Elliptic Discrete Fourier Transforms of Type II

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Introduction

- Fourier analysis is one of the most frequently used tools in signal/image processing and communication systems.
- The N-point discrete Fourier transformation can be defined in the real space R^{2N} by the block-wise matrix. Each $2x2$ block of this matrix represents the Givens rotation, or multiplication by the twiddle factors.

  These coefficients are roots of the unit and represent the Givens rotations by angles $\varphi_k=(2\pi/N)k,\ k=0:(N-1).$
- We introduce a concept of the elliptic DFTs in the real space, which are defined by 2-point transformations different from the Givens rotations.

  Matrices of these transformations describes the movement of points around ellipses.

Transformation: C^N to R^{2N}

- Consider the transformation of the signal

$$f=(f_0,f_1,f_2,\ldots,f_{N-1})\to\overline{f}=(r_0,i_0,r_1,i_1,r_2,i_2,\ldots,r_{N-1},i_{N-1})$$

and components of the vector $\overline{f}$ as

$$\bar{f}_k=(\bar{f}_{2k},\bar{f}_{2k+1})'=(r_k,i_k)'=(\mathrm{Re}\, f_k,\mathrm{Im}\, f_k)'.$$

The N-point DFT as $2N$-point in real space R^{2N}

$$\overline{F}_p=\begin{bmatrix}R_p\\ I_p\end{bmatrix}=\sum_{k=0}^{N-1}\begin{bmatrix}\cos\varphi_k & \sin\varphi_k\\ -\sin\varphi_k & \cos\varphi_k\end{bmatrix}^p\begin{bmatrix}r_k\\ i_k\end{bmatrix}=\sum_{k=0}^{N-1}T^{kp}\bar{f}_k,\quad p=0:(N-1),$$

has the following matrix:

$$[F_{N-b}]=\begin{bmatrix}I & I & I & \ldots & I\\ I & T^1 & T^2 & \ldots & T^{N-1}\\ I & T^2 & T^4 & \ldots & T^{N-2}\\ I & \ldots & \ldots & \ldots & \ldots\\ I & T^{N-1} & T^{N-2} & \ldots & T^1\end{bmatrix}.$$

$T^{k_1+k_2}=T^{k_1}T^{k_2},\ \forall k_1,k_2,$
$(T^0=T^N=I).$

Elliptic DFT of type I

T-generated N-block discrete transformation, or the N-block T-GFT is defined by the block-wise DFT type matrix

$$H = (H_{(n,m)} = T^{nm})_{n,m=0:(N-1)}$$

T is a matrix *2x2*, det T=1, and it defines a one-parametric group with period N.

Case *T=W*: *The N-block W-GFT (N-block DFT)*

Example: Given the angle $\varphi = 2\pi / N,$ consider the matrix

$$T = T(\varphi) = \begin{bmatrix} \cos\varphi & \cos\varphi - 1 \\ \cos\varphi + 1 & \cos\varphi \end{bmatrix} = \cos\varphi \cdot I + \sin\varphi \cdot R$$

$$R = R(\varphi) = \begin{bmatrix} 0 & -\tan(\varphi/2) \\ \cot(\varphi/2) & 0 \end{bmatrix}, \quad (\det R = 1, \quad R^2 = -I).$$

$$T^N(\varphi) = I.$$

$$T = T_{\frac{2\pi}{7}} = \begin{bmatrix} 0.6235 & -0.3765 \\ 1.6235 & 0.6235 \end{bmatrix}, \quad \det T = 1.$$

Elliptic DFT of type II

For new EDFT, the construction of ***N***th roots of the identity matrix is based on the specific projection operators which have an analytical formula for the general ***N*** case.

Case *(N=2):* Let $a=(a_1,a_2)$ and $b=(b_1,b_2)$ be 2-D vectors both with the norm one, $||a||=||b||=1$.

Let H be the matrix that satisfies the equations:

$$Ha' = b', \quad Hb' = a'. \qquad \Rightarrow \quad H^2 = I.$$

$$H = H(a,b) = v_1'a + v_2'b$$

$$H = \frac{1}{1-s^2}[(b'a + a'b) - s(a'a + b'b)] \qquad s = (a,b) = ab' = ba'.$$

Example 3

Vectors *a=(1,-2)* and *b=(3,-4)*; the matrix is

$$H = H\left(\frac{a}{\sqrt{5}}, \frac{b}{5}\right) = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix}, \quad H^2 = \mathrm{I}, \quad \det H = -1.$$

The matrix *H* can also be defined from equations

$$Ha' = b', \qquad Hb' = a'$$

in a different way. By adding and subtracting these equations, we obtain

$$H = \frac{1}{\| y_1 \|} y_1' y_1 - \frac{1}{\| y_2 \|} y_2' y_2$$

where $y_1=a+b$ and $y_2=a-b$ are orthogonal vectors.

Case (N=3)

Let a_1, a_2, and a_3 be 2-D vectors, $||a_k||=1$, $k=1,2,3$.

Let H be the matrix that satisfies the equations:

$$Ha_1' = a_2', \quad Ha_2' = a_3', \quad Ha_3' = a_1' \qquad \Rightarrow \quad H^3 = I.$$

We consider the matrix in the form

$$H = H(a_1, a_2, a_3) = v_1'a_1 + v_2'a_2 + v_3'a_3 \qquad (\det H = 0)$$

$$H = v_1'y_1 + v_2'y_2$$

where $y_1 = a_1 - a_2$ and $y_2 = a_2 - a_3$.

$$H = \frac{1}{1-s^2}[(y_2' + s(y_1' + y_2'))y_1 - (sy_2' - (y_1' + y_2'))y_2]$$

$$s = (a_1, a_2) = a_1 a_2'.$$

Exampe 4

Vectors $a_1=(1,2)$, $a_2=(6,5)$, and $a_3=(-1,-2)$ then

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{\sqrt{89}}, \frac{a_3}{\sqrt{5}}\right) = \begin{bmatrix} -2.3806 & 2.5932 \\ -1.6530 & 1.3806 \end{bmatrix}, \quad \det H = 1,$$

$$H^3 = I, \quad \text{and} \quad I + H + H^2 = 0.$$

In the real space R^6 the block transform matrix is

$$X = \begin{bmatrix} I & I & I \\ I & H^1 & H^2 \\ I & H^2 & H^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -2.3806 & 2.5932 & 1.3806 & -2.5932 \\ 0 & 1 & -1.6530 & 1.3806 & 1.6530 & -2.3806 \\ 1 & 0 & 1.3806 & -2.5932 & -2.3806 & 2.5932 \\ 0 & 1 & 1.6530 & -2.3806 & -1.6530 & 1.3806 \end{bmatrix}$$

$\det(X) = 27 = 3^3$ and $X^4 = 9I$ as for the 3-point DFT.

General N>2 case

Given vectors a_1 and a_2, let $y_1 = a_1 / ||a_1||$ and $y_2 = a_2 / ||a_2||$. Consider the matrices $P_{n,m}$

$$P_{1,1} = y_1' y_1, \quad P_{1,2} = y_1' y_2, \quad P_{2,1} = y_2' y_1, \quad P_{2,2} = y_2' y_2.$$

$$P'_{n,m} = P_{n,m}, \qquad P^2{}_{n,m} = P_{n,m}, \qquad P_{n,m} P_{p,k} = s^{|m-p|} P_{n,k}.$$

The *Nth* root of the identity matrix is defined as

$$H = \frac{1}{1-s^2}\left[s(P_{1,1} - P_{2,2}) - P_{1,2} + P_{2,1} + 2\cos(\frac{2\pi}{N})(P_{2,2} - sP_{2,1}) \right]$$

matrix S **matrix Q**

$$s = s_{1,2} = (a_1, a_2) = a_1 a_2'.$$

$$H = S + 2\cos(\frac{2\pi}{N})Q, \qquad S^2 = -I, \quad Q^2 = Q, \quad QS + SQ = S.$$

Exampe 5: 15th root of the identity matrix

Vectors $a_1=(-1,2)$, $a_2=(3,4)$, and $N=15$:

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{5}\right) = \begin{bmatrix} 1.0068 & 1.1742 \\ -0.1483 & 0.8203 \end{bmatrix}, \quad \det H = 1,$$

$$H^{15} = I, \quad \text{and} \quad I + H + H^2 + \ldots + H^{14} = 0.$$

The product $s=(a_1,a_2)=0.4473$ and the matrices ***S*** and ***Q*** in the representation of the matrix ***H***

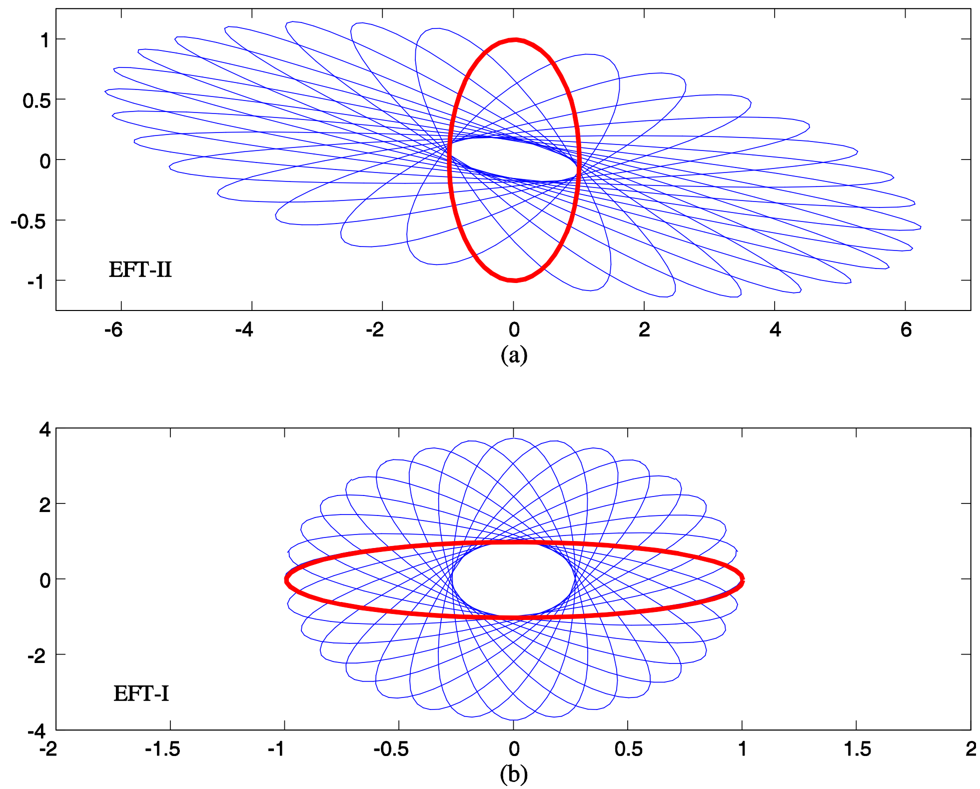
$$H = S + 2\cos(\frac{2\pi}{15})Q = S + 1.8271Q$$

equal

$$S = \begin{bmatrix} 0.0894 & 0.6261 \\ -1.6100 & 0.0894 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.60 & 0.30 \\ 0.80 & 0.40 \end{bmatrix}.$$

Movement of the circle

The successive movement of the unit circle by the groups of motion H^n, when $n=0:15$.



EDFT-II

EDFT-I

Figure 1.

Movement of the point (1,0)

The successive movement of the point *(1,0)* by the group of motion H^n, when $n=0:N$.

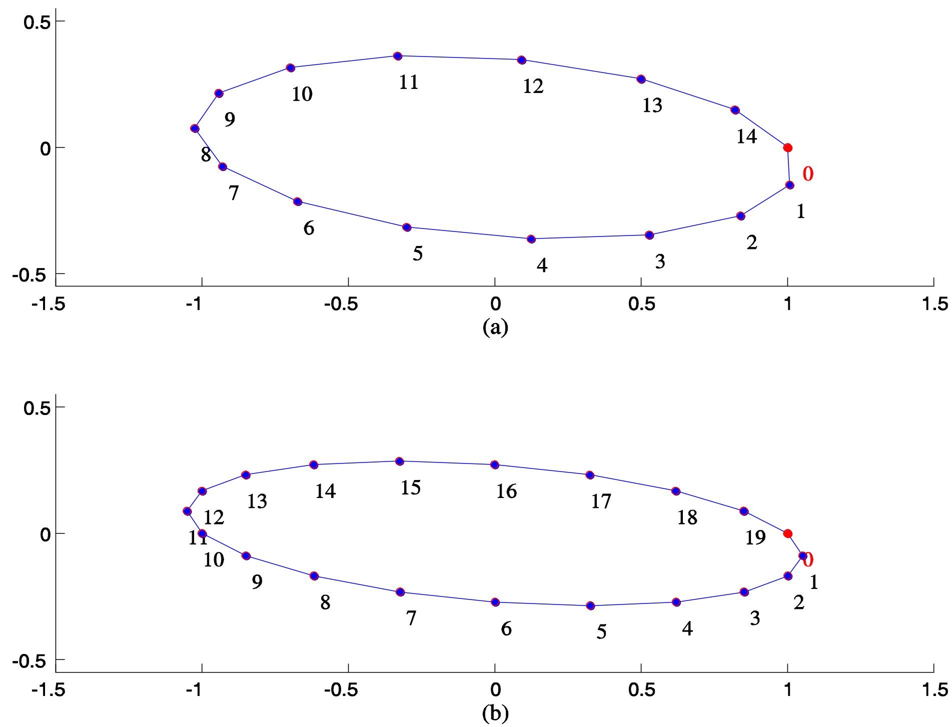
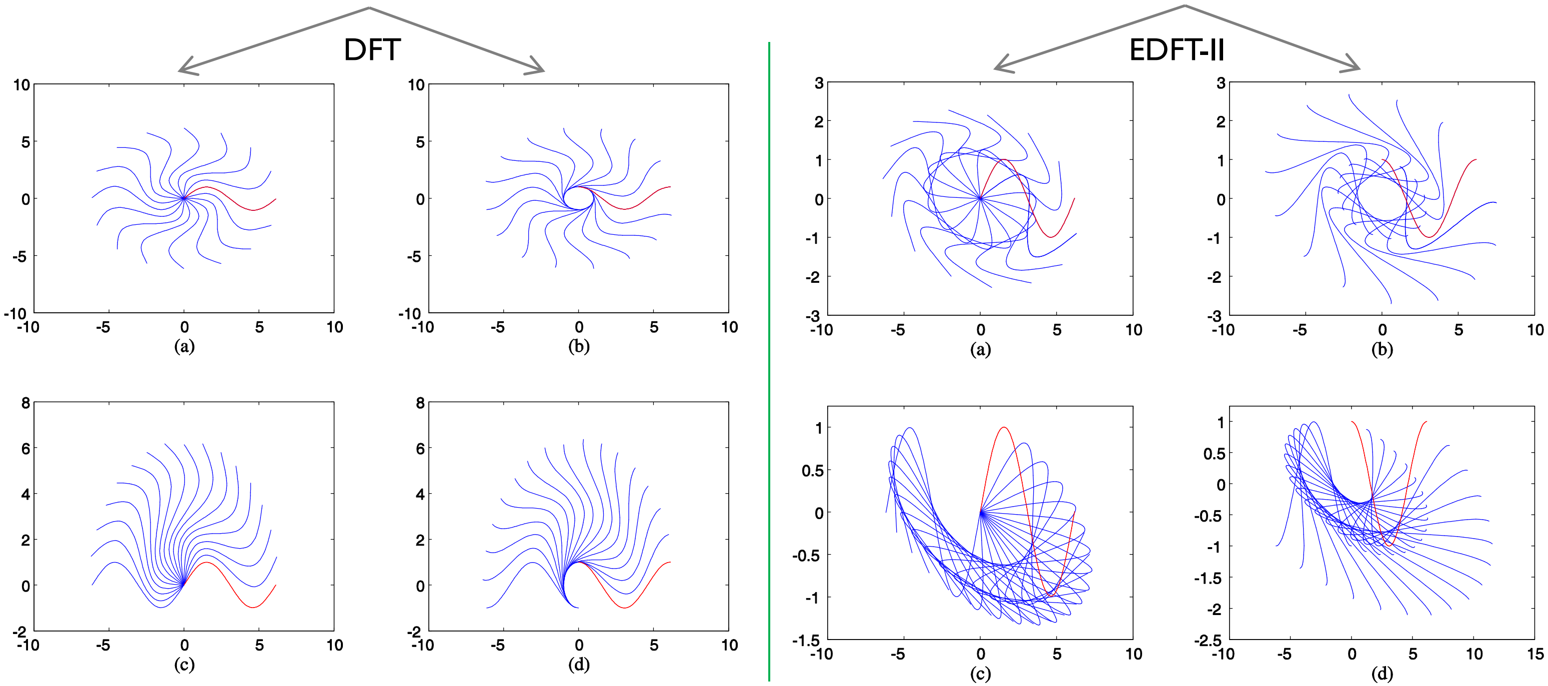


Figure 2: (a) $N=15$ and (b) $N=20$.

Movement of sine and cosine waves

Waves *sin(2πt)* and *cos(2πt)*, *t* is in *[0,2π]*, when applying two groups of motion:

$$\{W_N^k; k = 0:(N-1)\} \quad and \quad \{H^k; k = 0:(N-1)\}$$



Figures 3 and 4: (a,b) $N=16$ and (c,d) $N=32$.

Experimental Results

? It is interesting to know if the EDFT-II can distinguish the carrier frequencies of the sine waves as the DFT does, or better.

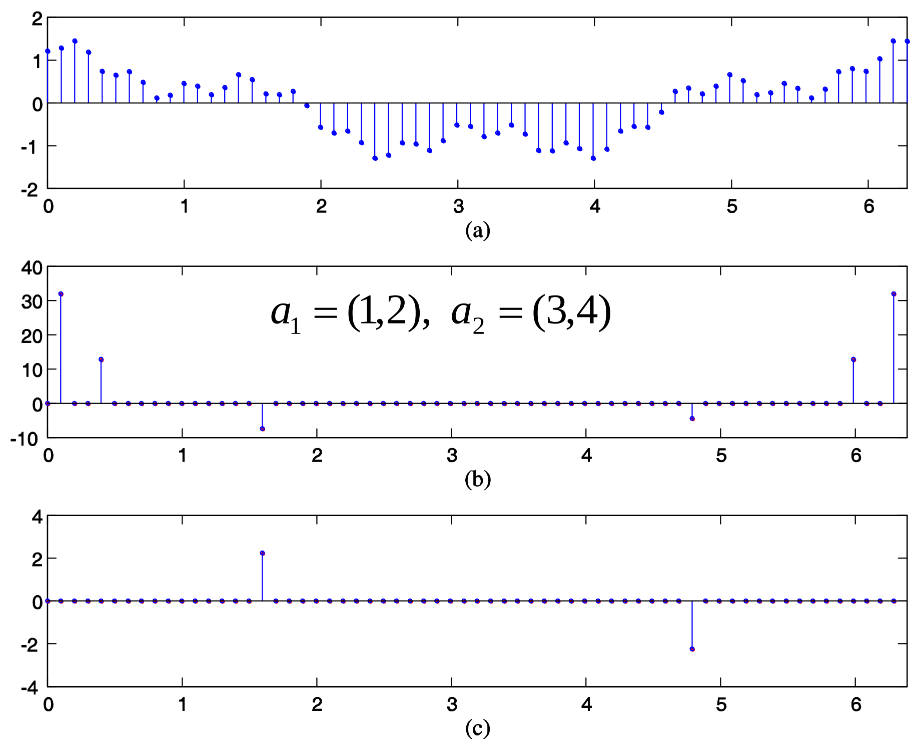
- The EDFTs of type I possess this property, and the imaginary part of these transforms is more sensitive to such frequencies than that for the DFT.
- The EDFTs of type II are also able to distinguish the carrier frequencies of the waves in a different degree which depends on the vector-generators a_1 and a_2.

Example 6: N=64

The discrete wave is sampled from the signal

$$x(t)=\cos(\omega t) + 0.4\cos(4\omega t) - 0.2\cos(16\omega t-0.4),$$

$\omega=\pi/32$, and t runs 64 points in the interval $[0,2\pi]$.



x(t)

Maximums at

p=1,4,16,N-p

Re(EDFT)

p=16,N-p=48

Im(EDFT)

Figure 5: Signal and the 64-block EDFT-II.

Example 7: N=64

The vectors $a_1=(1,2)$, $a_2=(12,2)$; the basic matrix

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{148}\right) = \begin{bmatrix} 1.4762 & 0.3647 \\ -0.6607 & 0.5142 \end{bmatrix}.$$

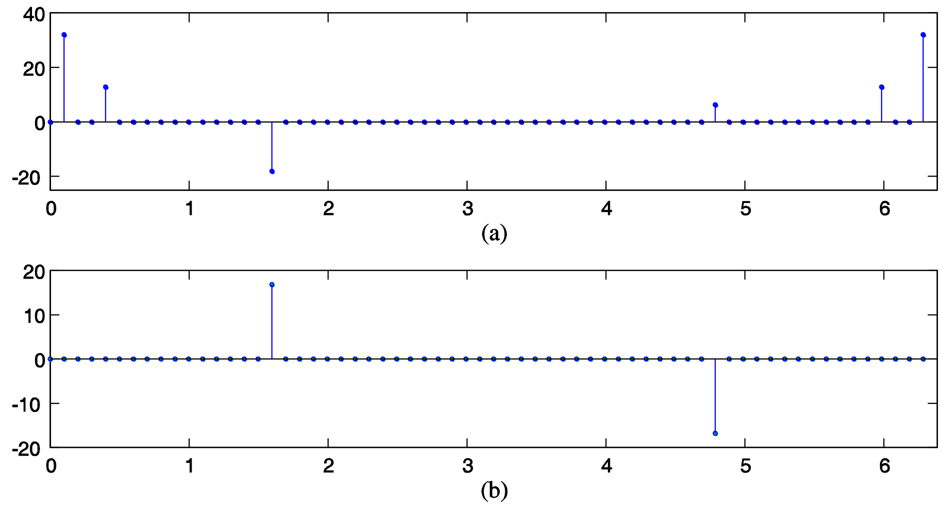


Fig 6: Real and imaginary parts of the 64-block EDFT-II.

- *The amplitudes of the components of the EDFT at the frequency-points p=16 and 48 become big. This transform distinguishes the carrier frequencies of the signal x(t) better than the EDFT with vectors* $a_1=(1,2)$ *and* $a_2=(3,4)$ *and better than the DFT.*

Comparison with the DFT

The vectors a_1=(1,2), a_2=(12,2); the basic matrix

$$H = H\left(\frac{a_1}{\sqrt{5}}, \frac{a_2}{148}\right) = \begin{bmatrix} 1.4762 & 0.3647 \\ -0.6607 & 0.5142 \end{bmatrix}.$$

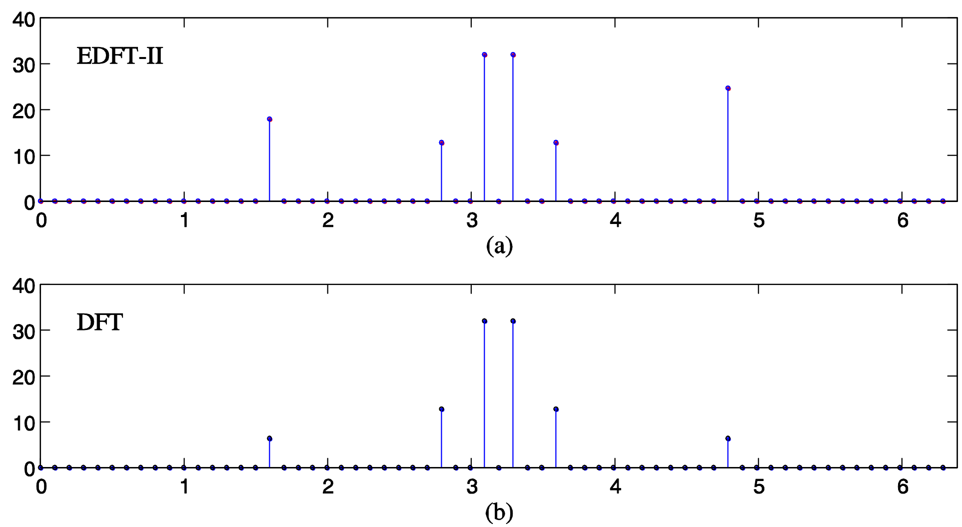


Fig 7: Amplitude spectrum of the 64-block EDFT-II and the 64-point DFT.

• *This transform distinguishes the carrier frequencies of the signal x(t) better than the DFT.*

Example 8: signal of length 256

The vectors $a_1=(3,5)$, $a_2=(3,4)$.

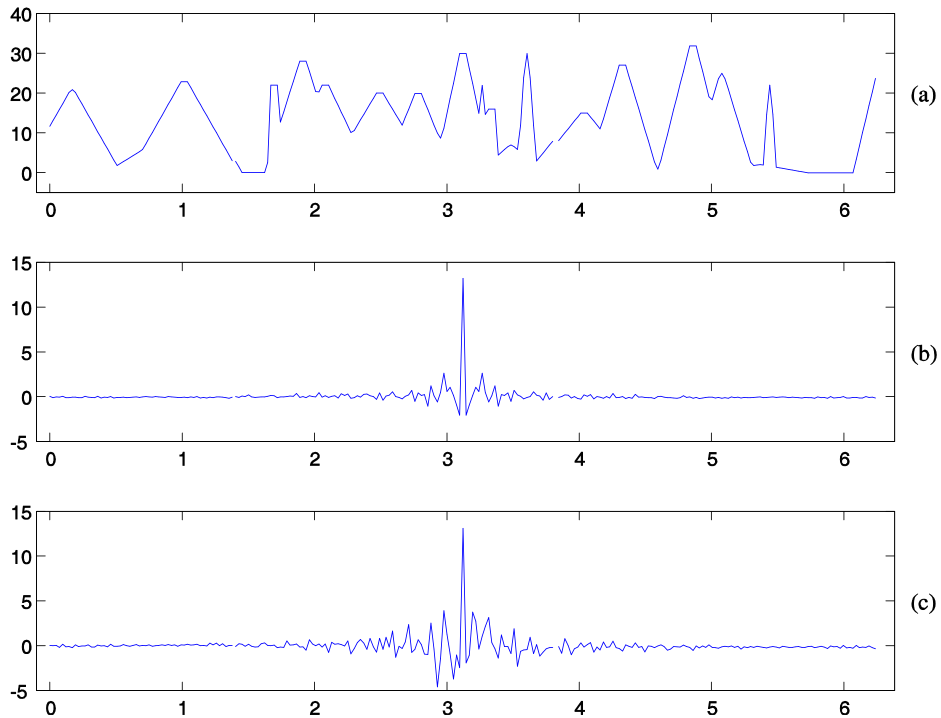


Fig 8: Signal and real parts of the DFT and 256-block EDFT-II.

- *Vectors a_1 and a_2 together are referred to as a key which allows for reconstructing the original signal from its spectrum represented by the elliptic DFTs. By changing this key we can vary the original form of the signal.*

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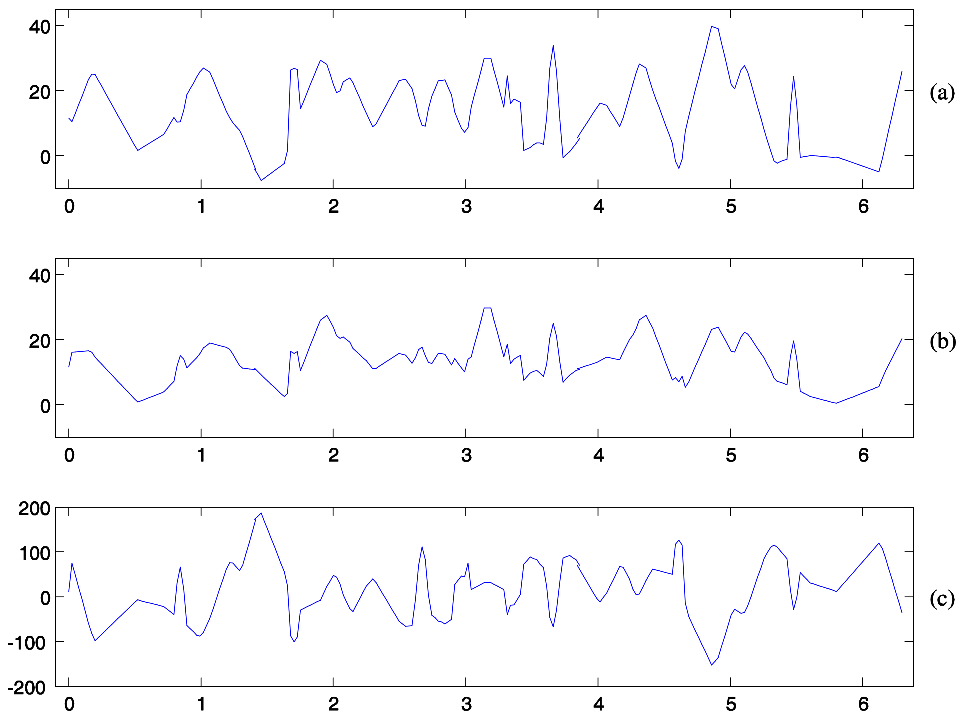
Example 9: inverse DFT/EDFT

Consider the inverse DFT of the EDFT when the vector generators are unknown:



(a) $a_1=(3,5),\ a_2=(3,4)$

(b) $a_1=(-1,2),\ a_2=(3,4)$

(c) $a_1=(-1,2),\ a_2=(3,4)$

Fig 9: Inverse DFTs of the 256-block EDFTs of type II.

- *Vectors a_1 and a_2 together are referred to as a key. By changing this key we can vary the original form of the signal.*

2-D separable EDFT

EDFTs with the same or different vector-generators can be used for representing and processing images in frequency domain.

Example 10: The tree image is processed row-column wise by the same 1-D EDFTs generated by vectors $a_1=(1,2)$ and $a_2=(3,4)$.

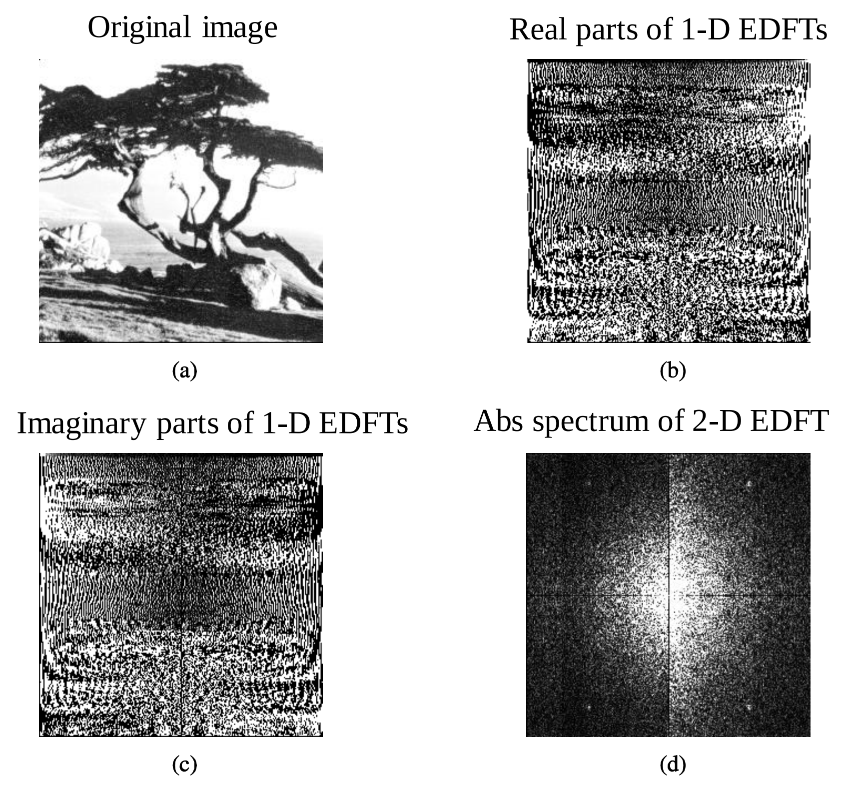


Fig 10: 2-D 256-block EDFT-II.

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2-D separable EDFT with the Key

Consider the following key for image processing:

$$K_{256,(3)}=\{1,2,3,4,80,3,5,3,4,160,1,7,1,-3,16\}$$

▪ The first 80 rows are processed by the 256-block EDFT with vectors (1,2) and (3,4). The next 160 rows by the EDFT with vectors (3,5) and (3,4). The remaining 16 rows by the EDFT with vectors (1,7) and (1,-3).

▪ The same key can be used for processing all columns in the second stage of calculation of the 2-D EDFT.

case 2-D EDFT: a_1=(1,2), a_2=(4,3)

(a)

case 2-D EDFT: (a_1,a_2) from *K*

(b)

Fig 11: Inverse 2-D DFT of 2-D EDFT-II.

▪ The probability of finding this key and obtaining the original image from the 2-D EDFT in b is almost zero, especially when the range of the vector-generators is large.

Conclusion

- In this paper, we have generalized the concept of the N-point DFT, by introducing the block-type elliptic DFT of type II in the real space R^{2N}. The class of the 1-D EDFT is parameterized by two vectors and defined for any order N.
- We believe that the proposed parameterized EDFT of type II can be used in different areas in signal and image processing, such as filtration, enhancement, and encryption.

Acknowledgment

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Art

References: *EDFT*

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Enhancement by the 2-D EDFT

(a) (b)

Fig 13: Bridge image and its enhancement by the 2-D EDFT-I.

The image size is 256x256, and we perform the 2-D separable EDFT by processing the rows and then columns by the 1-D 256-block EDFT with parameter $\varphi_1=\pi/6$, and then we calculate the inverse 2-D separable EDFT by processing the columns and rows by the inverse 1-D 256-block EDFT with parameter $\varphi_2=1.5\varphi_1$. The result, which is an enhanced image is shown in b.