New Discrete Unitary Haar-Type Heap Transforms

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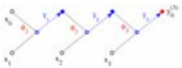
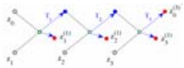




Abstract

This paper introduces a new class of the discrete Haar-type heap transformations (DHHT) which are induced by input signals and use the path similar to the traditional Haar transformation. Transformations are fast and performed by simple rotations, can be composed for any order, and their complete systems of basis functions represent themselves variable waves that are generated by signals. The 2^r-point discrete Haar transform is the particular case of the proposed transformations, when the generator is the constant sequence {1,1,1,...,1}. These transforms can be used in many applications and improve the results of the Haar transformation.

Introduction

The discrete heap transformations (DHT) are defined by complete systems of functions which are referred to as waves moving through the field which is generated by given signals. In other words, the described transformations are unitary and induced by input signals. We here stand on the case, when the transformations are real and defined by the generators through the path similar to the Haar transformation. The selection of generators allows for tuning the proposed transformations to certain classes of signals, and it is considered a very important aspect in application of the transformations. We assume that the new class of transforms, which we call the Haar-type heap transforms can be used in many applications together with the Haar transform. Our preliminary results show, that in many cases the application of the DHHT provides better results than the DHT does.

Energy-preserving heap transforms

The N-point discrete heap transform of the vector $\mathbf{z}=(z_0,z_1,...,z_{N-1})'$ is calculated by:

$$T[\mathbf{z}]=(z_0^{(N-1)},z_1^{(1)},...,z_{N-1}^{(1)})',\quad \text{while}\quad T[\mathbf{x}]=(y_0^{(N-1)},0,...,0).$$

When processing the vector-generator **x**, the first pair of components, (x_0, x_1), is transfered to the vector $(y_0,0)'$ as the point (x_0,x_1) is rotated to the point $(y_0,0)$ of the horizonal line. Then, on the next kth step of calculations, when k>1, similar rotations are accomplished over vectors $(y_0^{(k-1)},x_k)'$, with a new value of the first component on the (k-1)th step:

$$\begin{bmatrix} y_0^{(k)} \\ 0 \end{bmatrix}=\begin{bmatrix} \cos\varphi_k & -\sin\varphi_k \\ \sin\varphi_k & \cos\varphi_k \end{bmatrix}\begin{bmatrix} y_0^{(k-1)} \\ x_k \end{bmatrix},\quad \tan(\varphi_k)=-\frac{x_k}{y_0^{(k-1)}},\quad (y_0^{(0)}=x_0)$$

$$y_0^{(k-1)}=(\mathbf{T}_{k-1}\cdots\mathbf{T}_2\cdot\mathbf{T}_1\mathbf{x})_0=\sqrt{x_0^2+x_1^2+...+x_{k-1}^2},\quad k=1:(N-1).$$

Each DHT has the unique angular representation: $\mathbf{x}\to\mathbf{A_x}=\{||\mathbf{x}||,\varphi_1,\varphi_2,...,\varphi_{N-1}\}$

1. Discrete Haar Transform

The basic transformation **T** is defined by the simple binary orthogonal matrix 2×2:

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix},\qquad \varphi=\frac{\pi}{4},\ \text{i.e.}\ T=T_{\frac{\pi}{4}}.$$

The energy of the input is distributed between the components of the output as follows:

$$\begin{bmatrix} E[x_0]=x_0^2 \\ E[x_1]=x_1^2 \end{bmatrix}\to\begin{bmatrix} E[y_0]=\frac{x_0^2+x_1^2}{2}-x_0x_1 \\ E[y_1]=\frac{x_0^2+x_1^2}{2}+x_0x_1 \end{bmatrix}$$

The Haar transform can be calculated by a few stages. In the first stage, the pairs of the vector are processed as follows:

$$\begin{bmatrix} y_{2k} \\ y_{2k+1} \end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_{2k} \\ x_{2k+1} \end{bmatrix},\quad k=1,...,(N/2-1),$$

and y_{2k} is recorded as the kth heap. We obtain the following vector with N/2 heaps:

$$\mathbf{y}^{(1)}=(\mathbf{y}_1^{(1)},\mathbf{y}_2^{(1)})=(\underbrace{y_0,y_2,y_4,...,y_{N-2}}_{N/2\ \text{heaps}},y_1,y_3,y_5,...,y_{N-1}).$$

The process of calculation of the Haar transform is continued, but the second part $y_2^{(1)}$ is not used for the further calculations. N/4 transformations $T_{\pi/4}$ are applied over the first half of the obtained vector, in other words, over the heaps

$$\mathbf{y}_1^{(1)}=(y_0,y_2,y_4,...,y_{N-2}).$$

Result: N/4 heaps are calculated in the output

$$\mathbf{y}_1^{(1)}\to\mathbf{y}^{(2)}=(\mathbf{y}_1^{(2)},\mathbf{y}_2^{(2)})=(\underbrace{y_0^{(1)},y_2^{(1)},...,y_{N/2-2}^{(1)}}_{N/4\ \text{heaps}},y_1^{(1)},y_3^{(1)},...,y_{N/2-1}^{(1)}).$$

Continuing this process $\log_2(N-2)$ times more, until only one heap stays, we obtain the traditional N-point discrete Haar transform of the vector **x**

$$\mathbf{x}\to H[\mathbf{x}]=(y_1^{(r)},y_2^{(r)},\mathbf{y}_2^{(r-1)},...,\mathbf{y}_2^{(3)},\mathbf{y}_2^{(2)},\mathbf{y}_2^{(1)}).$$

2. Haar-type heap transforms

By using the described above approach for calculating different heaps, we can define a unitary transformation **H** induced by a vector **x**, when the basis transformation $T\varphi_k$ are generated and used instead of $T_{\pi/4}$. All transformations $T\varphi_k$ will be generated by inputs as is done for the DHT. The path for **H** is the same as for the Haar transformation. The process starts with two components (x_0,x_1), and the kth heap y_{2k} is calculated by

$$\begin{bmatrix} y_{2k} \\ 0 \end{bmatrix}=\begin{bmatrix} \cos\varphi_{k+1} & -\sin\varphi_{k+1} \\ \sin\varphi_{k+1} & \cos\varphi_{k+1} \end{bmatrix}\begin{bmatrix} x_{2k} \\ x_{2k+1} \end{bmatrix},\quad \varphi_2=-\arctan\frac{x_{2k}}{x_{2k+1}},\quad k=0,1,...,\frac{N}{2}-1.$$

As a result, we obtain the following new vector with N/2 heaps,

$$\mathbf{y}^{(1)}=(\underbrace{y_0,y_2,y_4,...,y_{N-2}}_{N/2\ \text{heaps}},0,0,0,...,0).$$

The process of calculation is continued, and new transformations $T\psi_n$, $n=1:(N/2-1)$, are generated by the obtained heaps $\mathbf{x}^{(1)}$. The calculation results in N/4 new heaps:

$$\mathbf{x}^{(1)}=(y_0,y_2,y_4,...,y_{N-2})\to\mathbf{y}^{(2)}=(\underbrace{y_0^{(1)},y_2^{(1)},...,y_{N/2-2}^{(1)}}_{N/4\ \text{heaps}},0,0,...,0)$$

and more N/8 new heaps on the following stage and so on, until we receive only one heap. The transformation thus is generated by $N/2+N/4+...+2+1=N-1$ basic transformations $\{\{T\varphi_k\},\{T\psi_n\},...\}$, whose angles define the angular representation of the transform. This is the discrete Haar-type heap transformation (DHHT) generated by the vector **x**.

Example 1: In the $N=4$ case, we have the following matrices of the DHHT:

$$\mathbf{H}_1=\begin{bmatrix} 0.5000 & 0.5000 & 0.5000 & 0.5000 \\ -0.5000 & -0.5000 & 0.5000 & 0.5000 \\ -0.7071 & 0.7071 & 0 & 0 \\ 0 & 0 & -0.7071 & 0.7071 \end{bmatrix},\quad \mathbf{x}=(1,1,1,1),$$

$$\mathbf{H}_2=\begin{bmatrix} 0.3162 & 0.6325 & 0.6325 & 0.3162 \\ -0.3162 & -0.6325 & 0.6325 & 0.3162 \\ -0.8944 & 0.4472 & 0 & 0 \\ 0 & 0 & -0.4472 & 0.8944 \end{bmatrix},\quad \mathbf{x}=(1,2,2,1).$$

Example 2: Let $N=8$, and let generator be $\mathbf{x}=(2,1,1,3,2,1,3,2)$. The matrix of the DHHT generated by $\mathbf{x}$ can be written in form **H=DM**, where **M** is the integer orthogonal matrix

$$\mathbf{M}=\begin{bmatrix} 2 & 1 & 1 & 3 & 2 & 1 & 3 & 2 \\ 12 & 6 & 6 & 18 & -10 & -5 & -15 & -10 \\ 4 & 2 & -1 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -26 & -13 & 15 & 10 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 \end{bmatrix},\quad \mathbf{D}=\text{diag}\begin{Bmatrix} 0.1741 \\ -0.0318 \\ -0.1826 \\ 0.0292 \\ -0.4472 \\ 0.3162 \\ -0.4472 \\ -0.2774 \end{Bmatrix}.$$

Example 3: 16 basis function of the 16-point DHHT are given in (a), when the generator is the sampled cosine function defined at 16 equidistant points in the interval [0,π]. The waves of the DHHT are referred to as moving waves which change during the movement.

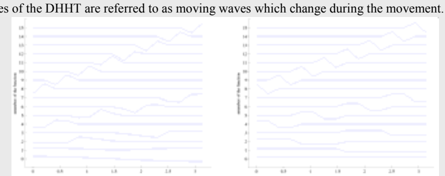


Fig. 1: (a) Variable basis functions of the DHHT. (b) Standing basis functions of the DHT.

When applying this transformation to an input vector **z,** the calculations are performed on each stage, in accordance with the composition of the DHHT. On the first stage, all basic transformations $T\varphi_k$ are applied as follows:

$$\begin{bmatrix} t_{2(k-1)} \\ t_{2k-1} \end{bmatrix}=\begin{bmatrix} \cos\varphi_k & -\sin\varphi_k \\ \sin\varphi_k & \cos\varphi_k \end{bmatrix}\begin{bmatrix} z_{2(k-1)} \\ z_{2k-1} \end{bmatrix},\quad k=1:N/2.$$

Then, the first part of the result

$$\mathbf{t}^{(1)}=(\underbrace{t_0,t_2,t_4,...,t_{N-2}},t_1,t_3,t_5,...,t_{N-1})$$

is processed similarly by the basic transformations $T\psi_n$, $n=1:N/4$. As a result, other $N/4$ heaps are calculated in the output. On this stage, the input **z** is transformed as

$$\mathbf{z}\to\left(\underbrace{\underbrace{t_0^{(1)},t_2^{(1)},...,t_{N/2-2}^{(1)}},t_1^{(1)},t_3^{(1)},...,t_{N/2-1}^{(1)}},t_1,t_3,t_5,...,t_{N-1}\right)$$

Illustrtation of DHHTs

Figure 2 shows the signal **z** of length 512 in part a, its Haar-type heap transform in b. The generator **x** is the sampled wave cos(t), t is in [0,4π], and the DHT in c.

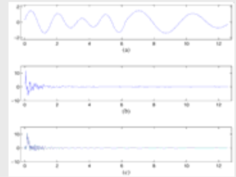


Figure 2: (a) The discrete-time signal **z**, (b) the **x**-induced DHHT of **z**, (c) DHT of **z.**

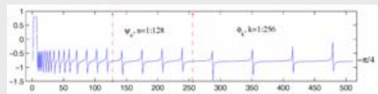


Figure 3: The angular representation $\mathcal{A}_\mathbf{x}=(||\mathbf{x}||,\varphi_1,\varphi_2,...,\varphi_{N/2},\psi_1,...,\psi_{N/4},...)$

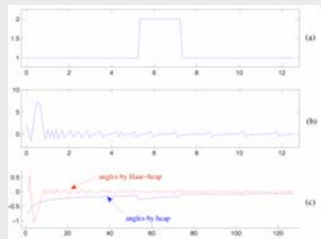


Figure 4: (a) Input rectangle signal **z**, (b) the DHHT of **z,** and (c) the angular representation generated by the DHHT and DHT (The generator **x** is defined by x(t)=cos(t), t is in [0,4π].)

DHHT of an order N≠2^r

There are many ways to compose a few heaps by reducing the process of transform composition to the $N=2^r$ case. For instance, when $N=5$, we can propose the following process:

$$\mathcal{H}_5:\{x_0,x_1,x_2,\underbrace{x_3,x_4}\}\xrightarrow{I_3\oplus H_2}\{\underbrace{x_0,x_1,x_2,y_0^{(3)}},y_1^{(3)}\}\xrightarrow{H_4\oplus 1}\{y_0,y_2,y_1,y_3,y_1^{(3)}\}.$$

The last two components are rotated first, and then the first three components together with the obtained heap $y_0^{(3)}$ are processed as for the four-point DHHT. We obtain the following matrix:

$$\mathbf{H}_5=(\mathbf{H}_4\oplus 1)(\mathbf{I}_3\oplus\mathbf{H}_2)=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

- For N=6, we can consider the following process:

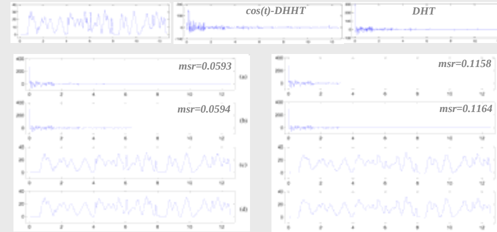
$$\mathcal{H}_6:\{x_0,x_1,\underbrace{x_2,x_3,x_4,x_5}\}\xrightarrow{I_2\oplus H_4}\{\underbrace{x_0,x_1,y_0^{(2)},y_2^{(2)}},y_1^{(2)},y_3^{(2)}\}\xrightarrow{H_4\oplus I_2}\{y_0,y_2,y_1,y_3,y_1^{(2)},y_3^{(2)}\}$$

which leads to the following matrix composition: $\mathbf{H}_6=(\mathbf{H}_4\oplus\mathbf{I}_2)(\mathbf{I}_2\oplus\mathbf{H}_4)$

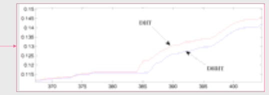
- For N=7, we can calculate $\mathbf{H}_7=(\mathbf{H}_6\oplus 1)(\mathbf{I}_5\oplus\mathbf{H}_2)$, or $\mathbf{H}_7=(\mathbf{H}_5\oplus\mathbf{I}_2)(\mathbf{I}_3\oplus\mathbf{H}_4)$.

Experimental results

DHHT can be used, for instance, for signal compression. For instance, we consider the simple process of truncating L components of the transform of a random signal **z**.



The mean-square-root error curves for signal reconstruction after truncating L coefficients of the 512-point DHHT and DHT, where L is in [256,480],



Characteristics of basis waves

Let **z** be a real vector of dimension N, which is defined at time-points $t_0,t_1,...,t_{N-1}$ of an interval $[t_0,t_{N-1}]$. We assume $||\mathbf{z}||=1$. The numbers $p_k=z_k^2$ can be considered as probabilities of z_k, k=0:(N-1). Each basis function, or row $m_n(k)$ of the matrix of the DHHT is referred to as a wave moving in the field of the generator **x** of this transform. Therefore, we can apply for these waves the concepts of the centroid and mean width of the wave:

$$\bar{m}_n=\sum_{k=0}^{N-1}t_kp_k=\sum_{k=0}^{N-1}t_km_n^2(k),\qquad d(m_n)=\sqrt{\sum_{k=0}^{N-1}(t_k-\bar{m}_n)^2m_n^2(k)},\quad n=0:(N-1).$$

Example 4: Let N=32 and the generator be **x**=(1,1,…,1)’. The centroid of the basis waves of the 32-point DHT are shown in Fig. 5 in part a, along with the mean widths in b. For waves of numbers 17– 32, the centroid moves uniformly with the step Δ=1/8, for waves Of numbers 9-16, 5-8, and 3- 4, Centroids move with steps 2Δ, 4Δ, and 8Δ, respectively.

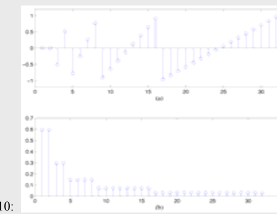


Figure 10:

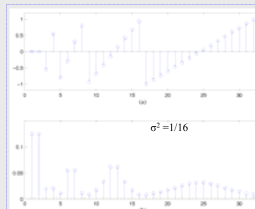


Figure 11: (a) Centroids and (b) widths of the basis waves of the 32-point DHHT generated by the Gaussian vector **x** with zero mean and variance 1.

Example 5: N=32 and the generator **x** is defined by the Gaussian function $g(t)=\exp(-t^2/\sigma^2)$ sampled at 32 points in [-1,1]. The centroid and mean widths of the basis waves of the 32-point **x**-induced DHT are shown in Fig. 6. The widths of these waves change as The Gaussian function.

Difference of DHHT and DHT:
In the wavelet theory, the functions $\psi(t)=k(a)\psi((t-b)/a)$ are used – plane waves. In the heap transform, the decomposition of the signal and its reconstruction are performed by functions $k(a,t)\psi((t-b)/a)$. The system of functions is generated by inputs.