Gradients, Means, and Image Reconstruction

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OUTLINE

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Abstract

In this paper, the inverse gradient operators are considered that allow for calculating the original image from its gradients, such as gradients in the horizontal and vertical directions.

Different gradient operators are considered and the image reconstruction from the gradients is presented.

The presented method of image reconstruction is also applied for mean operators.

Examples with images processed by different gradients are given

Image Gradients

Different image gradient operators can be defined to calculate changes in the intensity of the image in different directions for grayscale and color images.

The list of such gradients includes the Roberts, Sobel, Prewitt, and Laplacian gradients.

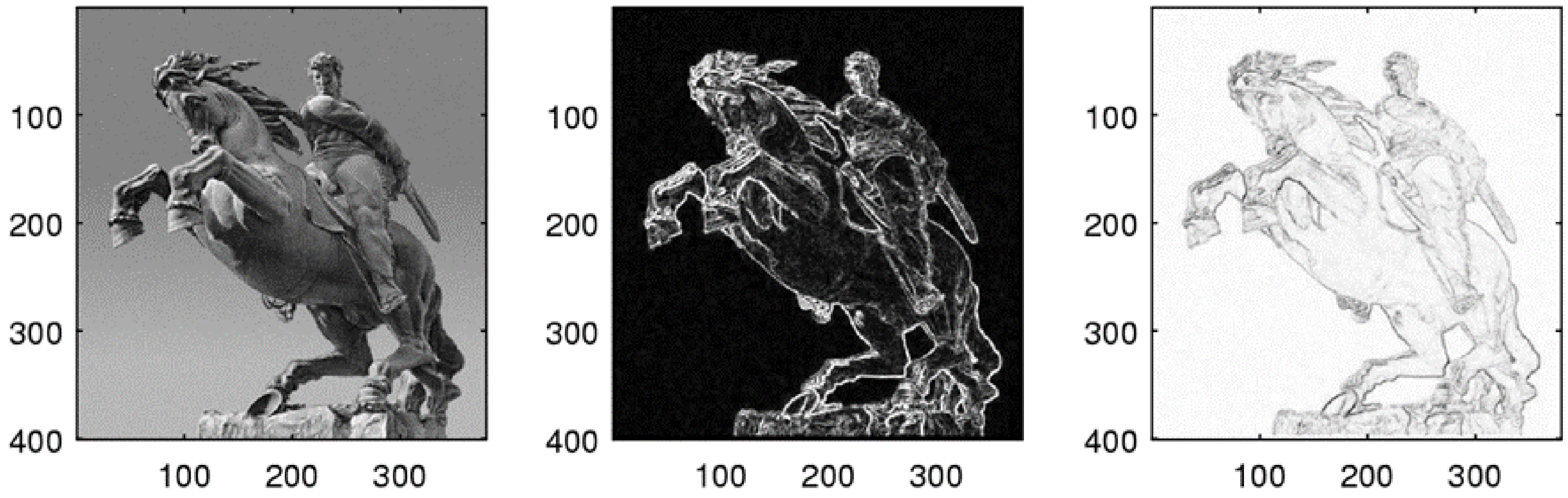
Image gradients allow to see clearly many or all edges in images.

A gradient image contains information of location of important points in the image, such as counters and other points where a significant change of intensity occurs in certain directions.

Image gradients are important and used in different applications of imaging, for extracting the useful information from images.

Image Gradients

We discuss a few gradient operators with examples and describe the process of full image reconstruction from the gradients. As an example, Fig. 1 shows the 379×400 grayscale image in part (a) and its gradient image in part (b), and in negative format in part (c).



(a) (b) (c)

Fig. 1 (a) The image, (b) the gradient image (without thresholding), and (c) its negative image.

SIMPLE 2-D GRADIENTS IN MATRIX FORM

We consider a grayscale image $\boldsymbol{f} = \{f_{n,m}\}$ of size $N \times M$ pixels and the gradient operations, which in the horizontal and vertical directions are defined as the following differencing operators, respectively:

$$G_x\colon f_{n,m} \rightarrow G_x(\boldsymbol{f})_{n,m} = f_{n,m} - f_{n,m+1} \text{ or } f_{n,m} - f_{n,m-1}, \qquad (1)$$

and

$$G_y\colon f_{n,m} \rightarrow G_y(\boldsymbol{f})_{n,m} = f_{n,m} - f_{n+1,m} \text{ or } f_{n,m} - f_{n-1,m}\,. \qquad (2)$$

The images $G_x(\boldsymbol{f})$ and $G_y(\boldsymbol{f})$ are called *the horizontal (*or *row) gradient* and *the vertical (*or *column) gradient* of the image, respectively.

1. Gradient in Horizontal Direction

We consider the following convolution mask and its matrix:

$$M_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \underline{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G = G_x = {M_x}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{1} & 0 \\ 0 & -1 & 0 \end{pmatrix}. \qquad (3)$$

The center of this matrix with coefficient 1 is underlined; it is placed on the image with the center at the pixel (n, m) and is used for the transform

$$G(\boldsymbol{f})_{n,m} = f_{n,m} - f_{n,m+1}$$

as

$$(f_{n,m-1} \quad f_{n,m} \quad f_{n,m+1}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{1} & 0 \\ 0 & -1 & 0 \end{pmatrix} = (0 \quad f_{n,m} - f_{n,m+1} \quad 0).$$

It is clear that the entire image is processed by this gradient as the transform

$$G: \boldsymbol{f} \rightarrow \boldsymbol{f}\boldsymbol{G} = \boldsymbol{f}(1 - \boldsymbol{T}),$$

where $\boldsymbol{T}$ is the matrix of shifting with the coefficients $T_{n,m} = \delta_{n,m+1}$.

The matrix $\boldsymbol{G} = \boldsymbol{G}_x$ is the skew diagonal matrix, as for the $N = M = 6$ case

$$\boldsymbol{G} = 1 - \boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The determinant of the matrix $\boldsymbol{G}$ is 1 and the inverse is the lower triangular matrix

$$\boldsymbol{G}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

As an example, Fig. 2 shows the image $\boldsymbol{f}$ in part (a) and its gradient image $\boldsymbol{f}_x$ in part (b). The gradient image is shown in absolute scale and with the amplification factor of 4. The reconstruction of the image by its gradient image is shown in part (c); the error of reconstruction is zero.

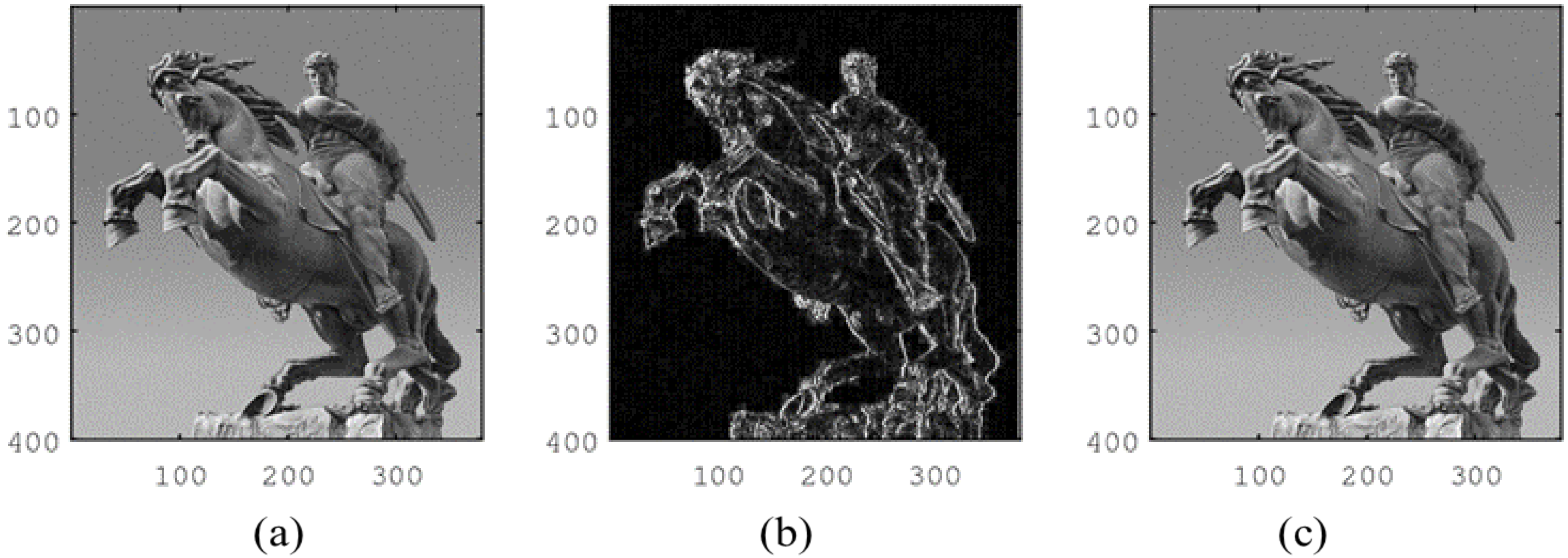


Fig. 2 (a) The original image, (b) the gradient image (without thresholding), and (c) the reconstructed image.

2 Gradient in Vertical Direction

We consider the following convolution mask and its matrix:

$$M_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \underline{1} & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad G = G_y = {M_y}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (4)$$

This matrix is used for the transform $G(f)_{n,m} = f_{n,m} - f_{n+1,m}$ and can be written as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{1} & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-1,m} \\ f_{n,m} \\ f_{n+1,m} \end{pmatrix} = \begin{pmatrix} 0 \\ f_{n,m} - f_{n+1,m} \\ 0 \end{pmatrix}.$$

The entire image is processed by this gradient as the transform $G: f \to \boldsymbol{Gf} = (1 - \boldsymbol{T})\boldsymbol{f}$, where $\boldsymbol{T}$ is the matrix of shifting with the coefficients $T_{n,m} = \delta_{n+1,m}$.

$$G(\boldsymbol{f})_{n,m} = f_{n,m} - (\boldsymbol{Tf})_{n,m} = f_{n,m} - \sum_{k=0}^{N-1} T_{n,k}\, f_{k,m} = f_{n,m} - f_{n+1,m}.$$

The matrix $\boldsymbol{G} = \boldsymbol{G}_y$ is the skew diagonal matrix and, for the $N = M = 6$ case,

$$\boldsymbol{G}_y = {\boldsymbol{G}_x}^T = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In general, the determinant of the matrix $\boldsymbol{G}_y$ is 1 and the matrix has the inverse. In the above case, the inverse matrix is the upper triangular matrix

$${\boldsymbol{G}_y}^{-1} = \left({\boldsymbol{G}_x}^{-1}\right)^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the image $\boldsymbol{f}$ in Figure 2(a), the gradient image $\boldsymbol{G}_y(\boldsymbol{f})$ is shown in Fig. 3 in part (a). The gradient image is shown in absolute scale and with amplification factor of 4 (and without thresholding). The reconstruction of the image by its gradient image is shown in part (b). The error of reconstruction is zero. The angle information of gradient images, atan2($\boldsymbol{G}_y(\boldsymbol{f})$,$\boldsymbol{G}_x(\boldsymbol{f})$), is shown in part (c).

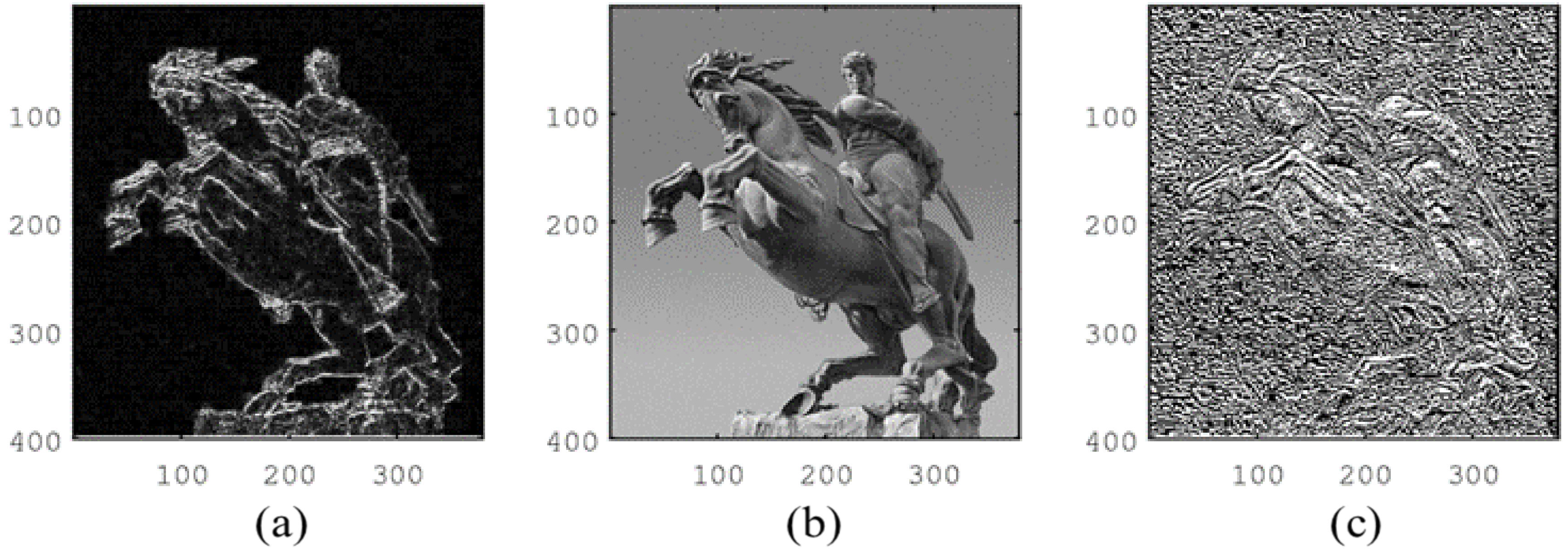


Fig. 3 (a) The gradient image, (b) the reconstructed image, and (c) the angles of the gradient images.

GRADIENT OPERATORS WITH 2×2 MASKS IN MATRIX FORM

In this section, we consider the horizontal and vertical gradients that are defined by the neighbors in the masks 2×2

$$M_x = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -\underline{1} & 1 \end{bmatrix} \quad \text{and} \quad M_y = -{M_x}^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -\underline{1} & -1 \end{bmatrix}. \qquad (5)$$

These masks correspond to the gradient operators that are defined respectively as follows:

$$(G_x f)_{n,m} = \left(f_{n-1,m+1} - f_{n-1,m} + f_{n,m+1} - f_{n,m}\right)/2$$

$$\left(G_y f\right)_{n,m} = \left(f_{n-1,m} - f_{n,m} + f_{n-1,m+1} - f_{n,m+1}\right)/2.$$

By introducing two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with coefficients

$$a_{n,m} = -\delta_{n,m} + \delta_{n,m+1} \quad \text{and} \quad b_{n,m} = \delta_{n,m} + \delta_{n,m+1}, \qquad (6)$$

we can write the following calculations:

$$(\boldsymbol{AfB})_{n,m} = 2\left(G_y f\right)_{n,m}.$$

In a similar way, one can show that

$$(\boldsymbol{BfA})_{n,m} = \sum_{p=0}^{N-1}\sum_{q=0}^{M-1} b_{n,p} f_{p,q} a_{q,m} = 2(G_x f)_{n,m}. \qquad (7)$$

Thus, the horizontal and vertical gradients can be accomplished by using the matrix multiplication as

$$G_x\colon \boldsymbol{f} \to \boldsymbol{f_x} = \frac{1}{2}\boldsymbol{BfA} \quad \text{and} \quad G_y\colon \boldsymbol{f} \to \boldsymbol{f_y} = \frac{1}{2}\boldsymbol{AfB}. \qquad (8)$$

The reversibility of these transformations depends on the inverse matrices of $\boldsymbol{A}$ and $\boldsymbol{B}$, which exits because the determinants of the matrices equal ± 1.

Therefore, the image can be reconstructed from its gradients as

$$\boldsymbol{f} = 2\boldsymbol{B}^{-1}(\boldsymbol{f_x}\boldsymbol{A}^{-1}) = 2\boldsymbol{A}^{-1}(\boldsymbol{f_y}\boldsymbol{B}^{-1}). \qquad (9)$$

As an example, Figure 4 shows the "cameraman" image of size 256×256 in part (a) and its gradient image $G_y(\boldsymbol{f})$ in part (b). The image reconstructed from the gradient image is shown in part (c).

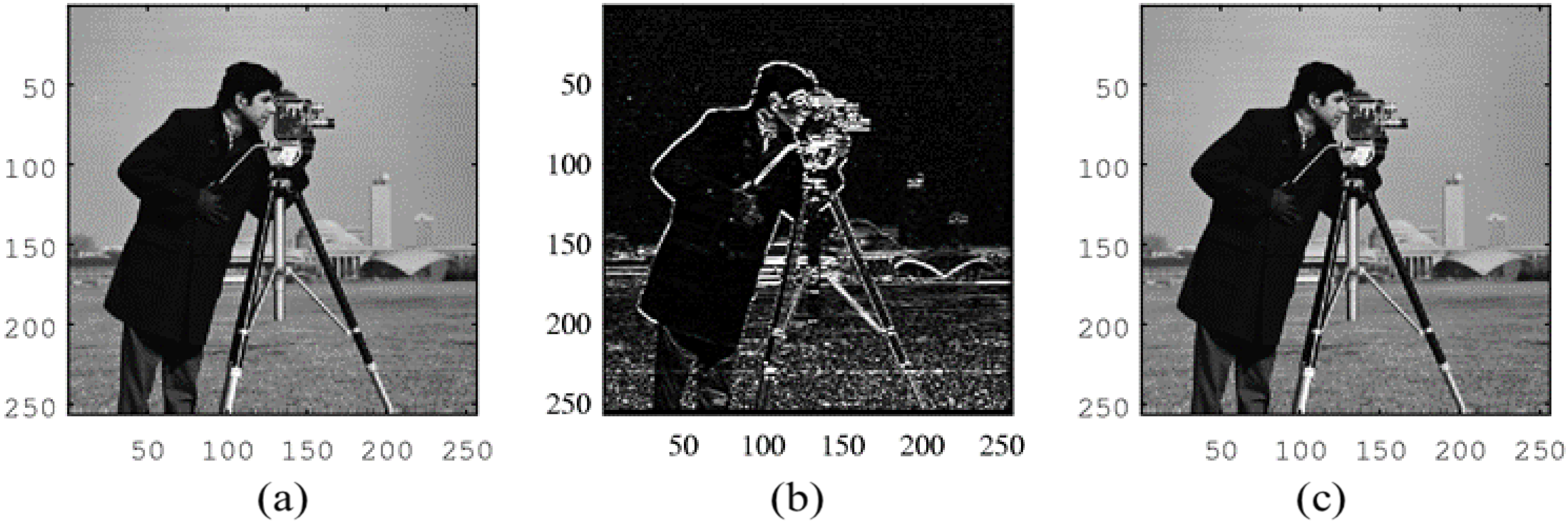


Fig. 4 (a) The original image, (b) the vertical gradient image (without thresholding), and (c) the reconstructed image.

We consider the Sobel operators with the matrices

$$G_x = \frac{1}{4}\begin{pmatrix} 1 & 0 & -1 \\ 2 & \underline{0} & -2 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad G_y = -G_x' = \frac{1}{4}\begin{pmatrix} -1 & -2 & -1 \\ 0 & \underline{0} & 0 \\ 1 & 2 & 1 \end{pmatrix}. \qquad (10)$$

Figure 5 shows the “peppers” image of size 512×512 in part (a) and the Sobel gradient image $G_x(\boldsymbol{f})$ in part (b), and the reconstructed image in part (c).

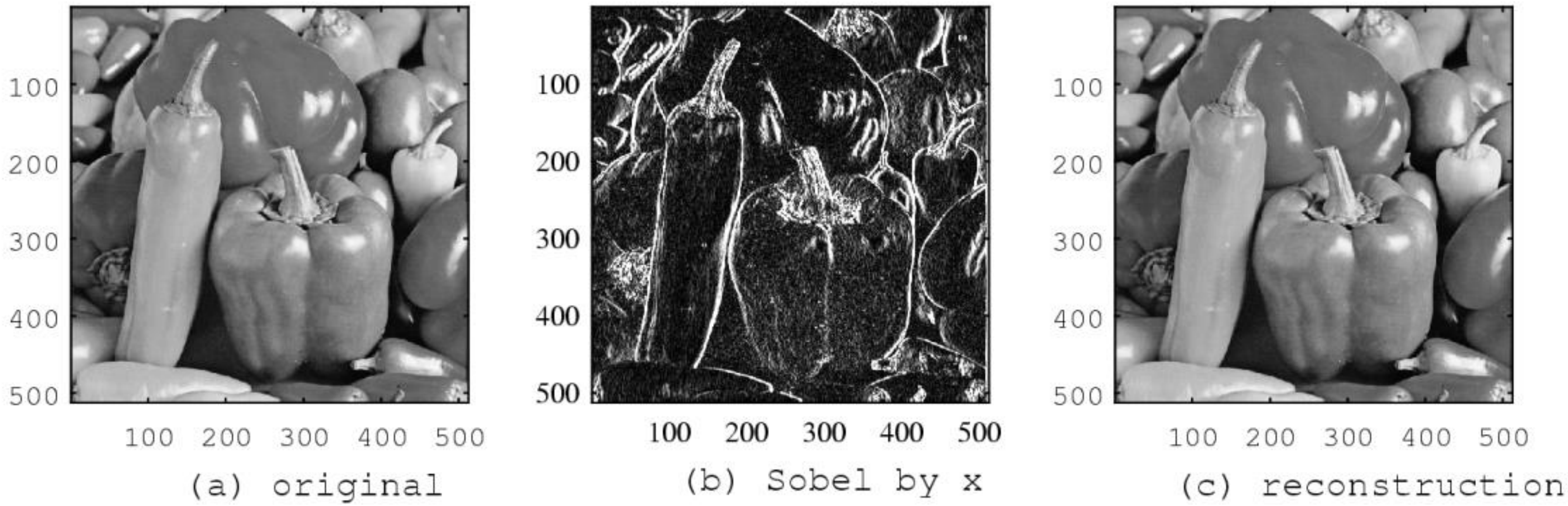


Fig. 5 (a) The original image, (b) the horizontal Sobel gradient, and (c) the reconstructed image.

Figure 6 shows the "peppers" image in part (a) and the Sobel gradient image $G_y(\boldsymbol{f})$ in part (b), and the image reconstructed from this gradient image in part (c).

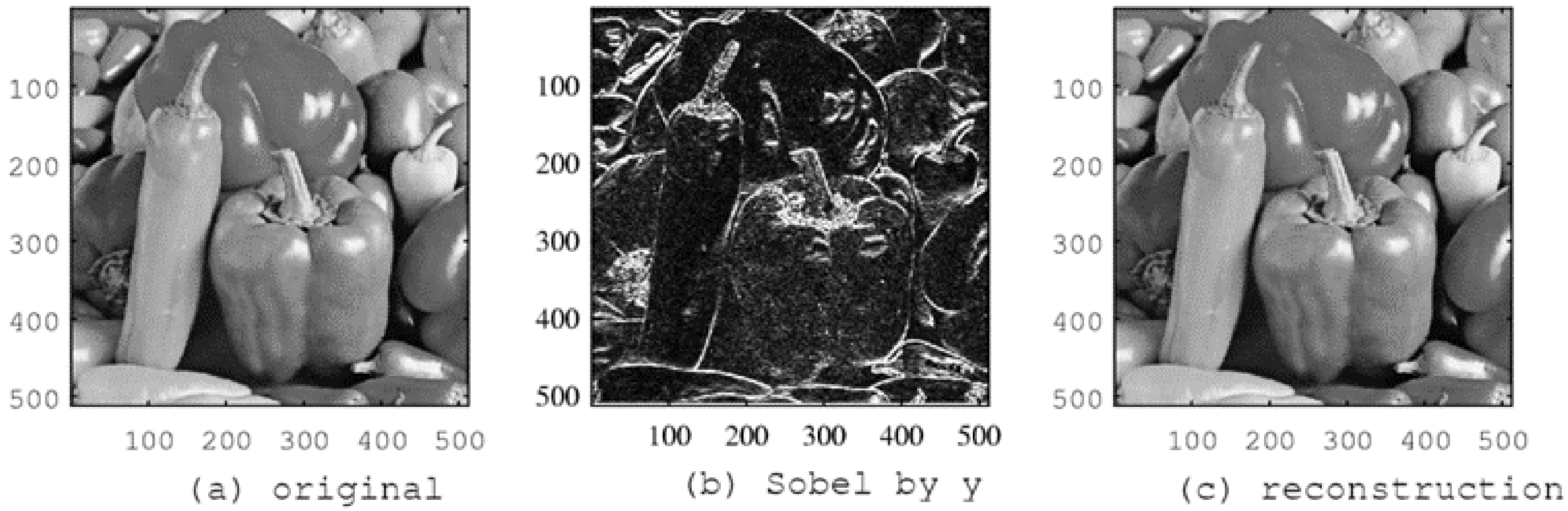


Fig. 6 (a) The original image, (b) the vertical Sobel gradient, and (c) the reconstructed image.

MEAN OPERATOR IN MATRIX FORM

We consider the operator of smoothing or averaging with the mask 3×4

$$M_{3,4} = \frac{1}{12}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \underline{1} & 1 & 1 & 1 \end{pmatrix}.$$

Thus, the image is processed as

$$\left(M_{3,4}\boldsymbol{f}\right)_{n,m} = \frac{1}{12}\begin{pmatrix} f_{n-2,m} + f_{n-2,m+1} + f_{n-2,m+2} + f_{n-2,m+3} \\ +f_{n-1,m} + f_{n-1,m+1} + f_{n-1,m+2} + f_{n-1,m+3} \\ +f_{n,m} + f_{n,m+1} + f_{n,m+2} + f_{n,m+3} \end{pmatrix}. \qquad (11)$$

We introduce two matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ with coefficients

$$c_{n,m} = \delta_{n,m} + \delta_{n,m+1} + \delta_{n,m+2}, \quad d_{n,m} = \delta_{n,m} + \delta_{n,m+1} + \delta_{n,m+2} + \delta_{n,m+3}.$$

The image averaging can be written in matrix form as

$$M_{3,4}\boldsymbol{f} = \frac{1}{12}\boldsymbol{C}(\boldsymbol{f}\boldsymbol{D}). \qquad (12)$$

The inverse matrices of $\boldsymbol{C}$ and $\boldsymbol{D}$ exit. The averaging operation is reversible.

Example 3: In the $N = 6$ and $M = 7$ case, the matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ equal

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The matrix $\boldsymbol{C}$ is of size 6×6 and matrix $\boldsymbol{D}$ is of size 7×7; $\det(\boldsymbol{C}) = \det(\boldsymbol{D}) = 1$. The inverse matrices are

$$\boldsymbol{C}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}, \quad \boldsymbol{D}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Therefore, the image can be reconstructed from its gradients as

$$\boldsymbol{f} = 12\boldsymbol{C}^{-1}\left[\left(M_{3,4}\boldsymbol{f}\right)\boldsymbol{D}^{-1}\right] = 12\boldsymbol{C}^{-1}\left[\left(\frac{1}{12}\boldsymbol{C}(\boldsymbol{f}\boldsymbol{D})\right)\boldsymbol{D}^{-1}\right]. \quad (13)$$

As an example, Fig. 7 shows the "clock" image of size 256×256 in part (a) and the smoothed image $M_{3,4}\boldsymbol{f}$ in part (b), and then the image reconstructed in part (c).

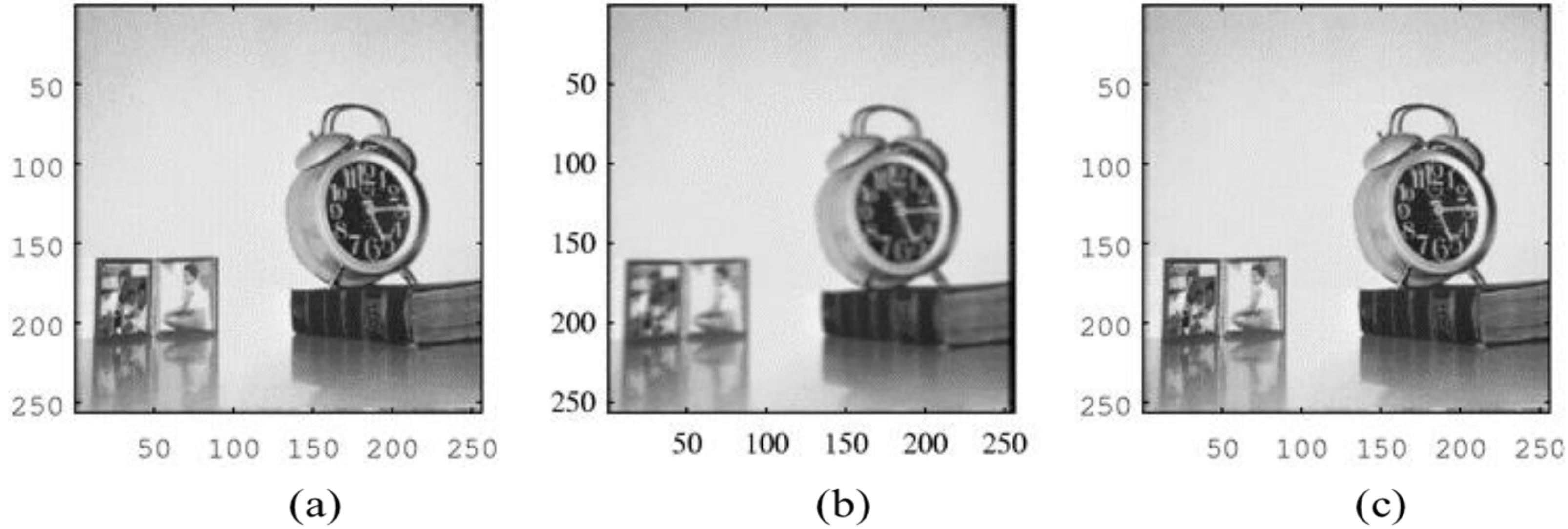


Fig. 7 (a) The image, (b) smooth image with the 3×4 mean operator, (c) reconstructed image.

SUMMARY

- We describe briefly the method of complete image reconstruction from the gradient and mean operations, by using the matrix representation of these operations.
- As examples, simple gradients are considered, and the presented method can be used for many well-known gradients defined by the Sobel, Prewitt, and Laplacian operators, as well different smoothing, or averaging operations over images.
- Similar calculations with complete image reconstruction from the smooth image can be accomplished when using other masks for mean operators with windows of size 3×3, 5×5, 7×7, and larger.
- Methods of image reconstruction from gradients can be used in different applications, including the image encoding, compression, and cryptography.

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