New Formula for Image Reconstruction from Projections by the 2-D Paired Transformation

Artyom Grigoryan and Nan Du
EE Dept. UTSA, San Antonio

Presented by ***Nan Du***

IEEE SMC 2009, San Antonio

OUTLINE

- Introduction
- Discrete Model of Image Reconstruction
- 2-D Paired Transform
  - ➢Concept of paired transformation
  - ➢Image Reconstruction using Paired transform (with and without calculating the 1-D and 2-D Fourier transforms)
  - ➢ Examples and demonstration for image 64 64.
- Conclusion
- References

Introduction

Fourier transform-based methods are widely used for practical applications of image reconstruction from projections.

We present a new effective formula for the inverse 2-D paired transform, which can be used for image reconstruction without calculating the Fourier transforms. The image is reconstructed directly from the defined paired image-signals which can be calculated from projection data.

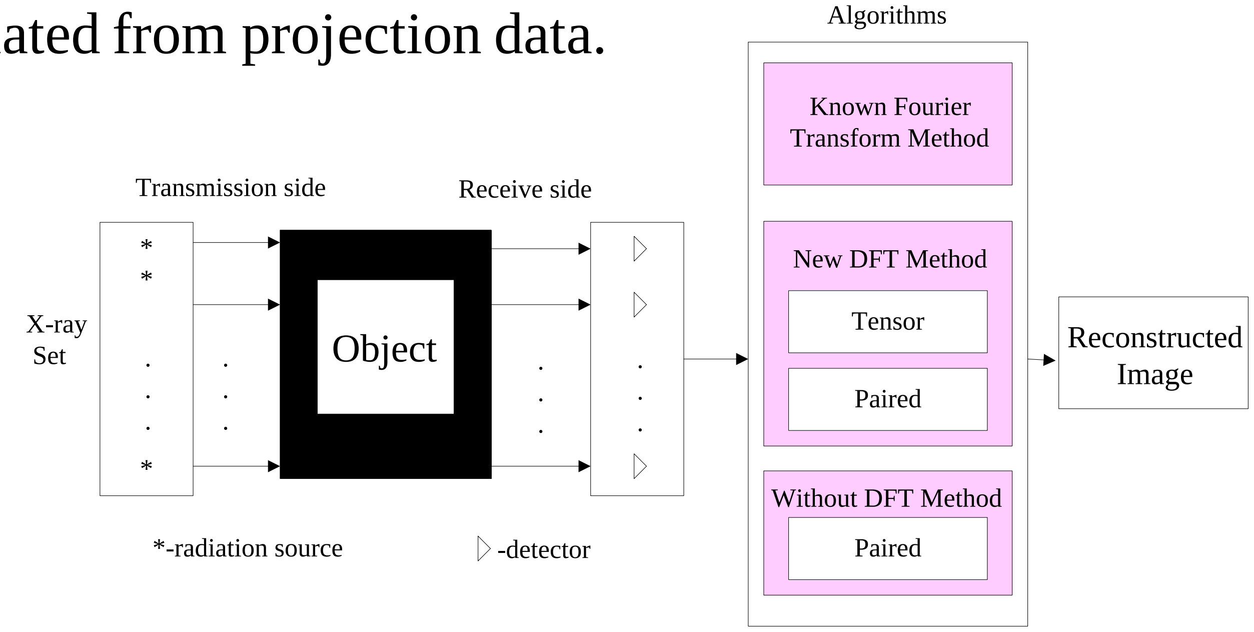


Fig. 1. Process of calculation of projection for parallel X-ray set

Model of Reconstruction

- **Discrete Model:**

Assuming:

- The reconstructed image $f(x, y)$ occupies the quadratic domain of dimension L by L, i.e., $x, y \in [0, L]$ on which the quadratic lattice $N \times N$ of image elements (IE) are marked.
- The absorption function of the (n,m)th image element, where n, $m=0$: (N-1), takes a constant value $f_{n,m}$.
- The radiation source and detector represent the points, and that the rays spreading between them are straight.

Measured value of the total attenuation energy along the *l*th ray, denoted by y_l :

$$y_l = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} a^l_{n,m} f_{n,m}$$

where $a^l_{n,m}$ is the length way of *l*th ray along the (*n*,*m*)th IE. The set of measurements y_l taken at a fixed direction is called a ***projection***.

Assuming the size of the image elements is small. For all *l=1*: *M* and *n*, *m=0*: (*N-1*):

$$a^l_{n,m} = \begin{cases} 1, & \text{if the } l\text{th ray intersects the } (n,m)\text{th IE} \\ 0, & \text{otherwise} \end{cases}$$

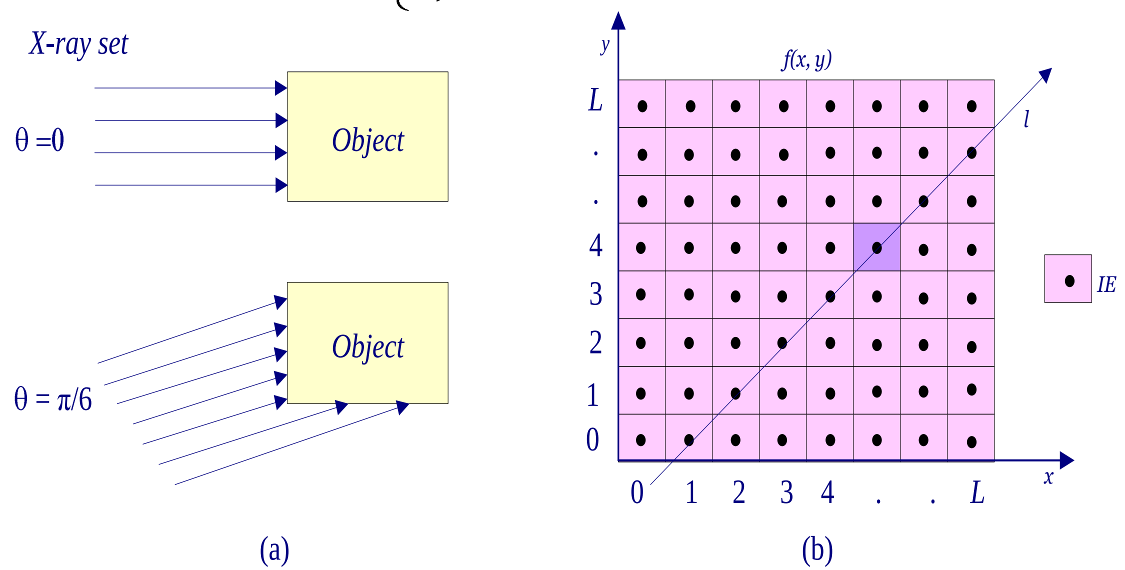


Fig. 2: (a) Parallel scanning scheme. (b) Cartesian lattice on the image domain.

2-D Paired Transform

Let $f = \{ f_{n,m} \}$ be an image of size N N, where $N = 2^r$ and $r > 1$

The N N-point 2-D DFT of the image:

$$F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{n,m} W^{np+ms}, \quad W = \exp(-2\pi j / N).$$

Frequency points are from the square lattice

$$X_{N,N} = \{(p,s);\ p,s = 0 : N-1\}$$

This 2-D DFT of the image can be split in such a way, that the whole domain of frequency-points is divided by subsets, and at frequency-points of each subset the 2-D DFT is represented by the 1-D DFT of a corresponding 1-D signal, which is called the *splitting-signal*, or *image-signal*.

Concept of the paired transform:

Given a triplet (p, s, t), where (p, s) is a frequency-point and t is an integer time-point from the interval [0,N-1]. The set of points (n, m) of the lattice in the image plane.

$$V_{p,s,t} = \{(n,m);\ n,m = 0:(N-1),\ np + ms = t \bmod N\}$$

and its characteristic function

$$\chi_{p,s,t}(n,m) = \begin{cases} 1, & \text{if } np + ms = t \bmod N \\ 0, & \text{otherwise} \end{cases}$$

The sets $V_{p,s,t}$ consist of points (n, m) allocated along a maximum of M=p + s parallel straight lines:

$$xp + ys = t + kN,\ \ k = 0:(p+s-1)$$

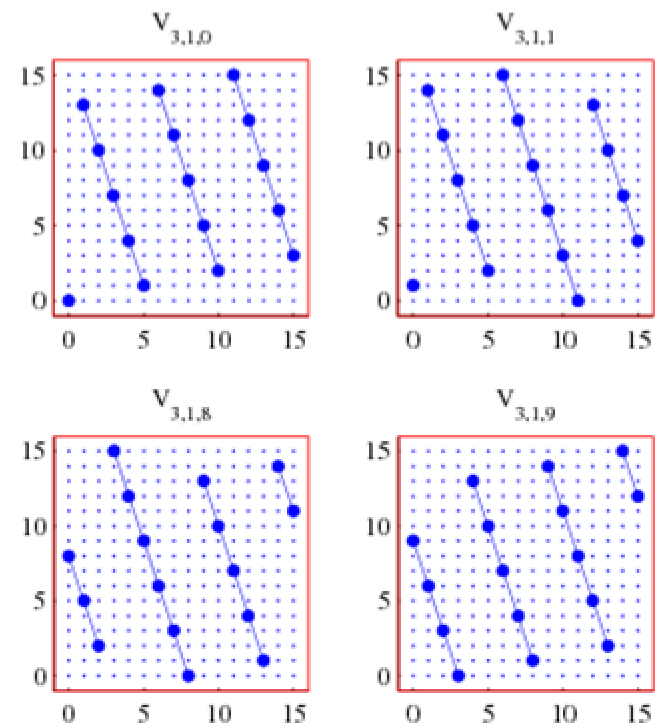


Fig.3. Example of $V_{p,s,t}$

Concept of the Paired transform

The following family of (*3N-2*) disjoint subsets is a partition of the lattice $X_{N,N}$,

$$\sigma' = (((T'_{2^k p,2^k s})_{(p,s)\in G_{r-k}})_{k=0:(r-1)}, \{(0,0)\})$$

The subsets equal

$$T'_{p,s} = \{(\overline{(2m+1)p}, \overline{(2m+1)s}); m = 0:(N/2-1)\}$$

$\bar{l} = l \bmod N$, sets of generators:

$$G_k = \{(p,1), p = 0:(2^k-1)\} \cup$$
$$\{(1,2s), s = 0:(2^{k-1}-1)\}$$

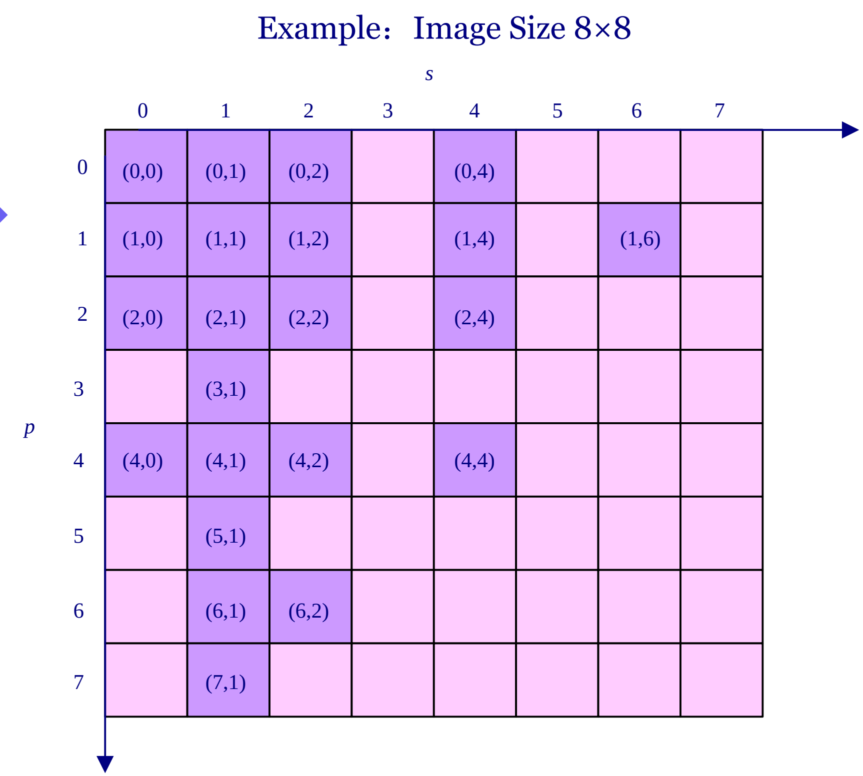


Fig. 4: All the generator for image (8x8)

The subset $T'_{p,s}$ associates with a *N/2*$^{k+1}$-point signal which is called the ***paired image-signal***,

$$f_{T'_{p,s}} = \{ f'_{p,s,0}, f'_{p,s,2^k}, f'_{p,s,2\cdot 2^k}, \ldots, f'_{p,s,N/2-2^k} \}$$

Concept of the paired transform

The component of the image-signal are calculated by

$$f'_{p,s,t} = \sum_{V_{p,s,t}} f_{n,m} - \sum_{V_{p,s,t+N/2}} f_{n,m} = f_{p,s,t} - f_{p,s,t+N/2}$$

where $t = 0, 2^k, 2\cdot 2^k, \ldots, N/2 - 2^k.$

Tensor transformation:

$$\chi_{N,N} : \{ f_{n,m} \} \to \{ f_{p,s,t}; (p,s) \in G_r, t = 0 : (N-1) \}$$

Given a frequency-point (p, s), the following holds:

$$F_{\overline{(2m+1)p},\overline{(2m+1)s}} = \sum_{t=0}^{N/2^{k+1}-1} (f'_{p,s,2^k t} W^t_{N/2^k}) W^{mt}_{N/2^{k+1}}, \quad m = 0 : (N/2^{k+1} - 1)$$

The 2-D DFT at frequency-points of $T'_{p,s}$ is defined by the $N/2^{k+1}$-point DFT of the modified splitting-signal

$$\{ f'_{p,s,0}, f'_{p,s,2^k} W^{2^k}, f'_{p,s,2\cdot 2^k} W^{2\cdot 2^k}, \ldots, f'_{p,s,N/2-2^k} W^{N/2-2^k} \}$$

Example 1: Tree Image

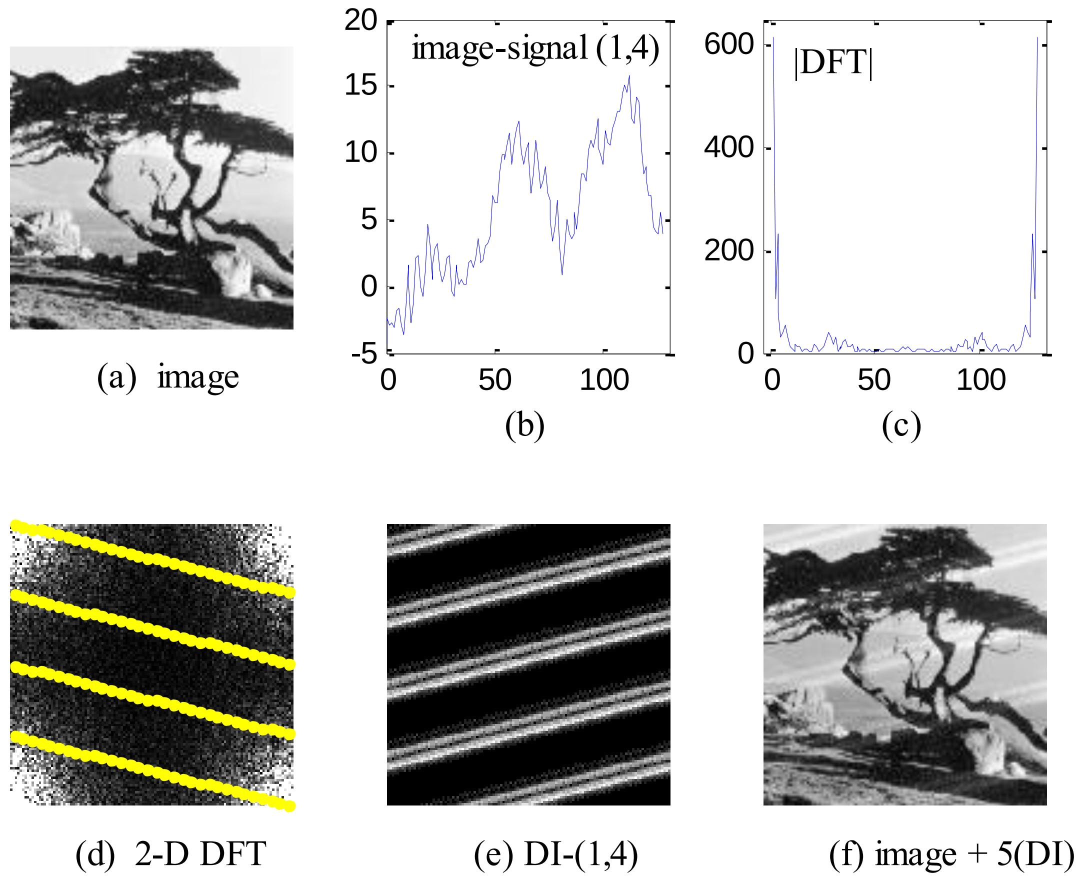


Fig. 5: (a) Original image 256 256, (b) the image-signal generated by frequency (1,4), (c) the magnitude of the 1-D DFT of this signal, (d) the 2-D DFT of the image and marked locations of the frequency-points of $T'_{1,4}$, (e) the direction image, and (f) the tree image with the amplified direction image (DI).

Example 2: Lena Image

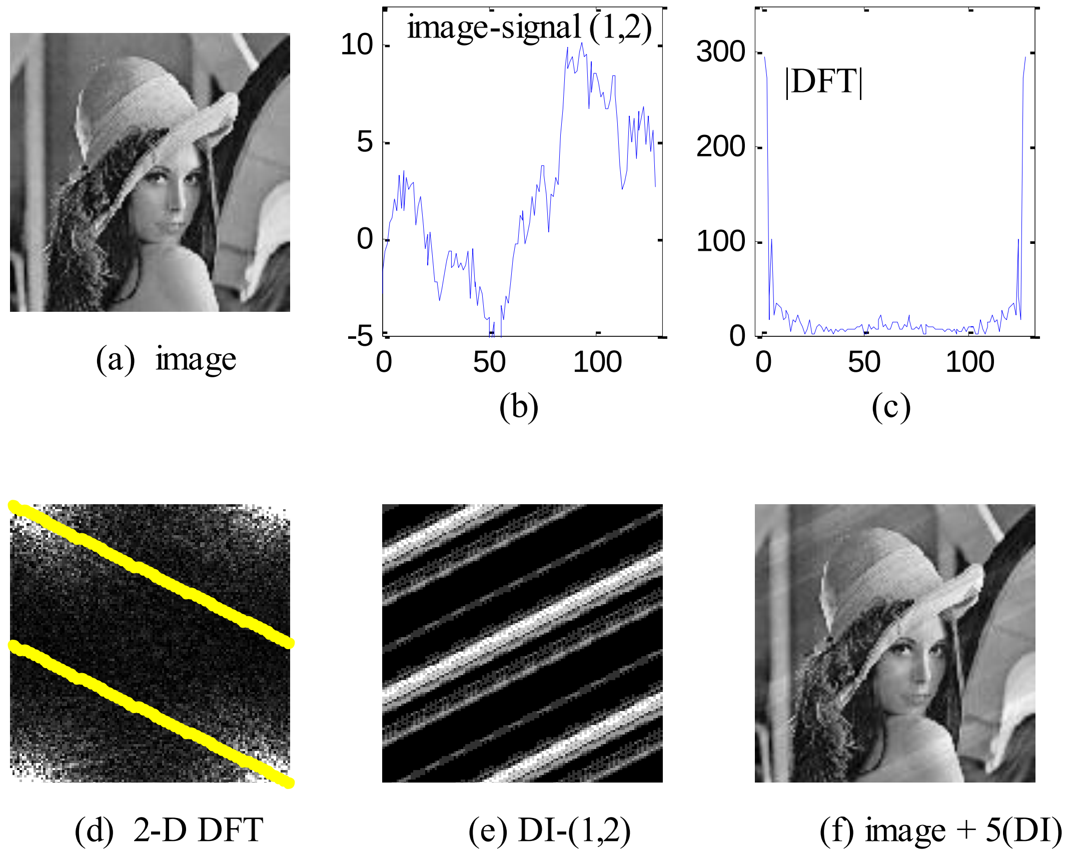


Fig. 6: (a) Original image 256 256, (b) the image-signal generated by frequency (1,2), (c) the magnitude of the 1-D DFT of this signal, (d) the 2-D DFT of the image and marked locations of the frequency-points of $T'_{1,2}$, (e) the direction image, and (f) the Lena image with the amplified direction image.

2-D Paired transform

The transformation:

$$\chi'_{N,N} : \{ f_{n,m} \} \to \{ f'_{p,s,t} ; (p,s,t) \in U \}$$

is called the **2-D *paired transformation***. U is the set of N^2 number-triplets $\{p,s,t\}$ of components of ($3N$-2) image-signals.

The paired transform is defined by the system that is extracted from the mathematical structure of the 2-D DFT. The complete system of paired functions is defined as

$$\chi'_{p,s,t}(n,m) = \chi_{p,s,t}(n,m) - \chi_{p,s,t+N/2}(n,m), \ \ (p,s,t) \in U.$$

Each paired function represents itself a 2-D plane wave, and the decomposition of the image by the paired functions is the decomposition of the image by the plane waves.

Statement 1

Given a frequency-point (p, s), components $f_{p,s,t}$, $t = 0 : N-1$, of the image-signal of a discrete image $f_{n,m}$ in tensor representation can be calculated directly from the projection data,

$$f_{p,s,t} = y(p,s,t)_1 + y(p,s,t)_2 + \ldots + y(p,s,t)_M$$

where $M=(p + s)$ and $y(p,s,t)_k = \sum_{(n,m)\in r(p,s,t)_k} f_{n,m}, \quad k = 1 : M.$

Statement 2

Given a frequency-point (p, s), the components $f'_{p,s,t}$ of the corresponding image-signal $f_{T'_{p,s}}$ of a discrete image $f_{n,m}$ can be calculated from projection data,

$$f'_{p,s,t} = y(p,s,t)_1 + y(p,s,t)_2 + \ldots + y(p,s,t)_M$$

$$- y(p,s,t+\frac{N}{2})_1 - y(p,s,t+\frac{N}{2})_2 - \ldots - y(p,s,t+\frac{N}{2})_M$$

where $t = 0, 2^k, 2 \cdot 2^k, \ldots, N/2 - 2^k$, $2^k = g.c.d(p,s)$, and $M=(p + s)$.

Example 3: Mask of Paired Functions (N=16)

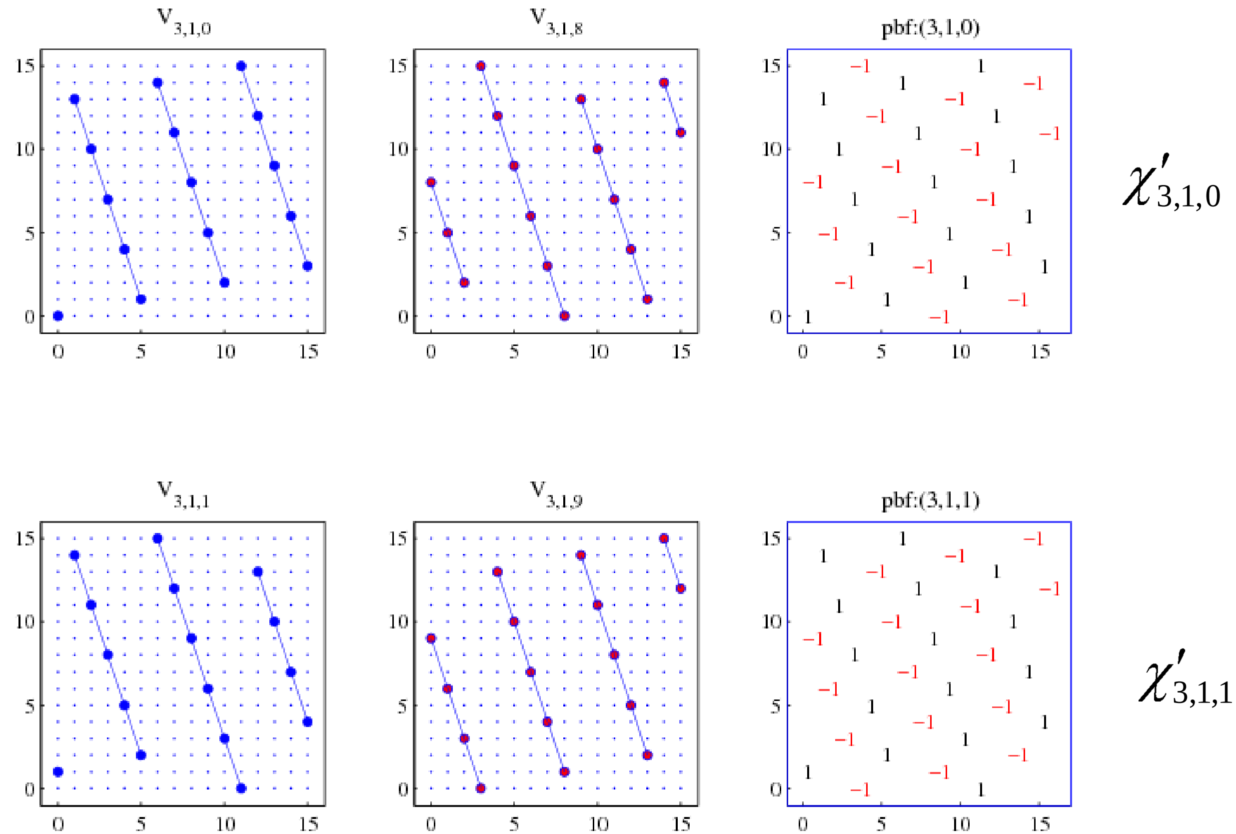


Fig. 7: Masks of the paired-functions.

All ‘1’s and ‘-1’s in the masks of paired functions $\chi'_{p,s,t}(n,m)$ lie on the parallel lines.

Algorithm of Image Reconstruction using Paired Transform (with calculating 2D-DFT.)

- Calculate all the generators $(p,s) \in G_k,\ k = 0:r$;
- For each generator, calculate paired image-signal;
- Modify the paired image-signal with twiddle coefficients;
- Perform 1-D DFT over the modified paired image-signal;
- Allocate $F_{\overline{(2m+1)p},\overline{(2m+1)s}}$ to fill 2-D DPT;
- Calculate 2-D inverse Fourier transform over $X_{N,N}$.

The reconstruction is complete, the result is original image

$f = \{ f_{n,m} \}$

Direction Image

The paired representation allows for inverting directly the projection data from frequency and time domain, without calculating both 1-D DFT and 2-D DFT. Let D_{p_1,s_1} be the following 2-D DFT composed only from the components of the 2-D DFT which lie on the given subset $T'_{p,s}$:

$$D_{p_1,s_1} = \begin{cases} F_{p_1,s_1}; & \text{if } (p_1,s_1) \in T'_{p,s}, \\ 0; & \text{otherwise.} \end{cases}$$

Case 1: $g.c.d(p,s)=1$, i.e. $(p,s) \in G_r$. The direction image of the 1st series is defined as:

$$d_{n,m} = d_{n,m;p,s} = \frac{1}{2N} f'_{p,s,(np+ms)\bmod N}, \quad (n,m) \in X_{N,N}.$$

The direction of these lines, or the projection is defined by coordinates of the frequency-point (p,s).

Example 4: Tree image of size 256×256

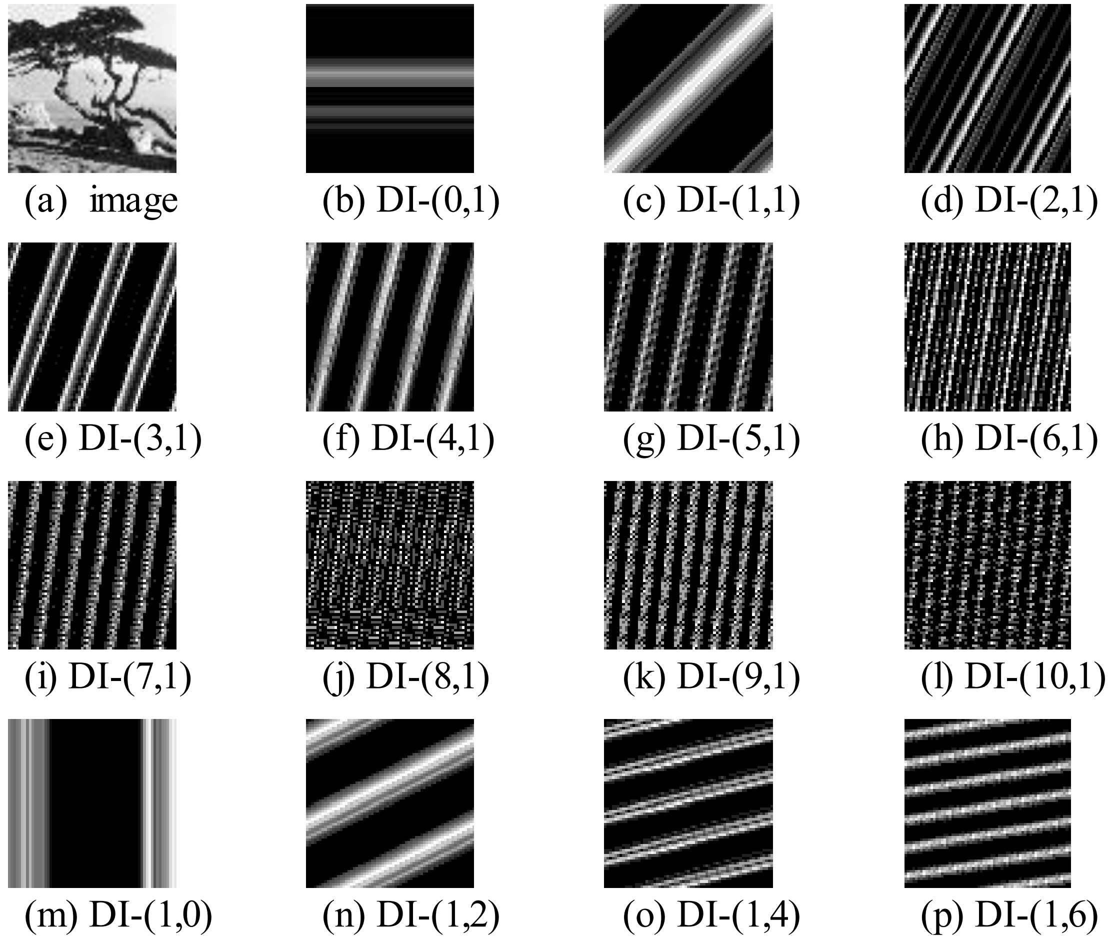


Fig. 8: (a) original image and (b)-(p) 15 direction image components of the image, generator *(p, s)=(0, 1), (1, 1), …, (10, 1), (1, 0), (1, 2), (1,4), (1,6)* . (All images have been scaled)

Example 5: Phantom image of size 256×256

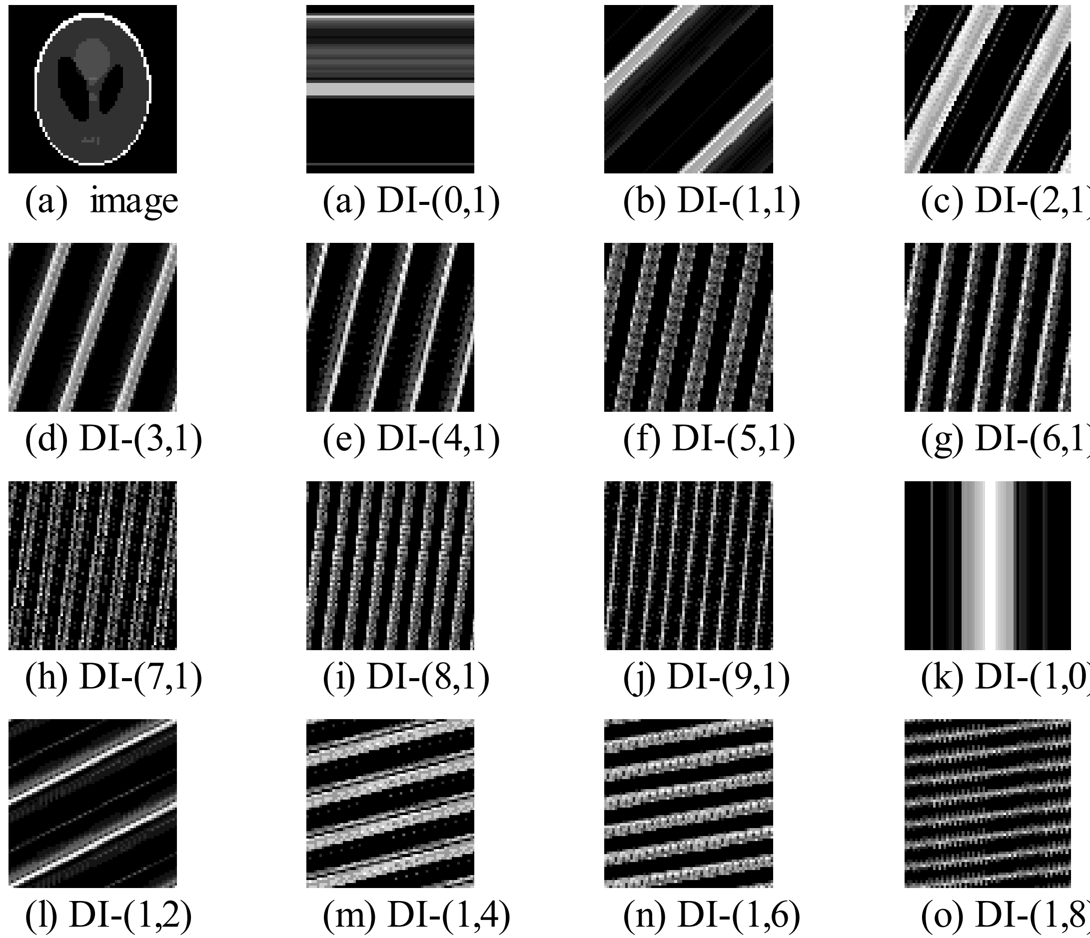


Fig. 9: (a) Original image and (b)-(p) 15 direction image components of the image, generator *(p, s)=(0, 1), (1, 1), …, (9, 1), (1, 0), (1, 2), (1,4), (1,6), (1,8)* . (All images have been scaled)

Case 2: $g.c.d(p,s)=2^k,\ (p,s)\in G_k,\ k\geq 1.$

The direction image of the kth series is defined as:

$$d_{n,m} \doteq d_{n,m;p,s} = \frac{1}{2^{k+1}N} f'_{p,s,(np+ms)\bmod N}, \quad (n,m)\in X_{N,N}$$

All (*3N-2*) subsets $T'_{p,s}$, with generators (*p*, *s*) from the set compose the partition of the grid $X_{N,N}$. The sum of all direction image $d_{n,m}$ equals the original image $f_{n,m}$.

Decomposition of the image :

$$f_{n,m} = \sum_{(p,s)\in J'_{N,N}} d_{n,m;p,s}$$

$$= \frac{1}{2N}\left(\sum_{k=0}^{r-1}\frac{1}{2^k}\sum_{(p,s)\in 2^k G_{r-k}} f'_{p,s,(np+ms)\bmod N} + \frac{1}{2^{r-1}}\sum_{(p,s)\in 2^r G_0} f'_{p,s,(np+ms)\bmod N}\right)$$

This is the formula of reconstruction of the image by its paired transform using operations of ***addition/subtraction*** and ***division by powers of two.***

Formula of Image Reconstruction

$$f_{n,m} = \frac{1}{2N} \sum_{k=0}^{r} \sum_{(p,s)\in 2^k G_{r-k}} [y(p,s,\overline{np+ms})_1 + y(p,s,\overline{np+ms})_2$$

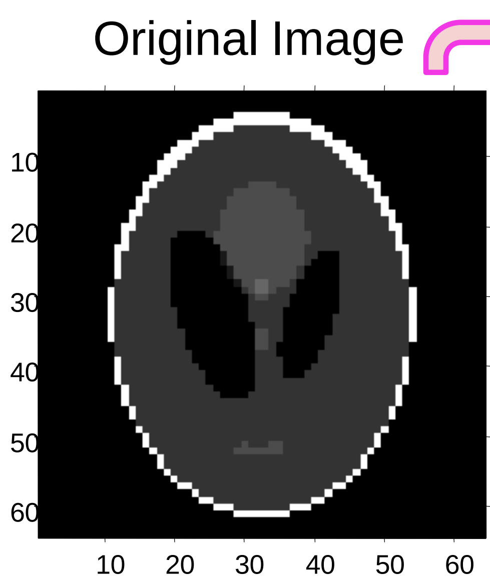
$$+ \ldots + y(p,s,\overline{np+ms})_M - y(p,s,\overline{np+ms}+\frac{N}{2})_1$$

$$- y(p,s,\overline{np+ms}+\frac{N}{2})_2 - \ldots - y(p,s,\overline{np+ms}+\frac{N}{2})_M]$$

where $M = p + s$.

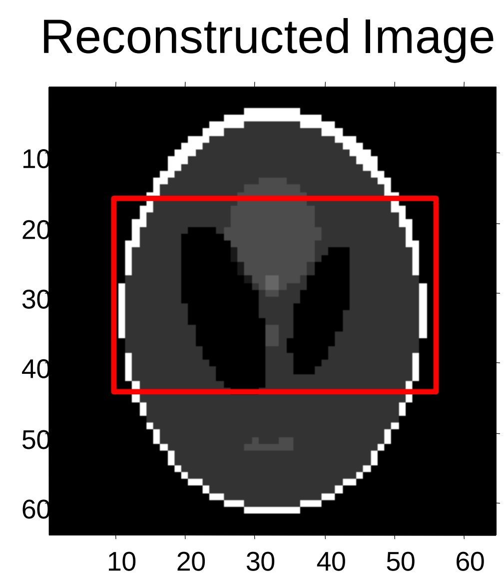
The problem of the image reconstruction from the projection by using the inverse 2-D paired transform has been solved.

Demo of Image Reconstruction (Phantom image of 64×64)



Demo: 2-D Fourier transform-based method*

Demo: Direct, or 2-D paired transform-based method*



*Avi files were removed

Conclusion

Based on the paired representation of the image, the problem of image reconstruction from its projections in the framework of the discrete model has been solved.

A new formula has been presented for the inverse 2-D paired transform, which can be used for solving the algebraic system described the measurement data of image reconstruction without using the 2-D Fourier transform technique. The image can be reconstructed directly from the image-signals which are calculated from projection data.

References

- R. Gordon, "A tutorial on ART (Algebraic reconstruction techniques)," *IEEE Trans. Nuclear Science, vol. 21, pp. 78*-93, 1974.
- A.M. Grigoryan and M.M. Grigoryan, "A new method of image reconstruction from projections," *Electronnoe Modelirovanie*, AS USSR, Kiev, 1986, vol. 8, no 6, pp. 74-77.
- A.M. Grigoryan, "2-D and 1-D multi-paired transforms: Frequency-time type wavelets," *IEEE Trans. on Signal Processing*, vol. 49, no. 2, pp. 344-353, Feb. 2001.
- A.M. Grigoryan, "Method of paired transforms for reconstruction of images from projections: Discrete model," *IEEE Trans. on Image Processing, vol. 12, no. 9, pp. 985-994*, Sep. 2003.
- A.M. Grigoryan and S.S. Agaian, *Multidimensional Discrete Unitary Transforms: Representation, Partitioning, and Algorithms*, Marcel Dekker, Inc., August 2003.

Thank you

Questions?