Multiresolution Signal Processing by Fourier Transform Time-Frequency Correlation Analysis

Julian U. Anugom and Artyom M. Grigoryan
Electrical and Computer Engineering
University of Texas at San Antonio

__Abstract:__ **In this paper, multiresolution signal processing is described, by the continuous Fourier transform, not the short-time Fourier transform. The inverse Fourier transform is defined by the integral Fourier formula which is referred to as the correlation of the function (signal) with cosine waveforms of various frequencies. This is a direct way to perform the time-frequency analysis of signals. The Fourier transform is described as the sum of wavelet-like transforms with the cosine analyzing function of one period. Properties of such transforms, including the inverse formula of reconstruction of the signal by the described wavelet-like transforms are described. Examples of application of such transforms to detect the exact location of a high-frequency signal are given.**

◦ It is common to refer to the traditional understanding of the Fourier transform as the decomposition of the function by cosine and sine waves running along the whole real line.

◦ The Fourier transform is considered as a transform without time resolution, meaning that it is very difficult to determine the time at which the sinusoidal waves occured in the signal.

◦ Our study shows that wavelet-like transforms are contained within the mathematical structure of the Fourier transform. It can be represented by an equivalent form of specified wavelets, such as the so-called A- and B-wavelets with fully scalable modulated windows.

◦ This paper presents a new look on the Fourier transform by considering its inverse formula, which is called the Fourier integral formula and which associates with the frequency-time analysis of the signal.

Integral Fourier Function

Let $f(t)$ be an absolute integrable function on the real line, R, that satisfies Dinni's condition at every point $t \in R$,

$$\int_{-\delta}^{\delta} \left| \frac{f(t+x) - f(t)}{x} \right| dx < \infty,\ \delta > 0 \qquad (1)$$

The function $f(t)$ can be represented as

$$f(t) = \frac{1}{\pi} \int_{0}^{\infty} d\lambda \int_{-\infty}^{\infty} f(x) \cos\left(\lambda(x-t)\right) dx.$$

This is the well-known integral Fourier formula. we consider the integral-function in the above decomposition of $f(t)$

$$F(\lambda, t) = \int_{-\infty}^{\infty} f(x) \cos\left(\lambda(x-t)\right) dx. \qquad (2)$$

The inverse Fourier transform is thus defined by

$$f(t) = \frac{1}{\pi} \int_{0}^{\infty} F(\lambda, t) d\lambda. \qquad (3)$$

Given a frequency λ, the function $F(\lambda, t)$ is periodic with period $2\pi/\lambda$.

Function $F(\lambda, t)$ is a frequency-time, or time-frequency representation of the function $f(t)$, which is defined as the correlation of the function with the scaled cosine wave $\cos(\lambda t)$ when t runs from $-\infty$ to ∞. The value of function $f(t)$ at point t is defined as the sum of all these correlations calculated at this point.

• Formulas (2) and (3) lead directly to the multiresolution signal processing. Indeed, the correlation is the basic operation in the wavelet theory that relates to wavelet transforms which provide a local, or time-scale analysis of the signal.

∘ The correlation in (2) leads to time-frequency analysis of functions by the Fourier transform. Indeed, let $\psi(t)$ be a function that is zero outside the interval $[-\pi, \pi)$ and coincides with the cosine function inside this interval,

$$\psi(x) = \begin{cases} \cos(x), \ x \in [-\pi, \pi) \\ 0, \ \text{otherwise.} \end{cases} \tag{4}$$

Let $\mathcal{A}$ be the family $\{\psi_{\lambda;t}(x) = \psi(\lambda(x-t)\}$ of the following time-scale and shift transformations

$$\psi_{\lambda,t}(x) = \begin{cases} \cos(\lambda(x-t), \ x \in [t - \frac{\pi}{\lambda}, t + \frac{\pi}{\lambda}) \\ 0, \text{ otherwise} \end{cases}$$

where $\lambda > 0$ and $t \in (-\infty, \infty)$.

<u>Process of $F(\lambda, t)$ component formation:</u>

For the partition of the time line by specific intervals of length $2\pi/\lambda$ with the centers being integer multiples by $2\pi/\lambda$ and shifted by t

$$I_n = I_n(\lambda, t) = \left[t + \frac{(2n-1)\pi}{\lambda}, t + \frac{(2n+1)\pi}{\lambda} \right)$$

the following holds:

$$\begin{aligned} F(\lambda, t) &= \sum_{n=-\infty}^{\infty} \int_{I_n} f(x) \cos(\lambda(x-t))\, dx \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \psi_{\lambda,t_n}(x) dx \end{aligned}$$

where $t_n = t_n(\lambda, t) = t + n(2\pi/\lambda)$, $n = 0, \pm 1, \ldots$

The full Fourier correlation function equals to

$$F(\lambda, t) = \sum_{n=-\infty}^{\infty} \Psi(\lambda, t, n) \qquad (5)$$

where the transformation $\Psi(\lambda, t, n)$ is defined by

$$\Psi(\lambda, t, n) = \Psi(\lambda, t_n) = \int_{-\infty}^{\infty} f(x)\psi_{\lambda, t_n}(x)dx.$$

If $\lambda = 0$, we consider $I_0 = (-\infty, +\infty)$ and

$$\Psi(\lambda, 0) = \int_{-\infty}^{\infty} f(x)dx = F(\lambda, 0).$$

The transform $\Psi(\lambda, t_n)$ is the cross-correlation of one period of the cosine waveform of frequency λ with the function $f(t)$ in the finite time interval $I_n(\lambda, t)$. The interval is located at time point t_n and has the length equal $2\pi/\lambda$. Thus $\Psi(\lambda, t_n)$ contains information if a cosine wave of frequency λ of short period within the interval of length $2\pi/\lambda$ is located (or, has been occured) in the signal at time point t_n. The length of the interval within which the wave is analyzed is inversely proportional to the frequency. The intervals begin at t.

The Fourier correlation function $F(\lambda, t)$ is the sum of correlations of one-period cosine wave $\psi(\lambda x)$ along the whole partition of the time line. As an example, Figure 1 shows the grid of 560 specified frequency-time points (λ_k, t_n) in the domain $[0, 200] \times [0, 1]$.

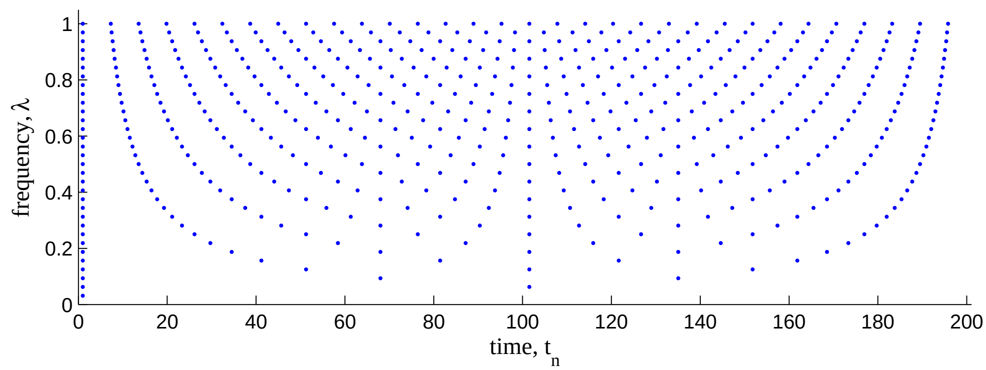


Fig. 1. Grid with 560 frequency-time points (λ_k, t_n), where $\lambda_k = k/32$ and the time points t_n start from point $t = 1$.

The reconstruction formula:

$$f(t) = \frac{1}{\pi} \int_0^\infty \sum_{n=-\infty}^{\infty} \Psi(\lambda, t, n)\, d\lambda, \;\; t \in (-\infty, \infty)$$

The transform

$$f(t) \rightarrow \{\Psi(\lambda, t, n); \lambda \in [0, \infty), n = 0, \pm 1, ...\}$$

describes the frequency-time analysis of the function $f(t)$. We call this transform the ψ-*resolution* of the function.

The ψ-resolution is described by the totality of 2D functions $\{\Psi(\lambda, t, n);\ n = 0, \pm 1, \pm 2, \ldots\}$, all values of which are calculated in the 2D frequency-time plane.

$\Psi(\lambda, t, n)$ is required to be calculated only for triples (λ, t, n) from the set

$$\Delta = \left\{(\lambda, t, n);\ t \in \left[0, \frac{2\pi}{\lambda}\right), \lambda > 0,\ n = 0, \pm 1, \ldots\right\}$$

We can consider $t' = t + 2\pi n/\lambda = t_n$ and define the isomorphism by

$$(\lambda, t') \to (\lambda, t, n) \to (\lambda, t_n).$$

In the subset of frequency-time points $[0, 120] \times [0, 1]$, Figure 2 illustrates this isomorphism in the form of partition of the semiplane by the subsets

$$\Delta_n = \left\{\left(\lambda, t_n = t + \frac{2\pi n}{\lambda}\right);\ t \in \left[0, \frac{2\pi}{\lambda}\right), \lambda > 0\right\}.$$

Wavelet transform and χ-resolution

The reconstruction formula of the function $f(t)$ from its integral wavelet transform with respect

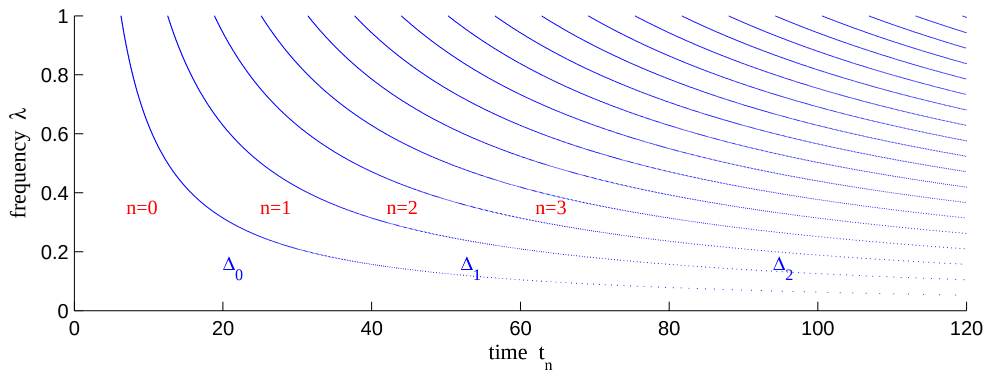


Fig. 2. Isomorphic map of the subsets Δ_n into the semiplane.

to some analyzing function $\phi(t)$:

$$f(t) = \frac{1}{C_\phi} \int\limits_0^\infty \left[\int\limits_{-\infty}^{\infty} T(\lambda, b)\phi\left(\lambda(t-b)\right) db \right] d\lambda.$$

Wavelet transform $T(\lambda, b)$ is a cross-correlation of $f(t)$ with a family of wavelets defined by functions $\phi(\lambda t)$ scaled by parameter $\lambda > 0$,

$$T(\lambda, b) = \sqrt{\lambda} \int\limits_{-\infty}^{\infty} f(t)\phi\left(\lambda(t-b)\right) dt,$$

where $b \in (-\infty, \infty)$. Constant C_ϕ is defined as

$$C_\phi = \int\limits_0^\infty \frac{|\widehat{\phi}(\omega)|^2}{\omega} d\omega < \infty. \qquad (6)$$

Comparison of the above formulas:

1. The wavelet transform is a redundant representation. All values of the wavelet transform $T(\lambda, b)$ are required, in order to calculate the original function $f(t)$ at any point t.

2. The ψ-resolution $\Psi(\lambda, t, n)$ has one additional discrete parameter n. However, for any given triple (λ, t, n), $\Psi(\lambda, t, n)$ is used only to calculate the original function at point t. This value is a cross-correlation of the function with $\psi(t)$ for the specified time-location $t_n = t_n(\lambda, t)$.

3. In the reconstruction of the original function from the wavelet transform, the cross-correlation of the transform with the analyzing function is used, as for the wavelet transform. The reconstruction of the function from the ψ-resolution does not require such a complex cross-correlation operation, but only the summation.

Example 1:

For frequency $\omega = 1.3$rad/s, consider the signal

$$f(t) = \cos(\omega t) + 0.5\sin(2\omega t - 0.25)$$

with the high-frequency sinusoidal signal of frequency 8rad/s with duration of 0.4s has been occurred in the signal $f(t)$ at two points

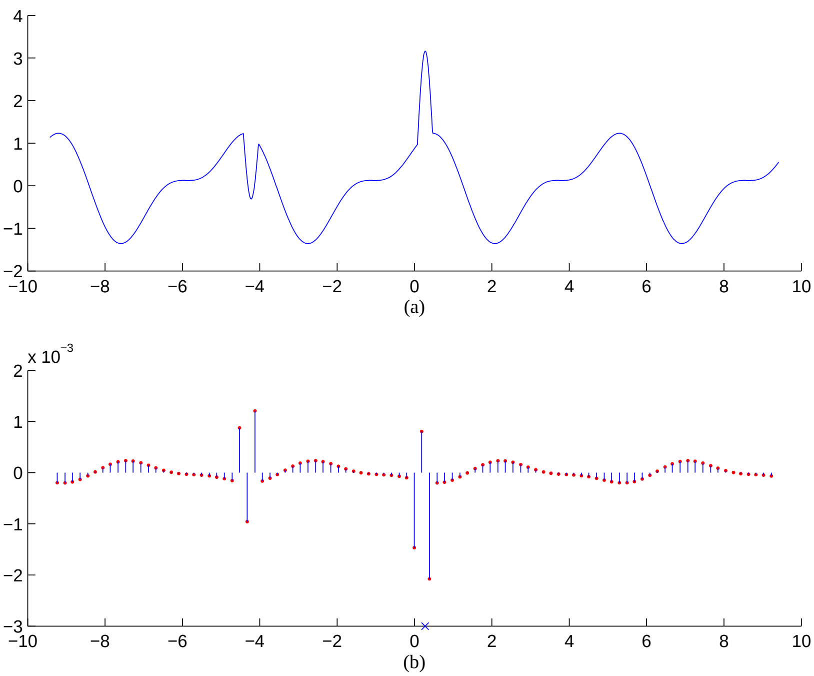


Fig. 3. (a) Original signal $f(t)$ plus a noise signal of duration 0.4s and (b) the cosine wavelet transform $\Psi(\lambda, t_n)$, when $\lambda = 32$rad/s, $t_n \in (-3\pi, 3\pi)$, and $t = 0.275$.

The transform $\Psi(\lambda, t_n)$, of the original signal change as a sinusoidal wave and the appearance of the noise signal causes the essential change in this wave at the locations of the noise.

Another example is shown in Fig. 4. In both the

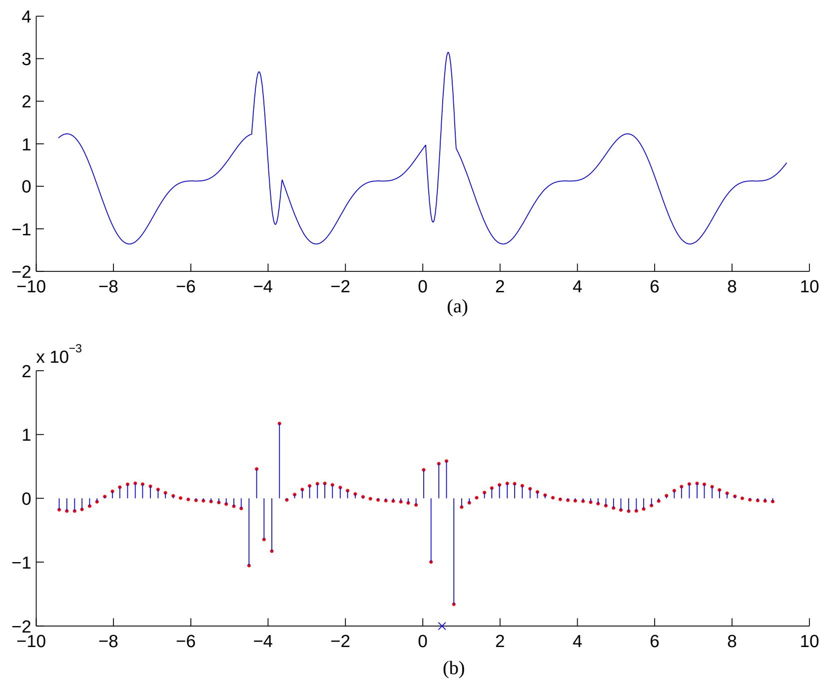


Fig. 4. (a) Original signal plus two noisy signals of duration 0.8s and (b) the cosine wavelet transform $\Psi(\lambda, t_n)$ when $\lambda = 8$rad/s and $t_n \in (-3\pi, 3\pi)$.

cases the transform detects the noisy signal at its two locations. Each pike of the noise signals can be determined by the transform $\Psi(\lambda, t_n)$.

It is interesting to note, that by changing the central point t of the intervals $I_n(\lambda, t)$, we can observe that the main change in the transform $\Psi(\lambda, t, n)$ occurs at points of discontinuity of the signal, i.e. at locations of peaks.

Fig. 5 shows the transform calculated for $t = 2, 0,$ and -4 in (a), (b), and (c), respectively. The major part of the transform is robust to the noise, when the time parameter changes.

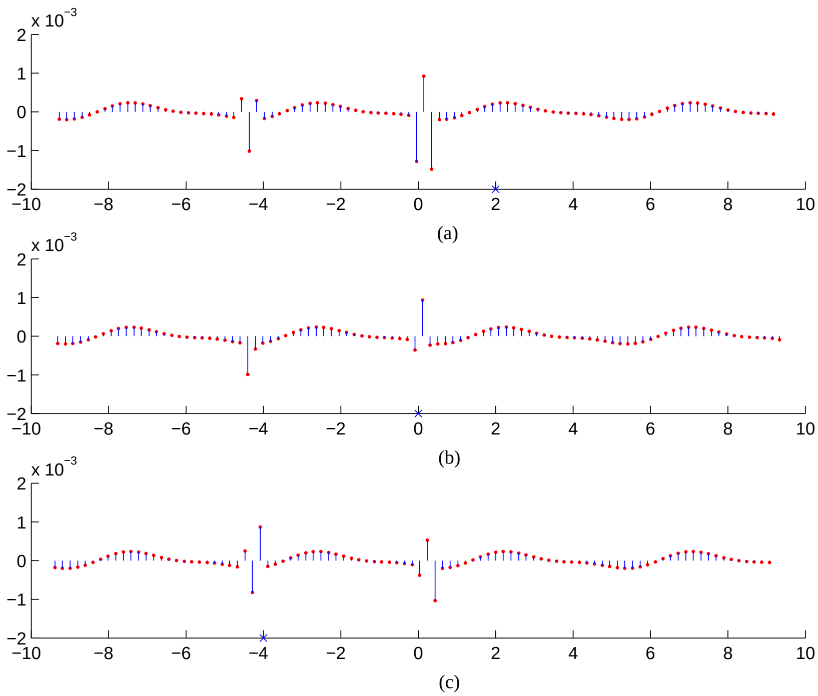


Fig. 5. The cosine wavelet transform $\Psi(\lambda, t_n)$ calculated for (a) $t = 2$, (b) $t = 0$, and (c) $t = -4$.

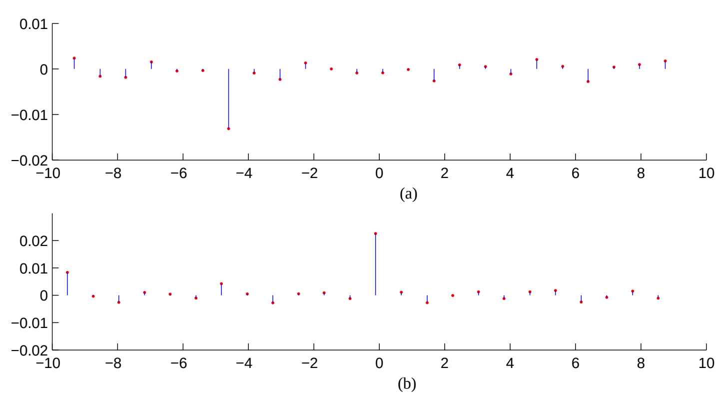
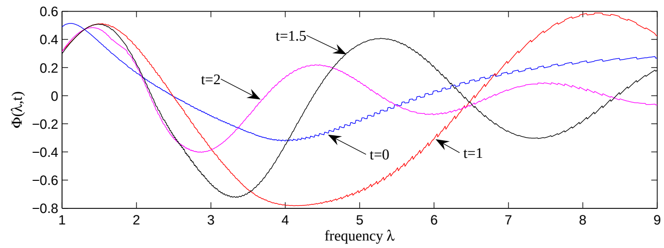


Fig. 6. The cosine wavelet transform $\Psi(8, t_n)$ calculated for (a) $t = -4.225$ and (b) $t = 0.275$.

The transform $\Psi(\lambda, t)$ calculated for three different points $t = 0, 1, 1.5$ and 2, when the frequency λ varies in the interval $[1, 9]$:



Below is the mesh of $\Psi(\lambda, t_n)$ calculated for points $t_n \in (-3\pi, 3\pi)$ and frequencies λ of the interval $(0, 32)$. The maximum at the point of the first peak increases when the frequency λ tends to the frequency of the noise signal.

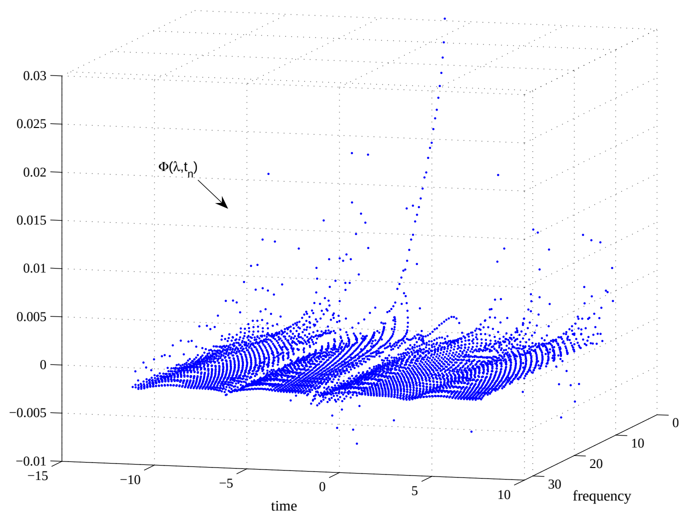


Fig. 7. The mesh of the transform $\Psi(\lambda, t_n)$.

Cosine and sine correlation-type transforms are based on the following is $\psi(\lambda x)$-and-$f(x)$ correlation

$$\Psi(\lambda, t) = \int_{-\infty}^{\infty} f(x)\psi_{\lambda,t}(x)dx \qquad (7)$$

where $(\lambda, t) \in [0, \infty) \times (-\infty, \infty)$.

The analyzing wavelet $\psi(x)$ has zero mean and vanishing moments of odd orders, i.e.

$$\int_{-\infty}^{\infty} x^n \psi(x)dx = \int_{-\pi}^{\pi} x^n \cos(x)dx = 0$$

where $n = 0$ and $2k+1$, $k \geq 0$. In the frequency domain, the function $\psi(t)$ is described by the sum of two shifted sinc functions

$$\hat{\psi}(\lambda) = \pi \left[\mathrm{sinc}(\pi(\lambda - 1)) + \mathrm{sinc}(\pi(\lambda + 1))\right]$$

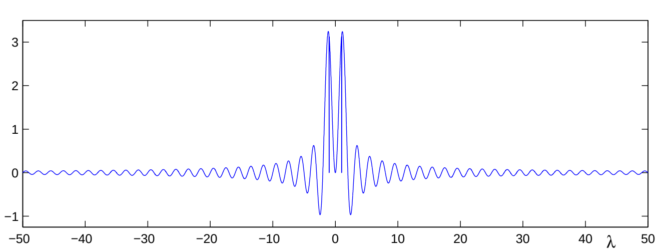


Fig. 8. The Fourier transform $\hat{\psi}(\lambda)$ of the function $\psi(x)$.

It is also not difficult to see, that, the admissibility condition (6) holds for the wavelet since, $C_{\hat{\psi}} < 2$. Figure 9 shows surface-plot of the cosine-wavelet transform of the waveform

$$f(t) = \cos(1.5t) + 0.5\sin(12t - 0.25),$$

versus frequency λ and location t. Frequency-time points (λ, t) are from the set $[0, 4]\times[-3\pi, 3\pi]$.

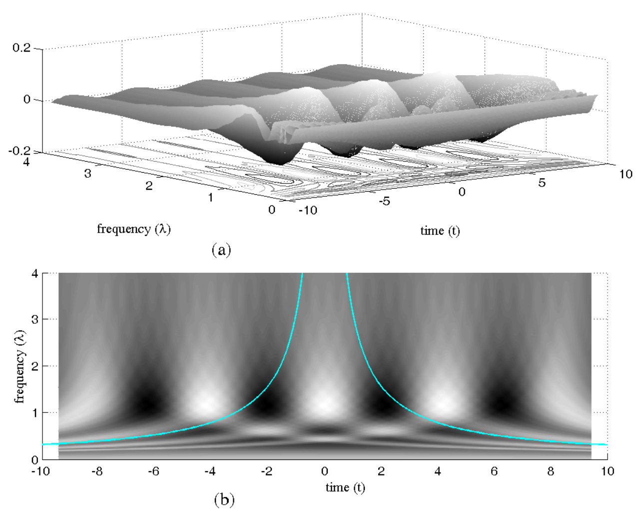


Fig. 9. Cosine wavelet transform plot of $f(t)$ in (a) 3-D view and (b) 2-D view with boundaries of set Δ_0.

We now consider the partition of the time line $(-\infty, \infty)$ by intervals I_n that begin at zero. In other words, for a given frequency $\lambda > 0$, let σ'

be the following partition

$$\sigma' = \sigma'(\lambda) = (I_n(\lambda, 0);\ n = 0, \pm 1, \pm 2, \ldots)$$

with centers of the intervals at the following set of time-points

$$c_n = c_n(\lambda) = n\frac{2\pi}{\lambda}, \quad n = 0, \pm 1, \pm 2, \ldots \quad (8)$$

If $\lambda = 0$, then we consider $I_0(\lambda, 0) = (-\infty, \infty)$.

The Fourier correlation function can be written as

$$\begin{aligned} F(\lambda, t) \ = \ & \cos(\lambda t) \int_{-\infty}^{\infty} f(x) \cos(\lambda x) dx \\ & + \sin(\lambda t) \int_{-\infty}^{\infty} f(x) \sin(\lambda x) dx. \end{aligned}$$

Together with $\psi(x)$, we define by $\varphi(x)$ the period of the sine function

$$\varphi(x) = \begin{cases} \sin(x),\ x \in [-\pi, \pi) \\ \qquad 0,\ \text{otherwise} \end{cases}$$

and consider the family $\mathcal{B} = \{\varphi_{\lambda;t}(x) = \varphi(\lambda(x-t)\}$ of time-scale and shift transformations of this function

$$\varphi_{\lambda;t}(x) = \varphi(\lambda[x - t]),$$

where $\lambda > 0$, and $t \in (-\infty, \infty)$. Then, the correlation function $F(\lambda, t)$ can be written as follows

$$F(\lambda, t) = \cos(\lambda t) \sum_{n=-\infty}^{\infty} \Psi(\lambda, c_n) + \sin(\lambda t) \sum_{n=-\infty}^{\infty} \Phi(\lambda, c_n).$$

$\Phi(\lambda, t)$ is the $\varphi(\lambda x)$-and-$f(x)$ correlation, or sine-wavelet transform of $f(x)$, which is defined by

$$\Phi(\lambda, t) = \int_{-\infty}^{\infty} f(x)\varphi_{\lambda,t}(x)dx.$$

For $\lambda = 0$, we consider that $\Phi(\lambda, \cdot) = 0$.

It is not difficult to see, that the Fourier transform of the function $f(x)$ can be defined as the sum of the pair of the cosine- and sine-wavelets calculated at time points c_n,

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} \left[\Psi(\lambda, c_n) - j\Phi(\lambda, c_n)\right].$$

The Fourier correlation function is actually the product of two vectors

$$F(\lambda, t) = (\cos(\lambda t), \sin(\lambda t)) \begin{pmatrix} \mathsf{Re}\, \hat{f}(\lambda) \\ \mathsf{Im}\, \hat{f}(\lambda) \end{pmatrix}. \quad (9)$$

The first vector is referred to as the coordinates of the point that rotates on the unit circle with frequency λ, and the second vector is the coordinates of the Fourier transform at this frequency. We can see that $F(\lambda, t) = \pm \mathsf{Re}\, \hat{f}(\lambda)$, when $t = 0$ and π/λ, and $F(\lambda, t) = \pm \mathsf{Im}\, \hat{f}(\lambda)$, when $t = \pm\pi/(2\lambda)$. In general, $|F(\lambda, t)| \leq |\mathsf{Re}\, \hat{f}(\lambda)| + |\mathsf{Im}\, \hat{f}(\lambda)|$.

The Fourier correlation function and transform:

$$F(\lambda, t) = |\hat{f}(\lambda)| \cos(\lambda t - \vartheta(\lambda))$$

$\vartheta(\lambda)$ is the phase of the FT at frequency λ.

Thus, for a fixed frequency λ, the Fourier correlation function represents a cosine wave of that frequency and the amplitude equal to the amplitude of the Fourier transform.

Conclusion A detail analysis of the Fourier transform shows that it is a transform providing time-frequency analysis of the signal. The integral Fourier formula leads to the definition of the signal as a superposition of the described correlation-type transforms with finite analyzing cosine and sine waves of various frequencies. The inverse Fourier transform based on such correlation transforms is very simple, when comparing with the formula of reconstruction of the signal by its wavelet transform.

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