MULTI-RESOLUTION OF FOURIER TRANSFORM

Artyom M. Grigoryan
Serkan Dursun

Department of Electrical Engineering
University of Texas at San Antonio

ABSTRACT

- Integral Fourier transform is described as a specific wavelet-like transform with fully scalable modulated window, but not all possible translation. The transform is defined by sinusoidal waves of half periods.

- A geometrical locus of frequency-time points for the proposed wavelet-like transform is derived and the function coverage is described and compared with short-time Fourier transform as well as with wavelet transform.

INTRODUCTION

- The Fourier transform is considered traditionally as a transform without time resolution, since the basis cosine and sine functions are defined everywhere on the real line. Each Fourier component depends on the global behavior of the signal. Wavelet analysis has been developed as multi-resolution signal processing, which is used effectively for signal and image processing, compression, computer vision, medical imaging. In wavelet analysis, a fully scalable modulated window is used for frequency localization. The window is sliding, and the wavelet transform of a part of the signal is calculated for every position. The result of the wavelet transform is a collection of time-scaling representations of the signal with different resolutions.

- In this paper, a new representation of the Fourier transform is described, by cosine and sine wavelet-like transforms with fully scalable modulated windows. The integral Fourier transform uses a specific collection of time-scaling representations of the signal with different resolutions. The transform can be also considered as a discretization of an integral wavelet transform. The Fourier transform uses the translations of windows not for every position. The transform provides the multi-resolution signal processing because cosine and sine type waveforms of every frequency are participated in Fourier analysis.

Fourier Transform Pairs

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- Used to analyze stationary signals
- No time resolution

Short Time Fourier Transform Pairs

$$F(t,\omega) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) e^{-j\omega\tau} d\tau$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega) g(t-\tau) e^{j\omega\tau} d\omega d\tau$$

- Used to analyze non-stationary signals
- Time resolution

$$g(t) = (\sqrt{\pi\sigma})^{-1} e^{-t^2/\sigma}$$ Window Gaussian Function

Why Not Fourier Transform ?

- We describe the integral Fourier transform by wavelet-like transforms with cosine and sine analyzing functions. The description differs from the well-known concept of the short-time Fourier transform or windowed Fourier transform.

  The short-time Fourier transform is based on a joint time-frequency signal representation and is defined above where $t \in (-\infty, +\infty)$. A time-sliding window function g(t) is used to emphasize *"local"* frequency properties. The window function is typically considered to be symmetric and with unit norm in the space of square-integrable functions.

  For instance, the Gaussian function

$$g(t) = (\sqrt{\pi\sigma})^{-1} e^{-t^2/\sigma}$$

with a symmetric finite support can be taken, where $\sigma \rangle 0$ is a fixed number defining a *"width"* of the window.

PROPOSED METHOD

Basis Functions

- Let $\varphi(t)$ and $\psi(t)$ be functions that coincide respectively with cosine and sine functions inside the half of period interval $[-\pi/2, \pi/2)$

$$\psi(t) = \begin{cases} \cos(t), & t \in [-\pi/2, \pi/2) \\ 0, & otherwise \end{cases}$$

$$\varphi(t) = \begin{cases} \sin(t), & t \in [-\pi/2, \pi/2) \\ 0, & otherwise \end{cases}$$

Time-Scale & Shift Transformation

- Consider family $\{\psi_{\omega;b_n}(t), \varphi_{\omega;b_n}(t)\}$ of time-scale and shift transformation of these functions

$$\psi_{\omega;b_n}(t) = \psi(\omega[t - b_n])$$

$$\varphi_{\omega;b_n}(t) = \varphi(\omega[t - b_n])$$

where

$t \in (-\infty, +\infty)$

ω varies along the real axis

b_n takes values of finite or infinite set

Fourier Transform F(w) decomposed by basis functions

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

$$I_n = [\frac{(2n-1)\pi}{2\omega}, \frac{(2n+1)\pi}{2\omega}), \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} \int_{I_n} f(t)\cos(\omega t)dt - j\sum_{n=-\infty}^{\infty} \int_{I_n} f(t)\sin(\omega t)dt$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n [\int_{-\infty}^{\infty} f(t)\psi(w[t - \frac{\pi}{\omega}n])dt - j\int_{-\infty}^{\infty} f(t)\varphi(w[t - \frac{\pi}{\omega}n])dt]$$

New Transforms of function f(t)

$$F_{\psi}(\omega, b_n) = \int_{-\infty}^{\infty} f(t)\psi_{\omega,b_n}(t)dt$$

$$F_{\psi}(0,0) = \int_{-\infty}^{\infty} f(t)dt$$

$$F_{\varphi}(\omega, b_n) = \int_{-\infty}^{\infty} f(t)\varphi_{\omega,b_n}(t)dt$$

$$F_{\varphi}(0,0) = 0$$

- We now introduce the above transforms of the function f(t) where $b_n = b_n(\omega) = (\pi/\omega)n$ when *n* is an integer. It is assumed that $b_n = 0, \text{ if } \omega = 0$

- $F_\psi(\omega, b_n)$ is the integral of the cosinusoidal signal of the half-period (π/ω) which is multiplied on f(t) waveform at location b_n. The signal $\psi(\omega t)$ is moving along f(t) waveform at locations b_n being integer multiples of (π/ω) The locations depend on the frequency ω. In a similar way, the transform
- $F_\varphi(\omega, b_n)$ is defined by the inner product of f(t) waveform with the sinusoidal signal $\varphi(\omega t)$ of the half-period (π/ω) when moving at the same locations b_n.

Composition of Fourier Transform by Introduced Transforms

$$F(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n F_\psi(\omega, b_n) - j \sum_{n=-\infty}^{\infty} (-1)^n F_\varphi(\omega, b_n)$$

Frequency-Time Plane

$$B = \{(\omega, b_n); \qquad \omega \in (-\infty, \infty),$$

$$b_n = \frac{\pi}{\omega} n, \quad n = 0, \pm 1, \pm 2, \ldots, \pm N(\omega)\}$$

$$N(\omega) = \left\{ \begin{array}{ll} 0, & \omega = 0 \\ \infty, & \text{in general} \\ finite, & \text{f(t) has finite support} \end{array} \right\}$$

Example #1

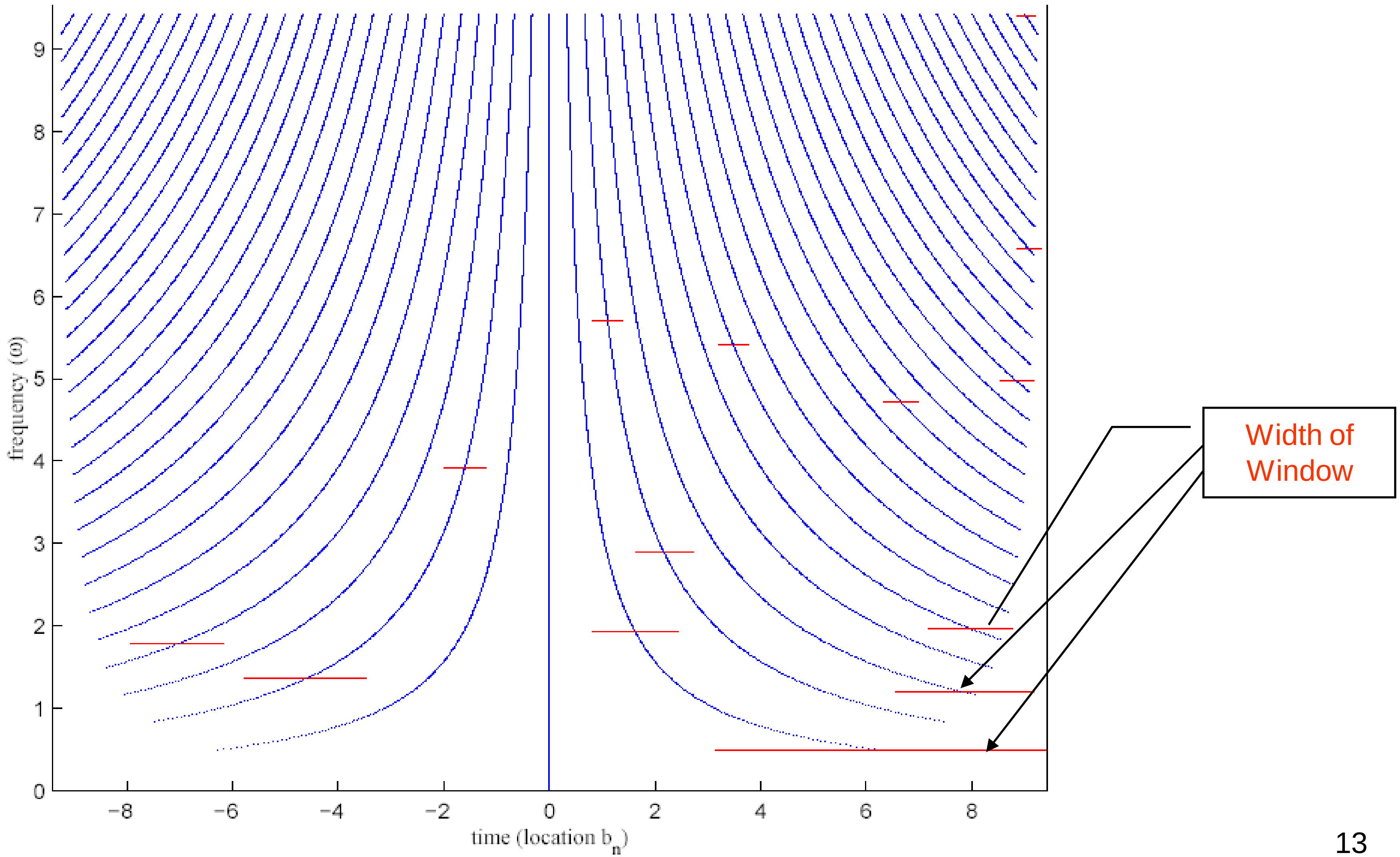
$$f(t) = \cos(\omega_0 t), \quad \omega_0 = 1.3, \quad t \in [0, 3\pi]$$

$$B = \{(\omega, b_n); \quad \omega \in (0, 3\pi),$$

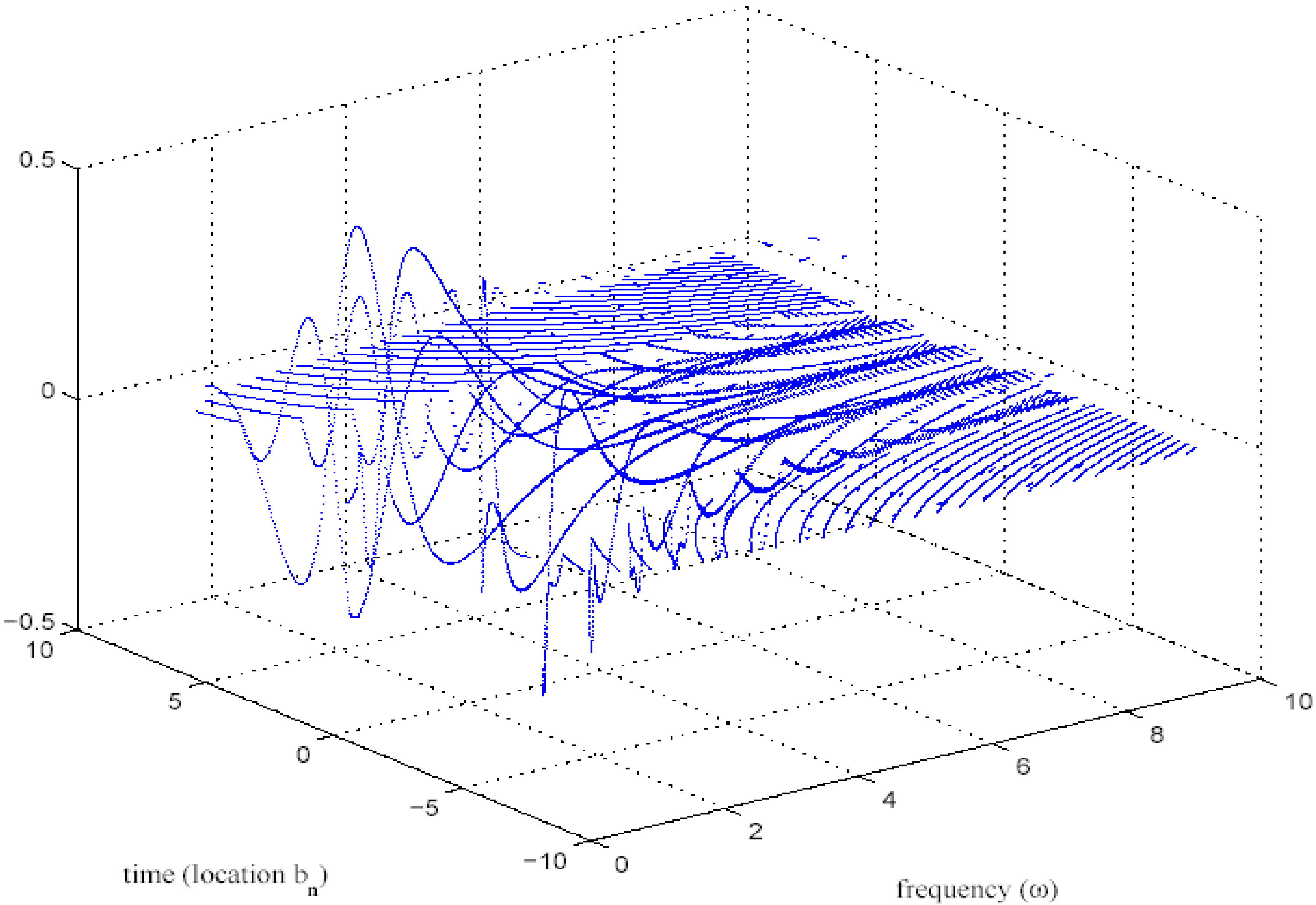
$$b_n = \frac{\pi}{\omega} n, \qquad n = 0, \pm 1, \pm 2, \ldots, \pm N(\omega)\}$$

$$N(\omega) = \begin{cases} 0, & \omega < 1/6 \\ \left\lfloor 3\omega - \frac{1}{2} \right\rfloor, & \omega \in (1/6, 3\pi) \end{cases}$$

Locus of time –frequency points for B-Wavelet Transform

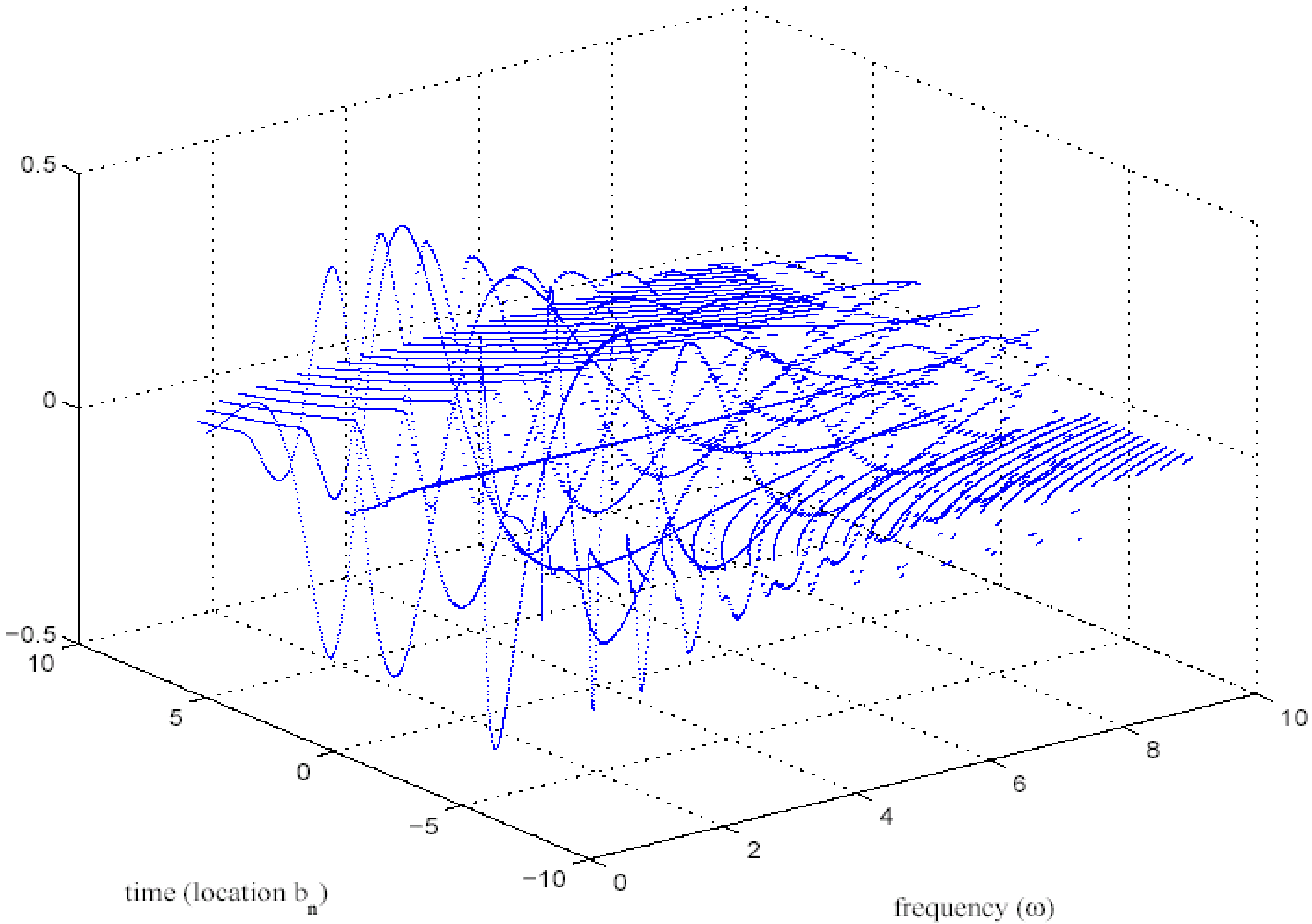


B-Wavelet by Cosine Analyzing Function



Wavelet transform plot of signal $f(t)=\cos(w_0 t)$ with cosine analyzing function

B-Wavelet by Sinus Analyzing Function



Wavelet transform plot of signal $f(t)=\cos(w_0 t)$ with sine analyzing function

Cosine and Sine B-wavelet Transform

$$f(t) \longrightarrow \{F_{\psi}(\omega, b_n), \quad (\omega, b_n) \in B\}$$

$$f(t) \longrightarrow \{F_{\varphi}(\omega, b_n), \quad (\omega, b_n) \in B\}$$

The representation of f(t) by the pair of transforms

$F_{\psi}(\omega, b_n)$ and $F_{\varphi}(\omega, b_n)$

are named to be cosine and sine B-wavelet transforms respectively.

- Set B can be considered as an *"optimal"* geometrical locus of frequency-time points for the integral Fourier transform defined by the B-wavelet transform. There is no need in calculation of the B-wavelet transform across the whole range of frequency-time points, but the points of B set. The integral Fourier transform can be derived from B-wavelet transform by

$$F(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n F_\psi(\omega, b_n) - j \sum_{n=-\infty}^{\infty} (-1)^n F_\varphi(\omega, b_n)$$

- The analysis of the introduced B-wavelet representation shows that the Fourier transform can be described as a pair of integral wavelet transforms sampled by only translation parameter. These wavelet transforms are defined as follows

Wavelet Transform defined by Proposed Basis Functions

$$T_\psi(\omega, b) = \int_{-\infty}^{\infty} f(t)\psi_{\omega,b}(t)dt$$

$$T_\varphi(\omega, b) = \int_{-\infty}^{\infty} f(t)\varphi_{\omega,b}(t)dt$$

$$(\omega, b) \in (-\infty, +\infty)$$

- Locus B of frequency-time points differs from grids used for sampling the short-time Fourier transform and the wavelet transform. For the short-time Fourier transform, a single window is used for all frequencies. The resolution of the analysis is the same at all locations in the time-frequency domain. When sampling the short-time Fourier transform,

$$F(t,\omega) \to F_{n,m} = F(nt_0, m\omega_0), \quad \mathrm{n, m} = 0, \pm 1, \pm 2, \ldots$$

a regular rectangular grid is used with time and frequency steps t_0 and ω_0 that satisfy the frame bound condition $t_0\omega_0 \le 2\pi$.
One can note that for the B-wavelet transform the condition $\omega b_1(\omega) = \pi$ holds for all frequencies ω.

- For traditional wavelet transforms, short high-frequency and long low-frequency windowed functions are used for all translations. The windows are overlapped, because of continuously shifting them, and the wavelet coefficients are therefore highly redundant. When sampling the wavelet transform T(a,b), two parameters are chosen, a time-step a_0 and location b_0. The frames are constructed by sampling the dilation exponentially $a = a_0^n$ and the translation b proportional to a_0^n,as follows

$$T(a,b) \to T_{a,b} = F(a_0^n, mb_0a_0^n), \quad \mathrm{n, m} = 0, \pm 1, \pm 2, \ldots$$

Example #2 : Detection of Noisy Signal

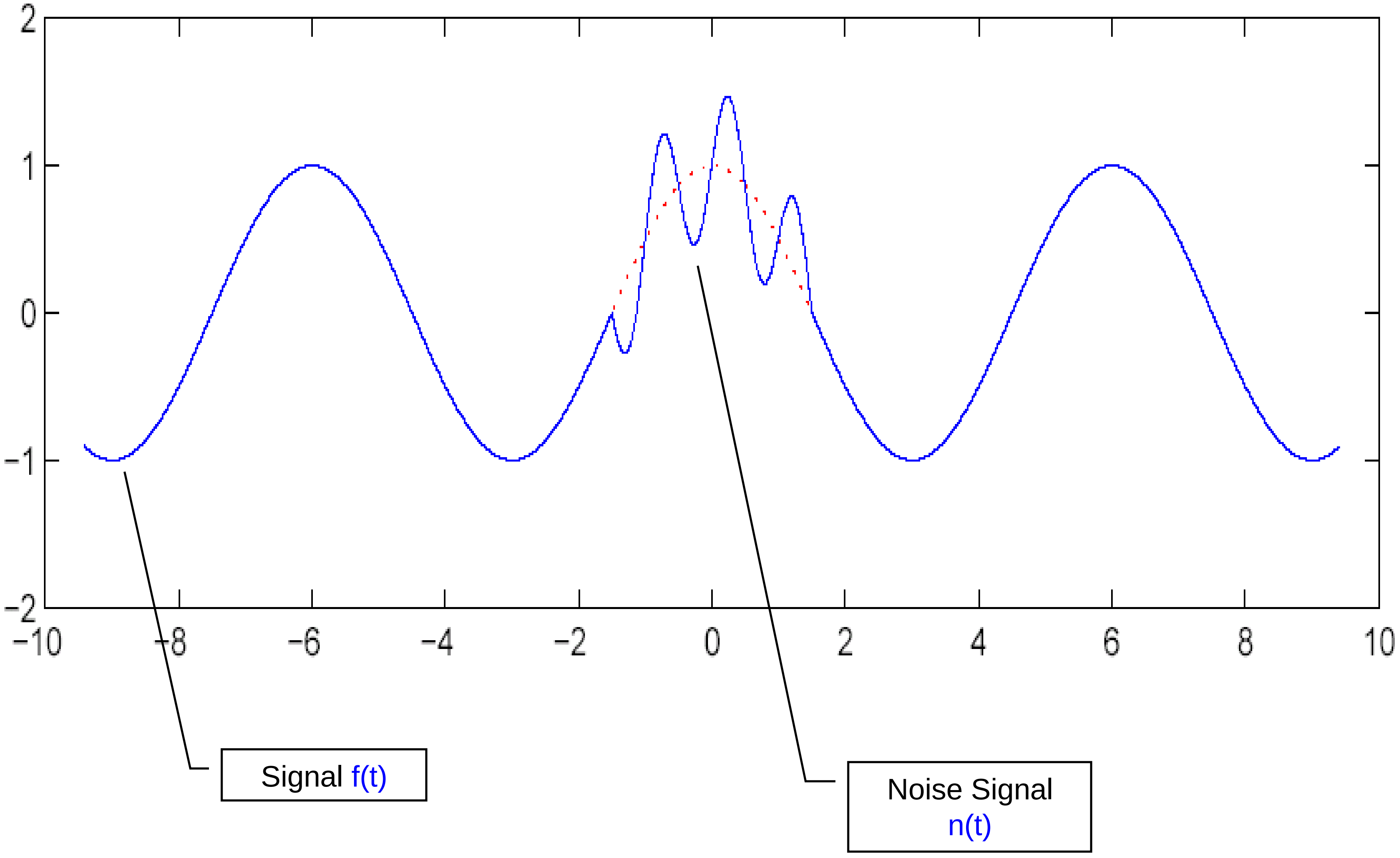
Original Signal

$$f(t) = \left\{ \cos(\omega_1 t), \qquad \omega_1 = \frac{\pi}{3} \right\}$$

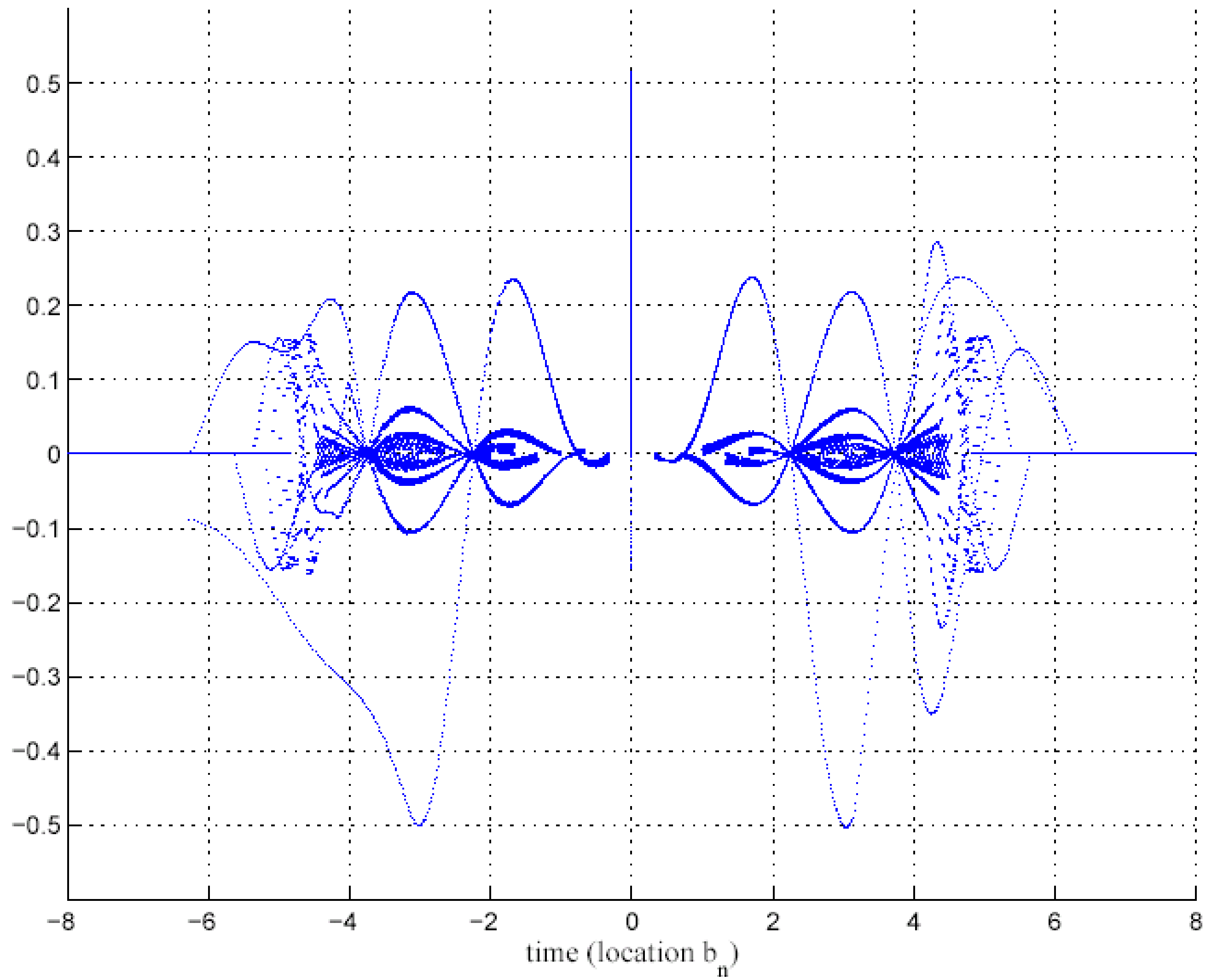
Noisy Signal

$$g(t) = \left\{ \begin{matrix} f(t) + n(t), & |t| < 1.5 \\ f(t), & 1.5 < |t| < 3\pi \end{matrix} \right\}$$

Noisy Signal g(t)

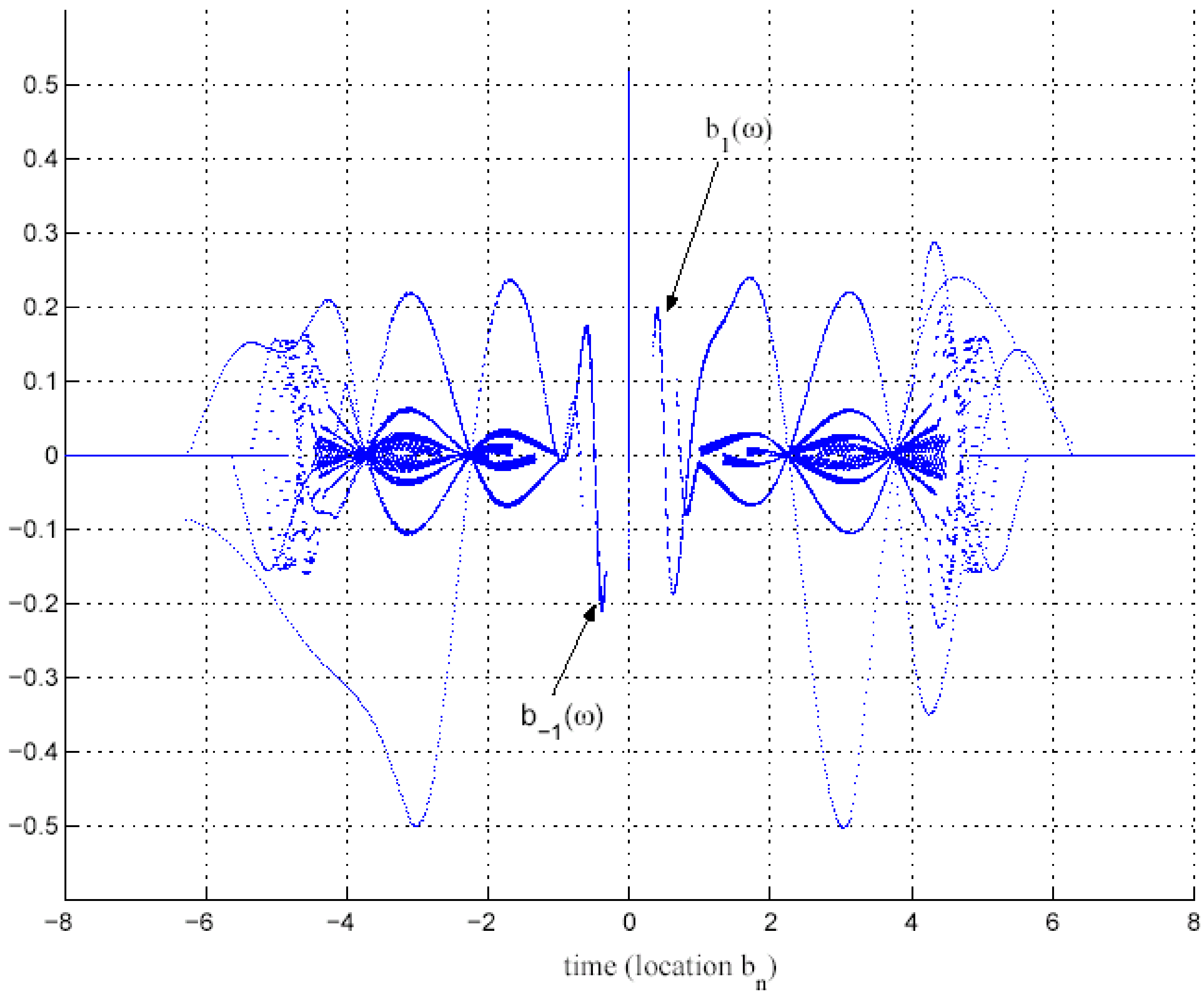


B-wavelet Transform of *f(t)*



Wavelet transform plot of signal f(t) with cosine analyzing function (projection on the time domain)

B-wavelet Transform of *g(t)*



Wavelet transform plot of signal g(t) with cosine analyzing function (projection on the time domain)

Conclusion

- A concept of the B-wavelet transform has been introduced and representation of the integral Fourier transform by this transform has been described.

  The B-wavelet transform is defined on a specific set B of points in the frequency-time plane. This transform uses a fully scalable modulated window but not all possible locations. We assume to study and develop the Fourier approach to signal analysis and examine practical applications of the proposed B-wavelet transform in signal processing.

References

- [1] D. Gabor, ”Theory of communication,” J. Inst. Elec. Eng., vol. 93, pp. 429-457, 1946.
- [2] O. Rioul andM. Vetterli, ”Wavelets and signal process-ing,” in IEEE Signal Processing Magazine, vol. 8, pp. 11-38, Oct. 1991.
- [3] A. Aldroubi and M. Unser, Wavelet in medicine and biology, Boca Raton: CRC Press, 1996.
- [4] S.G. Mallat, ”A theory for multiresolution signal decomposition: The wavelet representation,” IEEE Trans. on Pattern Anal. and Machine Intell., vol. 11, no. 7, pp. 674-693, 1989.
- [5] Y. Meyer, Wavelets. Algorithms and applications, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1993.
- [6] J.C. Goswami and A.K. Chan, Fundamentals of wavelets: theory, algorithms, and applications, New York: Wiley, 1999.
- [7] J.H. Halberstein, ”Recursive complex Fourier analysis for real time applications,” Proc. IEEE, vol. 54, p. 903, 1966.
- [8] W. Chen, N. Kehtarnavaz, and T.W. Spencer, ”An ecient recursive algorithm for time-varying Fourier transform,” IEEE Trans. On Signal Processing, vol. 41, no. 7, pp. 2488-2490, 1993.
- [9] S.G. Mallat, A Wavelet Tour of Signal Processing, San Diego: Academic Press, second dition, 1998.

Future Work

- The inverse Fourier transform can be also represented by the A*-wavelet transform defined on specific points in the time-frequency plane.
- The concept of the A-wavelet transform can be extended for representing other unitary transforms, for instance the Hartley and cosine transforms.
- We next assume to develop he theoretical relationship between the Fourier and wavelet approaches in signal processing.