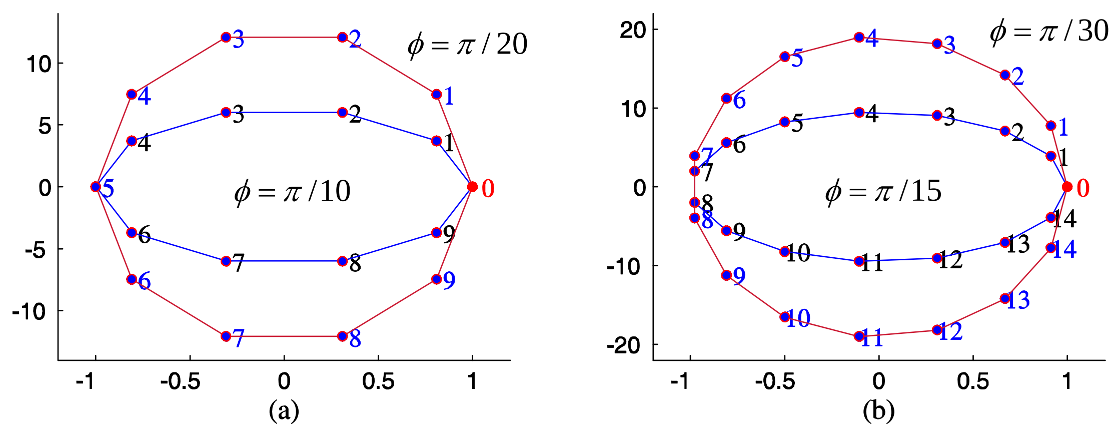


Fig 4: The scheme of the movement of the point *(1,0)* when (a) *N=10* and (b) *N=15*.

$$x^2 + \frac{y^2}{b^2} = 1, \quad b = \cot\left(\frac{\phi}{2}\right).$$

The ellipse does not depend on the angle φ .

General case

Matrix R satisfies the condition

$$R^2(\phi) = -I.$$

- Given $N > 1$ and matrix R such that $R^2 = -I$, we call the matrix

  $$C = C(\varphi) = \cos\varphi \cdot I + \sin\varphi \cdot R$$

  the generalized elliptic matrix (GEM).

  The matrix R has the form

  $$R = R(a,b,c) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

  when $a^2 + bc = -1.$

  Example:

  $$R = R(-1,2,-1) = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$$

N-block GEFT:

$C_\phi = C_\phi(\varphi)$ - generated matrix

$$X(C_\phi) = \begin{bmatrix} I & I & I & I & \dots & I \\ I & C_\phi^1 & C_\phi^2 & C_\phi^3 & \dots & C_\phi^{N-1} \\ I & C_\phi^2 & C_\phi^4 & C_\phi^6 & \dots & C_\phi^{2N-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ I & C_\phi^{N-1} & C_\phi^{2N-2} & C_\phi^{3N-3} & \dots & C_\phi^1 \end{bmatrix}$$

$$\varphi = \varphi_N = 2\pi / N.$$

parameterized by ϕ and φ.

The following property holds for the N-block GEFT as for the DFT:

$$R(x)_{N-k} = R(x)_k,\ I(x)_{N-k} = -I(x)_k$$

$$k = 1:(N/2-1),$$

where $R(x)$ and $I(x)$ denote the "real" and "imaginary" parts of the N-block GEFT of a real input x, respectively.

Example: 5-block GEFT

- For $N=5$, the 5-block GEFT has the following matrix:

$$X(C) = X(C_{\frac{\pi}{5}}(\frac{2\pi}{5})) = \begin{bmatrix} I & I & I & I & I \\ I & C^1 & C^2 & C^3 & C^4 \\ I & C^2 & C^4 & C^1 & C^3 \\ I & C^3 & C^1 & C^4 & C^2 \\ I & C^4 & C^3 & C^2 & C^1 \end{bmatrix},$$

$$C^1 = \begin{bmatrix} 0.3090 & -0.3090 \\ 2.9271 & 0.3090 \end{bmatrix}, \quad C^2 = \begin{bmatrix} -0.8090 & -0.1910 \\ 1.8090 & -0.8090 \end{bmatrix},$$

$$C^3 = \begin{bmatrix} -0.8090 & 0.1910 \\ -1.8090 & -0.8090 \end{bmatrix}, \quad C^4 = \begin{bmatrix} 0.3090 & 0.3090 \\ -2.9271 & 0.3090 \end{bmatrix},$$

$$C^5 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \det X(C) = 3125.$$

$$X^{-1}(C) = \frac{1}{5} X(C^{-1}) = \frac{1}{5} \begin{bmatrix} I & I & I & I & I \\ I & C^4 & C^3 & C^2 & C^1 \\ I & C^3 & C^1 & C^4 & C^2 \\ I & C^2 & C^4 & C^1 & C^3 \\ I & C^1 & C^2 & C^3 & C^4 \end{bmatrix}.$$

Example: 512-block GEFT

Consider $\varphi = 2\pi / 512$ and $\phi = \pi / 6$

and a "complex" signal of length 512

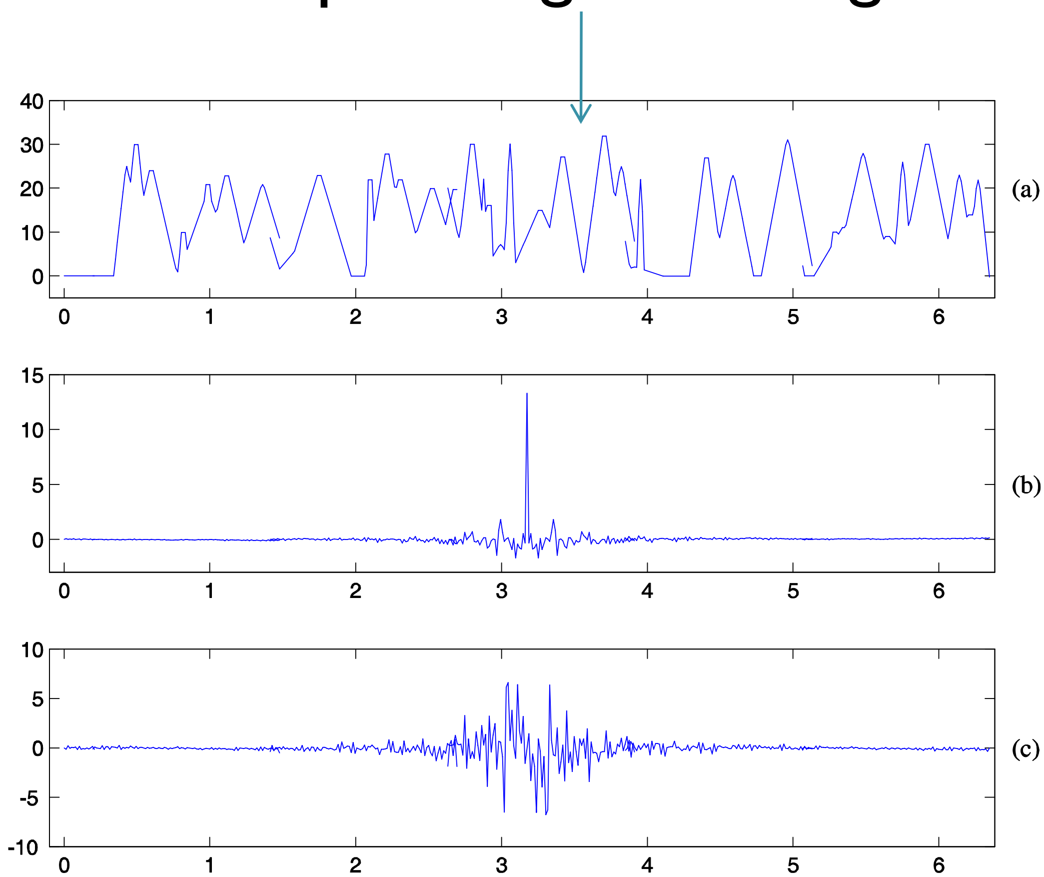
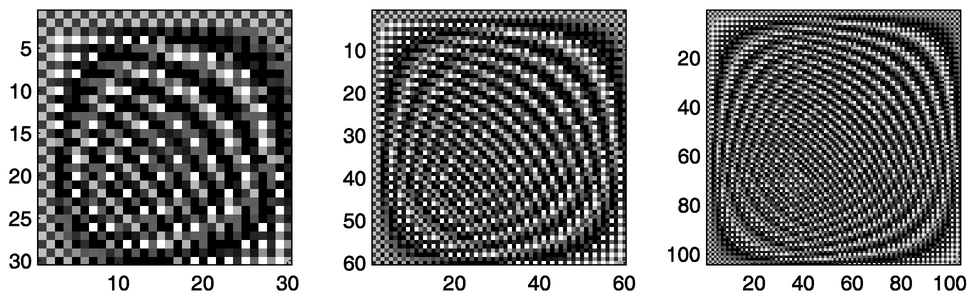


Fig 5: (a) Signal of length 512, (b) the real and (c) imaginary parts of the 512-block GEFT of the signal.

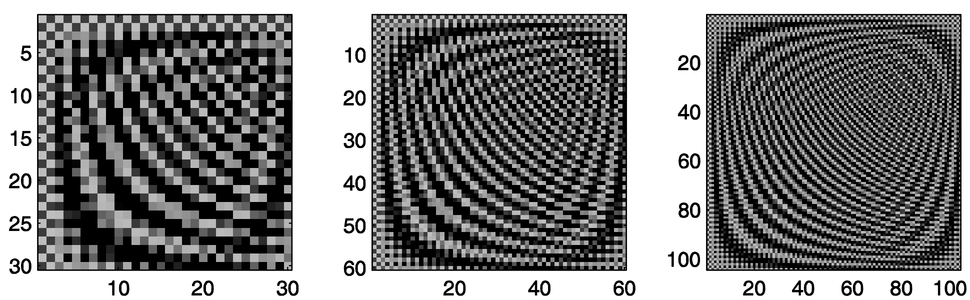
Comparison

- Images of matrices GEFT and DFT.



(a) N=15 (b) N=30 (c) N=52

Fig 6: The image of the matrix of the *N*-block GEFT when (a) *N=15*, (b) *N=30*, and (c) *N=52.*



(a) N=15 (b) N=30 (c) N=52

Fig 7: The image of the matrix of the *N*-block DFT when (a) *N=15*, (b) *N=30*, and (c) N=52.

Summary

By constructing matrices $C(\varphi)$ which are roots of the identity matrix, the N-block EFT has been defined whose block-matrices are composed by powers of $C(\varphi)$.

The set of matrices $\{C(\varphi_n); n-0:(N-1)\}$, where $\varphi_n = 2\pi n / N,$ compose a one-parametric group of order N.

The matrix $C(\varphi)$ is the matrix of rotation around the ellipse

$$\left(y+\frac{a}{b}x\right)^2+\frac{1}{b^2}x^2=r^2.$$

The ellipse turns into the circle of radius r when $a=0$ and $b=1$. Then C is the Givens rotation and the elliptic DFT is the traditional DFT.

References

1. A.M. Grigoryan and V.S. Bhamidipati, "Method of flow graph simplification for the 16-point discrete Fourier transform," *IEEE Trans. on Signal Processing*, vol. 53, no. 1, pp. 384-389, January 2005.

2. A.M. Grigoryan and M.M. Grigoryan, *Brief Notes in Advanced DSP: Fourier Analysis with MATLAB®*, Taylor & Francis Group / CRC Press, January 2009 (scheduled publication).

Example: 12-point Block Transform

Consider the integer matrix *3x3*

$$H=\begin{bmatrix}-1 & 0 & -2\\ 0 & -1 & 2\\ 3 & 2 & 1\end{bmatrix},\quad H^{-1}=\begin{bmatrix}5 & 4 & 2\\ -6 & -5 & -2\\ -3 & -2 & -1\end{bmatrix},\quad \det(H)=-1,$$

$$H^4=I \text{ and } I+H+H^2+H^3=0.$$

$$H_{4bl}=\begin{bmatrix}I & I & I & I\\ I & H^1 & H^2 & H^3\\ I & H^2 & I & H^2\\ I & H^3 & H^2 & H^1\end{bmatrix}$$

$$=\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1\\
\hline
1 & 0 & 0 & -1 & 0 & -2 & -5 & -4 & 0 & 5 & 4 & 2\\
0 & 1 & 0 & 0 & -1 & 2 & 6 & 5 & 0 & -6 & -5 & -2\\
0 & 0 & 1 & 3 & 2 & 1 & 0 & 0 & -1 & -3 & -2 & -1\\
\hline
1 & 0 & 0 & -5 & -4 & 0 & 1 & 0 & 0 & -5 & -4 & 0\\
0 & 1 & 0 & 6 & 5 & 0 & 0 & 1 & 0 & 6 & 5 & 0\\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1\\
\hline
1 & 0 & 0 & 5 & 4 & 2 & -5 & -4 & 0 & -1 & 0 & -2\\
0 & 1 & 0 & -6 & -5 & -2 & 6 & 5 & 0 & 0 & -1 & 2\\
0 & 0 & 1 & -3 & -2 & -1 & 0 & 0 & -1 & 3 & 2 & 1
\end{array}\right]$$

This is not Fourier-like transform: **$H^2 \neq I$.**

Example: 15-point Block Transform

Consider the complex matrix *3x3*

$$H = \begin{bmatrix} -0.1483 - 1.3080i & 0.0566 - 0.5002i & 0.5834 - 0.3570i \\ 0.3416 & -0.4714 & -0.8131 \\ 1.1971 + 1.3080i & 0.8862 + 0.5002i & -0.6893 + 0.3570i \end{bmatrix},$$

$$\det(H) = 1.$$

$$H^5 = I \text{ and } I + H + H^2 + H^3 + H^4 = 0.$$

$$H_{5bl} = \begin{bmatrix} I & I & I & I & I \\ I & H^1 & H^2 & H^3 & H^4 \\ I & H^2 & H^4 & H^1 & H^3 \\ I & H^3 & H^1 & H^4 & H^2 \\ I & H^4 & H^3 & H^2 & H^1 \end{bmatrix}$$

$$H_{5bl}^{-1} = \begin{bmatrix} I & I & I & I & I \\ I & H^{-1} & H^{-2} & H^{-3} & H^{-4} \\ I & H^{-2} & H^{-4} & H^{-1} & H^{-3} \\ I & H^{-3} & H^{-1} & H^{-4} & H^{-2} \\ I & H^{-4} & H^{-3} & H^{-2} & H^{-1} \end{bmatrix}$$

$$H_{5bl} \cdot H_{5bl}^{-1} = 5I.$$

This is a Fourier-like transform.