Discrete Unitary Transforms Generated by Moving Waves

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Abstract

This paper describes a new class of discrete heap transforms (DHT) which are unitary energy-preserving transforms and induced by input signals. These transforms have a simple form of composition and fast algorithms for any size of processed signals. We consider the heap transforms, defined by 2-D elementary rotations, as satisfying the given decision equations. The main feature of each DHT is the corresponding system of basis functions, which represent themselves a family of interactive waves which are moving in the field generated by the input signal. Properties and examples of DHT which we also call discrete signal-induced heap transforms, are described in detail.

Purpose

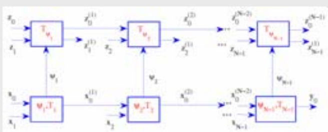
Development of novel unitary transforms generated by moving waves.

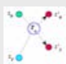
Introduction

Two main approaches to signal representation:
1st approach is based on the assumption that many signals can be described by certain complete systems of functions which are referred to as standing waves. For instance, when applying the Fourier transform, signals are defined as sums of cosine and sine waveforms of infinite (or finite) number of frequencies.
2nd approach is based on compactly supported window functions which cut the signal into parts, and transforms are analyzed for every part. This approach was first proposed for the short-time Fourier transform, when a single window was used for all frequency components. The window is translated by a chosen step along the time axis, to cover the entire time domain. In the wavelet theory this approach was developed, and fully scalable modulated windows for frequency localization were used. The window is sliding, and the transform of a part of the signal is calculated for every position. The result of the wavelet transform is a very redundant collection of time-scaling representation of the signal with different resolutions.

Definition and Examples of DsiHT

Let $\mathbf{x}=(x_0,x_1,x_2,...,x_{N-1})'$ be a vector-generator and $A=\{a_1,a_2,\dots,a_{N-1}\}$ to be given set of constants. Components of x are processed in sequence to x_0, x_1, ..., and x_{N-1}. The composition of the DsiHT is based on the special selection of a set of parameters which are initiated by the vector-generator through the so-called decision equations $\begin{cases} f(x,y,\varphi)=y_0 \\ g(x,y,\varphi)=a \end{cases}$
The transform is performed in a space of N-dimensional vectors z, but all required angles φ_k are found and the transformation is composed after solving the decision equations relative to a given vector-generator x.



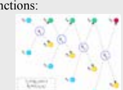


The transform of x results in a vector with the constant components of the set A

$$\mathcal{H}:\mathbf{x}\rightarrow H(\mathbf{x})=(y_0^{(N-1)},a_1,a_2,...,a_{N-1})'.$$

Example 1: Given a real number a, we consider the following functions:

$$f(x,y,\varphi)=x\cos\varphi-y\sin\varphi,\quad g(x,y,\varphi)=x\sin\varphi+y\cos\varphi,$$
$$H_\varphi:(x,y)\rightarrow(x\cos\varphi-y\sin\varphi,a)$$
$$\varphi=\arccos(a/\sqrt{x^2+y^2})-\arctan(x/y),$$
$$(\varphi=\arcsin(a/x),\ \text{if } y=0).$$



Example 2: Consider the DHT generated by the vector **h**=(1,1,1,…,1)' :

$$H(\mathbf{h})=||\mathbf{h}||\mathbf{e}_1=(||\mathbf{h}||,0,0,...,0)'=(\sqrt{N},0,0,...,0)'.$$

$$\mathbf{H}=\begin{bmatrix}0.5000&0.5000&0.5000&0.5000\\-0.7071&0.7071&0&0\\-0.4082&-0.4082&0.8165&0\\-0.2887&-0.2887&-0.2887&0.8660\end{bmatrix},\quad \begin{bmatrix}\varphi_1=-0.7854\\\varphi_2=-0.6155\\\varphi_3=-0.5236\end{bmatrix}.$$

Basis functions of the DHT, which are the row-waves of the non-normalized matrix **M**:

$m_n(t),\ n=0:3$

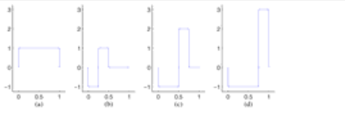


Fig. 8. Four non-normalized basis functions of the four-point DsiHT generated by the vector (1,1,1,1)'.

The process of formation of each row of **M** in terms of moving waves. The first row is the standing wave which is the generator. We compare with the second row of the matrix, $h_1(n)=[-1,1,0,0]$, the following continuous-time step function:

$$m(t)=m_0(t)=\begin{cases}-1; & \text{if } 0\le t<\Delta,\\ 1; & \text{if } \Delta\le t<2\Delta,\\ 0; & \text{otherwise,}\end{cases}\qquad \Delta=1/4.$$

The wave describes a process of breaking the generator into two different parts. In the following movement, this wave interacts with its previous form and composes new waves, or basis functions of the DHT. Indeed, we can associate to the 3rd row of **M**,

$$m_2(n)=[-1,-1,2,0]=[-1,1,0,0]+2[0,-1,1,0],\qquad m_2(t)=m(t)+2m(t-\Delta).$$
$$m_3(n)=[-1,-1,-1,3]=[-1,1,0,0]+2[0,-1,1,0]+3[0,0,-1,1]$$
$$m_3(t)=m(t)+2m(t-\Delta)+3m(t-2\Delta)=m_2(t)+3m(t-2\Delta).$$

Considering the normalized coefficients, these functions can be written as follows:

$$m_1(t)=\frac{1}{\sqrt{1+1^2}}m(t),\qquad m_2(t)=\frac{1}{\sqrt{1+2^2}}[m(t)+2m(t-\Delta)],$$
$$m_3(t)=\frac{1}{\sqrt{1+2^2+3^2}}[m(t)+2m(t-\Delta)+3m(t-2\Delta)].$$

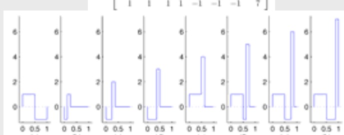
The wave associated with the nth row of the matrix M of the DHT is defined as:

$$m_n(t)=\frac{1}{\sqrt{\sum_{k=1}^{n-1}k^2}}\sum_{k=0}^{n}(k+1)m(t-k\Delta),\quad n=1:(N-1).$$

Waves $m_n(t)$ are formed due to superpositions of the first wave $m(t)$ with other $(n-1)$ shifted waves $m(t-k\Delta)$ magnified linearly by factors of $(k+1)$, $k=1:(n-1)$. In general, the form of interaction of waves is complicated and depends on the generator. The diagonal of **M** (or **H**) is referred to as a wall through which the waves can not pass, since their power diminishes to zero when they approach to the wall.

Example 3: In the $N=8$ case, when $\mathbf{h}=(1,1,1,1,-1,-1,-1,-1)'$, the non-normalized matrix generated by this vector equal

$$\mathbf{M}=\begin{bmatrix}1&1&1&1&-1&-1&-1&-1\\-1&1&0&0&0&0&0&0\\-1&-1&2&0&0&0&0&0\\-1&-1&-1&3&0&0&0&0\\1&1&1&1&4&0&0&0\\1&1&1&1&-1&5&0&0\\1&1&1&1&-1&-1&6&0\\1&1&1&1&-1&-1&-1&7\end{bmatrix}.$$



$$m_k(t)=\alpha m_{k-1}(t)+kw(t-(k-1)\Delta),\quad \alpha=(-1)^{h_k/h_{k-1}}$$

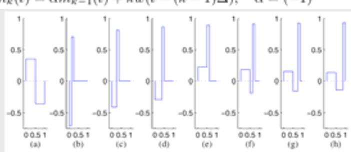


Figure 9: Non- and normalized basis functions of the DsiHT.

Process of motion and transformation of one basis function into another, when starting from the wave-generator. There are 3 stages which can be separated during this process.
- In the first stage, the statical stage, the generator itself is lying as the basis function.
- The second stage, the evolution stage, is related to the formation of a new wave.
- The last stage is the dynamical stage, when the new established wave is moving to the end of the path. This wave is composed by two parts, the first part resembles the generator and the second part, or a splash, is a static wave increasing by amplitude.

If we consider the normalized coefficients, then the amplitude of the mentioned splash tends to 1 when the generator vanishes in amplitude.

Example 4: The DsiHT generated by the vector h=(1,-1, 1,-1, 1,-1, 1,-1)'

$$\mathbf{H}=\mathbf{DM},\quad \mathbf{M}=\begin{bmatrix}1&-1&1&-1&1&-1&1&-1\\1&1&0&0&0&0&0&0\\-1&1&2&0&0&0&0&0\\1&-1&1&3&0&0&0&0\\-1&1&-1&1&4&0&0&0\\1&-1&1&-1&1&5&0&0\\-1&1&-1&1&-1&1&6&0\\1&-1&1&-1&1&-1&1&7\end{bmatrix},\quad \mathbf{D}=\mathrm{diag}\begin{Bmatrix}0.3536\\0.7071\\0.4082\\0.2887\\0.2236\\0.1826\\0.1543\\0.1336\end{Bmatrix}$$

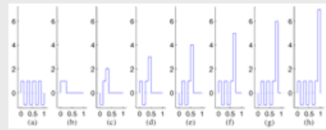


Figure 10: Eight non-normalized basis functions of the DsiHT.

We can note that the process of motion and transformation of basis functions can be described by three stages. The generator in the last stage is trying to restore itself, while moving the large splash to the end. One can see the maximum reconstruction of the generator in the last movement, when it gives the much power to the splash, whose amplitude increases linearly (in the non-normalized scale). We can also say, that the generator induces some field and then tries to pass through it, and that is fulfilled with the loss which occurs in the form of the high but narrow splash moving to the end of the path. If we consider the normalized basis-waves of the transformation, then the splash of the wave is moving with the established amplitude, and the generator in this wave is restored in the small amplitude, as shown in below:

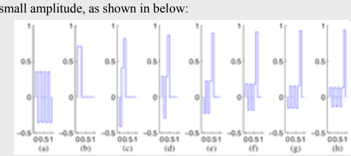


Figure 11: Eight basis functions of the DsiHT generated by the vector (1,-1,1,-1,1,-1,1,1,-1).

DsiHT in the 2nd generation

Analyzing the matrices of the heap transformations in non-normalized form, one can notice that the last row coincides with the first one at each point, except the last one, where the value $(N-1)$ is written. We call the vector composed by this row the dual vector. Thus,

$$m_{N-1}(n)=m_1(n),\quad h_{N-1}(n)=\alpha^{-1}h_1(n),\quad n=0:(N-2),\quad \alpha=h_1(0)/h_{N-1}(0).$$

If we consider the new heap transformation $\mathbf{H}_2$, generated by the dual-vector h_{N-1} (or m_{N-1}), then the matrix of this transformation will be similar to the original heap transformation **H** generated by the generator-vector h_1 (or m_1). For example, when $N=4$ and $\mathbf{h}=(1,-1,1-1)'$,

$$\mathbf{H}_{[1,-1,1,-1]}=\begin{bmatrix}0.5000&-0.5000&0.5000&-0.5000\\0.7071&0.7071&0&0\\-0.4082&0.4082&0.8165&0\\0.2887&-0.2887&0.2887&0.8660\end{bmatrix},\quad \mathbf{H}_{[1,-1,1,3]}=\begin{bmatrix}0.2887&-0.2887&0.2887&0.8660\\0.7071&0.7071&0&0\\-0.4082&0.4082&0.8165&0\\-0.5000&0.5000&-0.5000&0.5000\end{bmatrix}$$

$$\mathbf{H}_{[-0.2887,-0.2887,-0.2887,0.8660]}=\mathbf{H}_{[1,-1,1,3]}.$$

Movement by waves

The complete system of basis functions of the heap transformation can be described in the form of moving waves originated from the generator. Given $N>1$, we consider the matrix of the N-point discrete heap transform as a set of moving waves:

$$\mathbf{H}=\begin{bmatrix}h_{t=0}(n)\\h_{t=1}(n)\\h_{t=2}(n)\\\dots\\h_{t=N-1}(n)\end{bmatrix},\qquad h_{t=0}(n)=x(n)=\frac{\mathbf{x}}{||\mathbf{x}||}$$

- The starting time point is 0, when the wave rests. In this state of rest, the wave is the generator, and the generator is referred to as a field. When applying the heap transform to a signal $\mathbf{z}=(z_1,z_1,...z_{N-1})'$, the first component of the transform

$$y_0=x(0)z_0+x(1)z_1+...+x_{N-1}z_{N-1}$$

is considered as a process of solubility of the signal in the field. The signal falls into the field, and the value of y_0 defines the amplitude of solubility. The maximum solubility occurs for the generator itself.
- In the rest moments of time, the input signal is moving according to the laws of the field, which do not depend on the signal,

$$y_t=h_t(0)z_0+h_t(1)z_1+...+h_t(t)z_t,\quad t=2:(N-2).$$

y_t defines the amplitude of moving in the field at time t.
- In the last point, at the end of the movement, when the signal **z** gets out of the field, the amplitude describes the result of the movement, or the difference of the input signal with the field. The end of the movement is defined by the basis function which is similar to the generator.

$$y_{N-1}=h_{N-1}(0)z_0+h_{N-1}(1)z_1+...+h_{N-1}(N-1)z_{N-1},$$

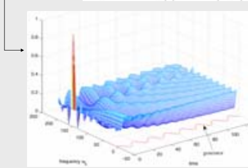
If the signal z=x, then there is no movement. The signal-generator gives the whole energy to the field at the beginning, $|y_0|=||\mathbf{x}||$, the signal dissolves completely and there is no movement because of the absence of energy. If an input signal z is composed of waves with frequencies as the generator x then each such component will be expressed at the beginning of the heap transform because of giving the maximum of its energy, and not moving much above the waves of the DHT.

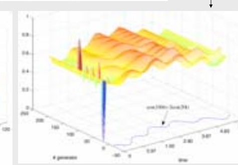
Example 6: The mesh of the 128-point DHT of cosine signals defined as:

$$\mathbf{x}\leftarrow x(t)=\cos(150\omega_0 t),\quad \omega_0=\tfrac{2\pi}{128}$$
$$\mathbf{z}\leftarrow z(t)=z_k(t)=\cos(\omega_k t),\quad \omega_k=k=0:200,\quad t\in[0,2\pi].$$

Example 7: The mesh of the 128-point DHTs of the cosine signal defined as:

$$\mathbf{z}\leftarrow\cos(100\omega_0 t)-2\cos(20\omega_0 t),\quad t\in[0,2\pi],$$
$$\mathbf{x}_k\leftarrow x_k(t)=\cos(\omega_k t),\quad \omega_k=k\omega_0,\quad k=0:200.$$





Example 8:

$$z(t)=\sin(4\pi t)+\sin(8\pi t)+2[\delta(t-t_1)+\delta(t-t_2)],\quad t\in[0,2\pi],$$
$$t_1=4.7124 \text{ and } t_2=4.8597$$

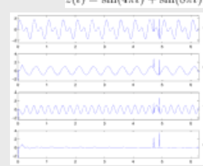


Figure 14: Input signal $z(t)$, the sin(8πt)-induced DHT of $z(t)$, $sin(4\pi t)$-induced DHT of $z(t)$, $sin(4\pi t)$-induced DHT of the signal in b.

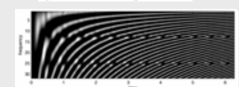


Figure 15: The image of DHTs of z(t), when generators are $x(t)=sin(\omega t)$, when $\omega=1:32$.