Mixed Fourier Transforms (MxFTs) and Image Encryption

Nan Du, Sree Devineni,

Artyom M Grigoryan

EE dept. UTSA, San Antonio TX, USA

Presented by Sree Devineni and Nan Du

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Introduction

Representations of signals help us to solve many problems in signal processing, image encryption, linear systems theory, etc. Understanding them is easy in the light of their interpretation as rotations which makes possible to define new transformations.

In this paper, we focus on a general concept of the mixed Fourier transformations (MxFT), when signals are transformed to the time-frequency domain, where the difference between of time and frequency is disappeared.

Mixed transformations allow for effective calculation of different roots of the Fourier and identity transformations, which can be used in signal processing, image filtration and encryption.

Interpretation of the CTFT as a rotation

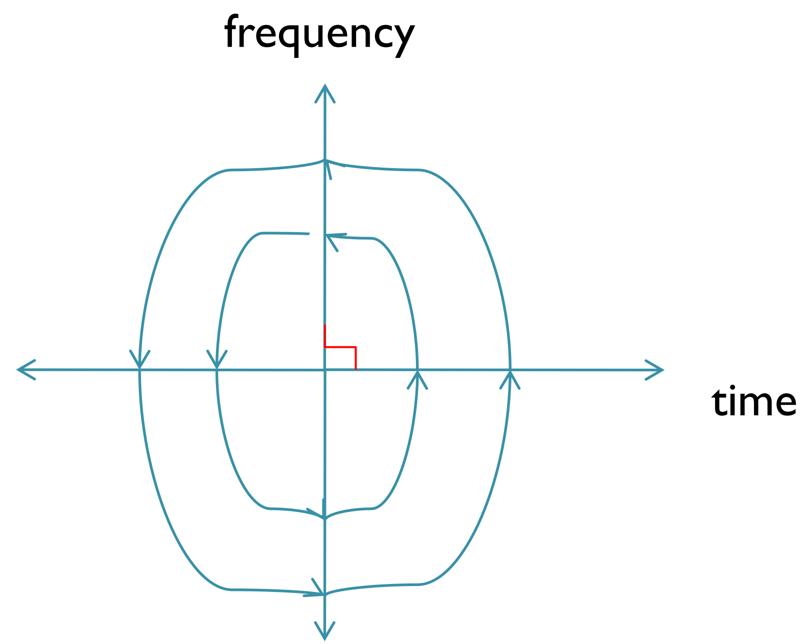
$x(t)$

$F[x(t)]=X(\omega)$

$F^2[x(t)]=x(-t)$

$F^3[x(t)]=X(-\omega)$

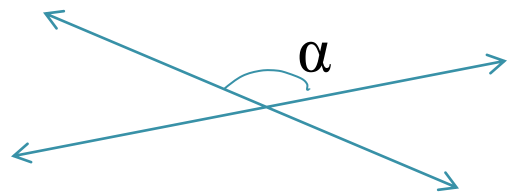
$F^4[x(t)]=x(t)$



- By transformation, a point in time domain space gets transformed to a point in frequency domain space whose spaces are “uncoupled”, represented as “perpendicular” axes.

Fractional FT

- Arbitrarily rotate time-frequency axes through a predefined angle ‘alpha’ in L^2 space



- Time-frequency are coupled generally.
- CTFT is a special case of FRFT where $\alpha=90^o$.
- FRFT has the following closed form expression:

$$F_\alpha(\omega) = (F_\alpha \circ f)(\omega) = \sqrt{\frac{(1-j\cot(\alpha))}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{j\pi Q_\alpha(\omega,t)} dt$$

$$Q_\alpha(\omega,t) = B_\alpha \omega^2 - 2C_\alpha \omega t + B_\alpha t^2$$

where $B_\alpha = \cot(\alpha)$ and $C_\alpha = 1/\sin(\alpha\pi/2)$

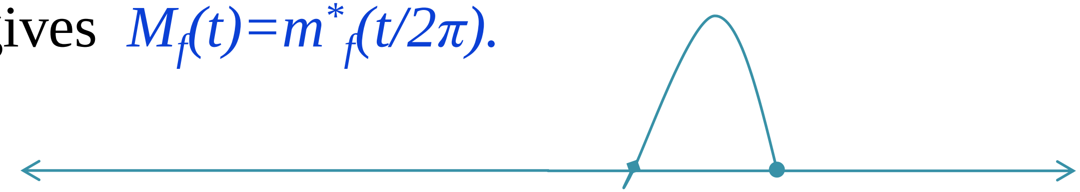
Mixed FTs

- Transform $f(t) \rightarrow m_f(t)$ followed by $M_f(t)=F[m_f(t)]$,

  e.g. simply $f(t) \rightarrow f(t)+F(t)$

  gives $M_f(t)=m^*_f(t/2\pi)$.

- in Space $M_{1,1}$



- Rotation by 180^o → time and frequency axis are the same.
- Generally, transform $f(t) \rightarrow a \cdot f(t)+b \cdot F(t)$
- in Space $M_{a,b}$

Properties

- FT of the mixed transformed signal is time-scaled by $1/(2\pi)$
- conjugate of the mixed transformed signal itself in $M_{1,1}$
- linear convolution between signals in $M_{1,1}$

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
Y(t) &= X(t)H(t) \\
&= x^*(t/2\pi)H(t) \\
&= x^*(t/2\pi)h^*(t/2\pi).
\end{aligned}
$$

Discrete mixed Transformation

- MxDFT is defined as a linear combination of the real signal *f(n)*, its DFT, and its DFT^2.
- Transform matrix:

$$S = aI + bF^2 + cF.$$

- Inverse Transform matrix:

$$S^{-1} = pI + sF^2 + tF,$$

$$p = \frac{a}{\Delta}, \quad s = \frac{b}{\Delta}, \quad t = \frac{c}{\Delta}, \quad (\Delta = a^2 + b^2).$$

512-point mxDFT

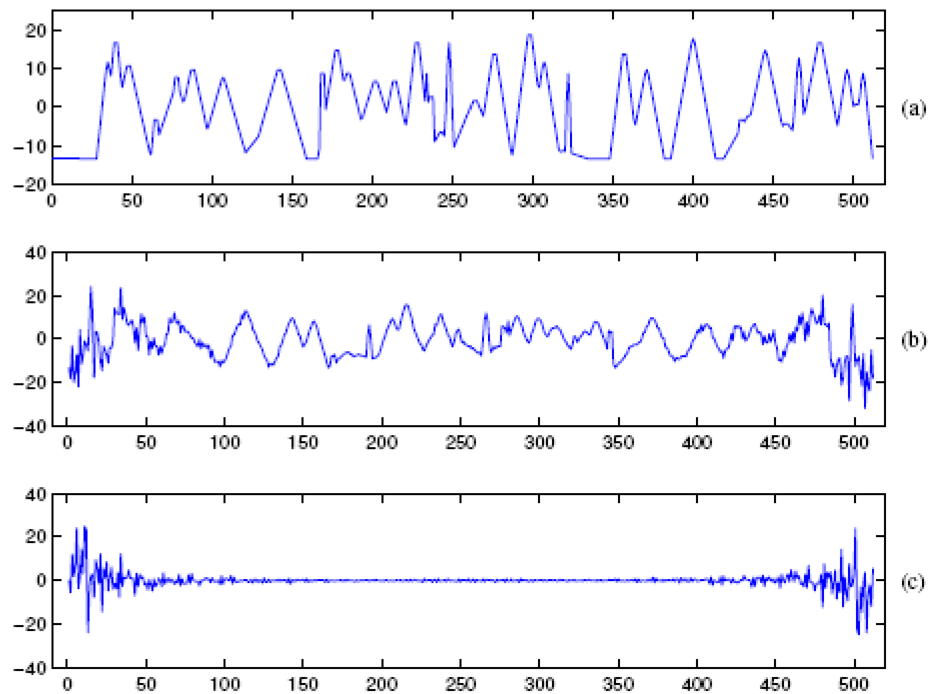


Fig.1: (a) Original signal of length 512, and (b) the real part and (c) imaginary part.

Applications

- Image encryption

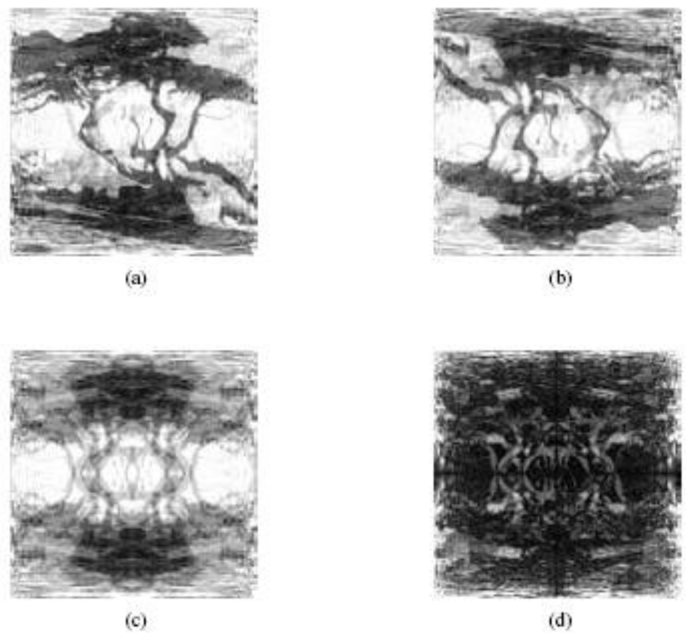


Fig. 2: Amplitude spectrums of the MxDFT, when (a) a = 0.2 and b = 0.8, (b) a = 0.75 and b = 0.25, (c)a = 0.25 and b = 0.25, and (d) a = 0.75 and b = −0.5.

Applications (contd…)

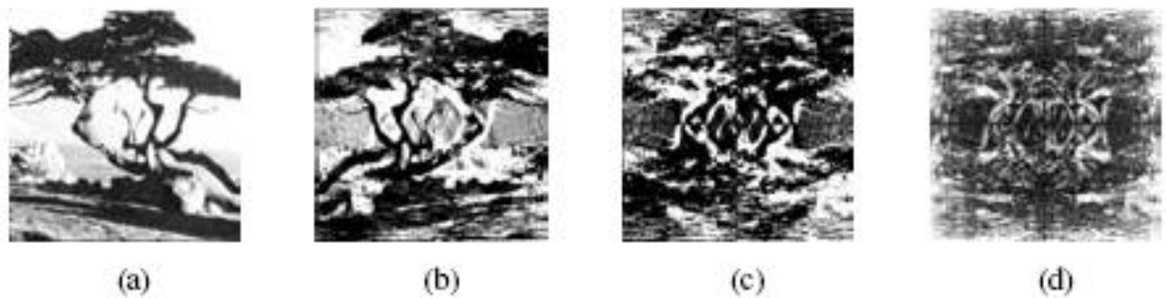


Fig.3.(a) The tree image f and 2-D mixed Fourier transforms (b) S1(f), (c) S2(S1(f)), and (d) S3(S2(S1(f))).

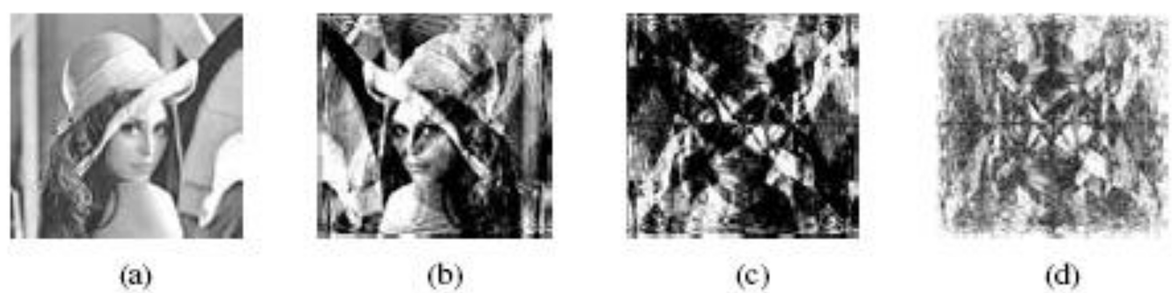


Fig.4.(a) Lena image f and 2-D mixed Fourier transforms (b)S1(f), (c) S2(S1(f)), and (d) S3(S2(S1(f))).

General Concept of the MxDFT

The mixed transformation with respect to the discrete Fourier transformation is defined by

$$S = aI + bE + cF + dF^*$$

where a, b, c and d are coefficients, I is the identity matrix, F is the matrix of the discrete Fourier transformation.

$$E = FF,\ \ EE = FFFF = I \ \ and\ EF = FFF = F^* = FE$$

Square roots of the Fourier transform: $\mathrm{S}^2 = F$.
The coefficients a, b, c and d are calculated by

$$a = \frac{s + w - jp + jt}{4} \quad b = \frac{s + w + jp - jt}{4} \quad c = \frac{s - w + p + t}{4} \quad d = \frac{s - w - p - t}{4}$$

$$s^2 = 1,\ w^2 = -1,\ \mathrm{t}^2 = -j,\ \mathrm{p}^2 = j.$$

The solution is not unique because of ambiguity of the square roots.

Case 1: [+,+,+,+]

Consider the following values of the square roots:

$$s=1 \qquad w=j \qquad p=\sqrt{j}=\frac{1+j}{\sqrt{2}} \qquad t=\sqrt{-j}=\frac{1-j}{\sqrt{2}}$$

Coefficients are obtained:

$$a=\frac{1}{4}(1+\sqrt{2}+j) \qquad b=\frac{1}{4}(1-\sqrt{2}+j) \qquad c=a^* \qquad d=b^*$$

Transformation defined by this set of coefficients is called *1st square root discrete Fourier transformation (1-SQ DFT)*

Case 2: [-,+,+,+]

For the following values of the square roots:

$$s=-1 \qquad w=j \qquad p=\sqrt{j}=\frac{1+j}{\sqrt{2}} \qquad t=\sqrt{-j}=\frac{1-j}{\sqrt{2}}$$

the coefficients of the transform S are calculated by

$$a=\frac{1}{4}(-1+\sqrt{2}+j) \qquad b=\frac{1}{4}(-1-\sqrt{2}+j) \qquad c=a^* \qquad d=b^*$$

The corresponding transformation is called the *2nd square root discrete Fourier transformation (2-SQ DFT)*

Square root of the inverse Fourier matrix

Replacing the coefficients c and d in the square root $F^{[1/2]}$

$$(F^*)^{1/2} = aI + bE + dF + cF^*$$

$$F^4 = I \quad \longrightarrow \quad F^{[1/2]} = I^{[1/8]}$$

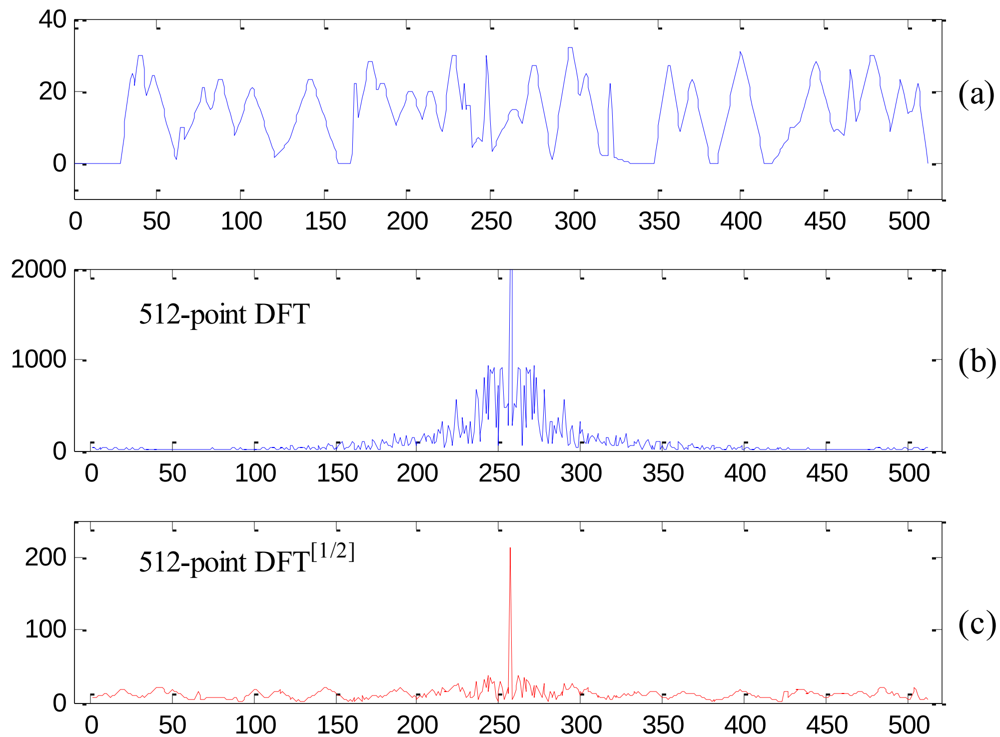


Fig. 6.
Original signal of length 512

The Fourier Transform

The square root Fourier transform.

$$x(t) = 2e^{-\frac{t}{16}} \sin\left(\pi t^2/64\right), \quad t \in [0,50].$$

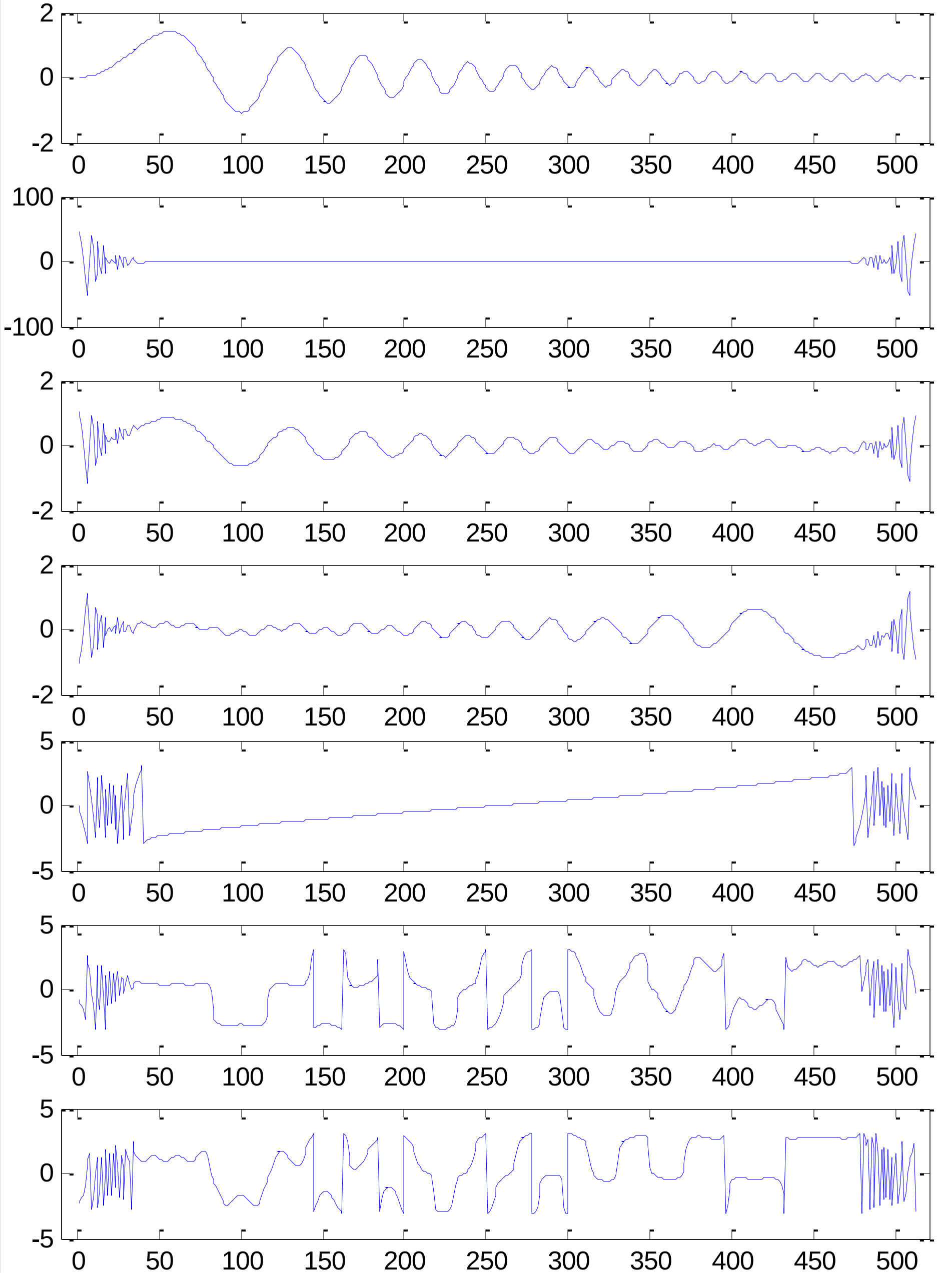


Fig. 6:

(a) Original signal of length 512

(b) real part of the DFT

(c) 1-SQ DFT

(d) 2-SQ DFT

(a) Phase of the (a) DFT

(b) 1-SQ DFT

(c) 2-SQ DFT.

Series of Fourier matrixes

A matrix S is represented by

$$S = a_0 I + a_1 F + a_2 F^2 + a_3 F^3$$

with real or complex coefficients a_k , $k=0,1,2,3$.

The square of this matrix can be written as

$$S^2 = (a_0 I + a_1 F + a_2 F^2 + a_3 F^3)^2 = \sum_{n=0}^{3} b_n F^n$$

Where the coefficients b_n are calculated by the cyclic convolution

$$b_n = \sum_{k=0}^{3} a_k a_{n-k \bmod 4}, \qquad n = 0,1,2,3$$

CASE 1: $S^2 = F$

The above cyclic convolution is written as

$$\sum_{k=0}^{3} a_k a_{n-k \bmod 4} = \delta_{n;1}\ , \qquad n = 0,1,2,3$$

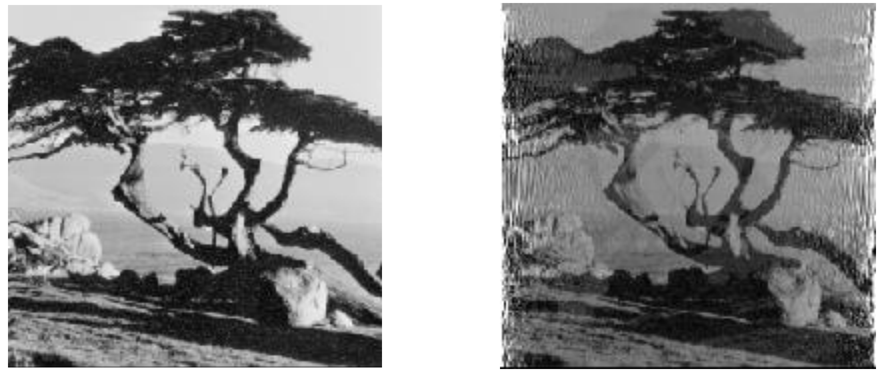
where $\delta_{1,1} = b_1 = 1$ and $\delta_{n,1} = b_n = 0$ if n=0,2,3. In the frequency domain, this convolution has a form

$$\left|A_p\right|^2 = e^{-\frac{2\pi j}{4}p}, \qquad p = 0,1,2,3$$

where A_p is the four-point discrete Fourier transform of the vector-coefficient $\{a_0, a_1, a_2, a_3\}$

$$A_0 = \pm 1,\ A_1 = \pm\frac{1}{\sqrt{2}}(-1+j),\ A_2 = \pm j,\ A_3 = \pm\frac{1}{\sqrt{2}}(1+j)$$

Images and their 2-D SQ-DFT



(a) (b)

Fig.7. (a) The tree image (b) 2-D SQ-DFT of the image

(a) (b)

Fig. 8: (a) The Lena image and (b) 2-D SQ-DFT of the image

CASE 2: $S^2 = I$

$S = I$, $S = E$,
?

$$S^2 = (\sum_{n=0}^{3} a_k F^k)^2 = \sum_{n=0}^{3} b_k F^k = I$$

$b_1 = b_2 = b_3 = 0$ and $b_0 = 1$, a_k can be found from cyclic convolution:

$$\sum_{n=0}^{3} a_k a_{n-k \bmod 4} = \delta_n, \quad n = 0, 1, 2, 3$$

where $\delta_n = 1$ if n=0, and $\delta_n = 0$ if *n=1,2,3*. In the frequency domain, this convolution has a form $|A_p|^2 = 1$, where p=0,1,2, 3 . The coefficients a_k can be defined by $\{A_p = \pm 1; \ p = 0,1,2,3\}$

$$A = (-1,1,1,1) \xrightarrow{F^{-1}} a_0 = 0.5, a_1 = a_2 = a_3 = -0.5$$

Square root of the identity matrix:

$$S = I^{[1/2]} = \frac{1}{2}(I - E - F - F^*)$$

Basic functions: N=4 case

$$I^{[1/2]} = \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

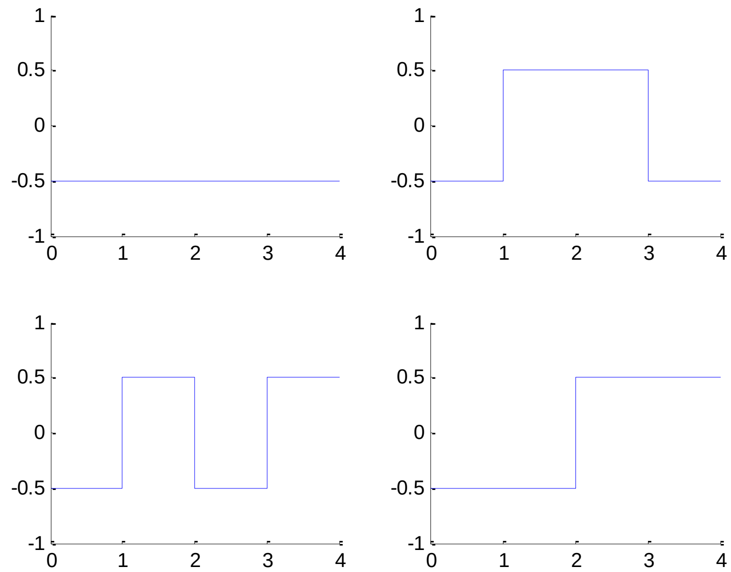


Fig.9.: The basis functions of the square root of the 4-point identity transformation *S*.

Basic functions: N=8 case

$$I^{[1/2]} = \frac{1}{2}\begin{bmatrix} -0.7071 & -0.7071 & -0.7071 & -0.7071 & -0.7071 & -0.7071 & -0.7071 & -0.7071 \\ -0.7071 & 0.5000 & 0 & 0.5000 & 0.7071 & 0.5000 & 0 & -1.5000 \\ -0.7071 & 0 & 1.7071 & 0 & -0.7071 & 0 & -0.2929 & 0 \\ -0.7071 & 0.5000 & 0 & 0.5000 & 0.7071 & -1.5000 & 0 & 0.5000 \\ -0.7071 & 0.7071 & -0.7071 & 0.7071 & -0.7071 & 0.7071 & -0.7071 & 0.7071 \\ -0.7071 & 0.5000 & 0 & -1.5000 & 0.7071 & 0.5000 & 0 & 0.5000 \\ -0.7071 & 0 & -0.2929 & 0 & -0.7071 & 0 & 1.7071 & 0 \\ -0.7071 & -1.5000 & 0 & 0.5000 & 0.7071 & 0.5000 & 0 & 0.5000 \end{bmatrix}$$

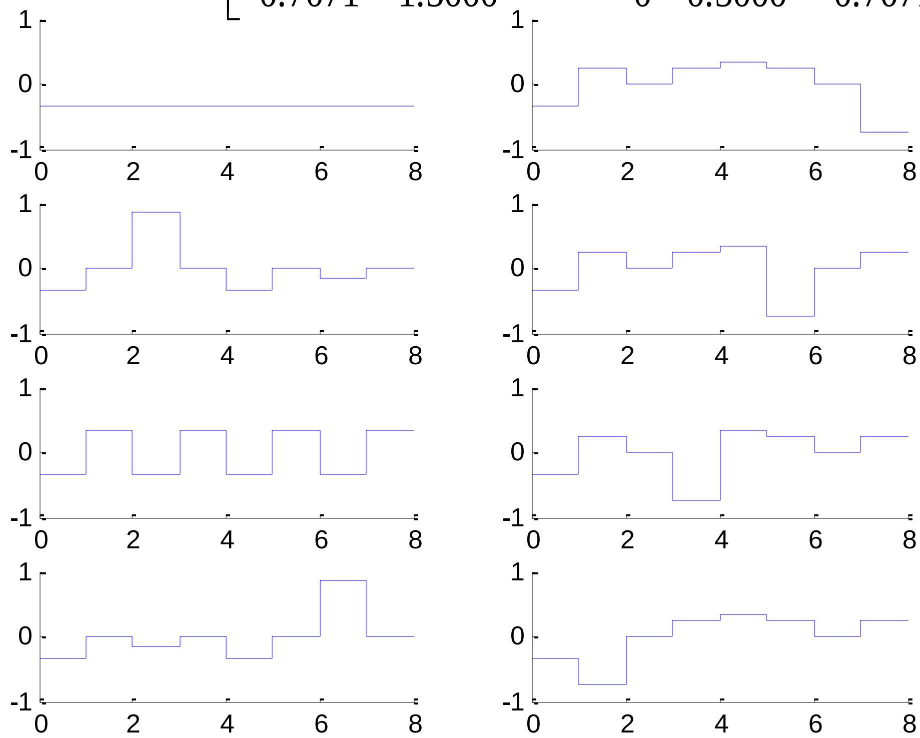


Fig.10.: The basis functions of the square root of the 8-point identity transformation S.

Square roots of the identity matrix

There are ***six*** square roots of the identity matrix.

$$S_0 = \pm I,\ S_1 = \pm E,\ S_2 = \pm \frac{1}{2}(I - E - F - F^*)$$

The discrete transform defined by S_2 is called the *square root of discrete identity transformation* (SR-DIT). The transform $S_1 = E$ is a square root of I , because the Fourier transform is the 4th degree root of the identity transform. S_2 contains the cosine transform

$$C = (F + F^*)/2$$

$$S_2 = \frac{1}{2}(I - E - F - F^*) = \frac{1}{2}(I - E) - \frac{F + F^*}{2} = \frac{1}{2}(I - E) - C$$

and the difference of transform equals

$$S_2 - S_1 = \frac{1}{2}(I - 3E) + C$$

$$x(t) = 2e^{-\frac{t}{16}} \sin(\pi t^2 / 64), \quad t \in [0,50]$$

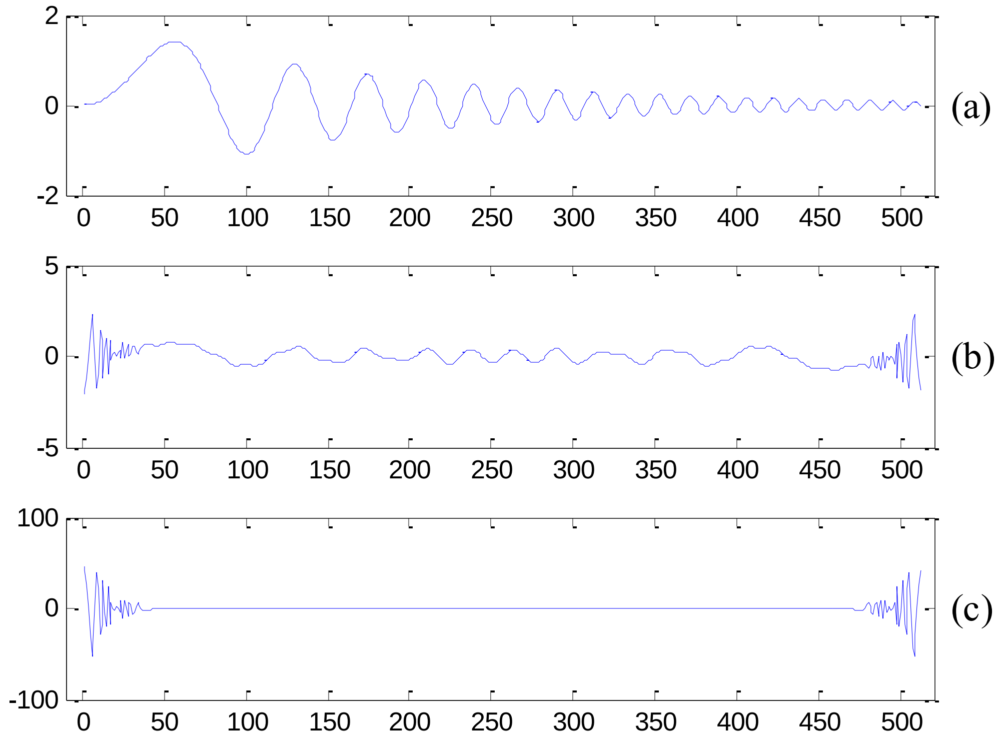


Fig. 11: (a) The original signal of length 512, (b) the square root of discrete identity transform, and (c) the real part of the square of the Fourier transform of this signal.

2-D SR-DIP

(a) (b) (c)

Fig. 12: (a) tree image in part, (b) 2-D SR-DIP of the image (c) second application of the square root over the image in b.

The second square root is the inverse to itself !

Conclusion

In this paper, the concept of the mixed Fourier transform in the continuous and discrete time cases have been considered. The mixed transform represents the signals and images in the time-frequency domain, where the concepts of time and frequency are united.

Mixed Fourier transformations can be used for calculating different roots of the Fourier and identity transformations, as well as other transformations, such as the Hadamard and cosine transformations.

Our preliminary experimental examples show that the described mixed and root transformations can be used for signal and image processing, especially for image encryption.

Thank you!

QUESTIONS?