Optimal Restoration of Multiple Signals in Quaternion Algebra

Artyom M. Grigoryan[a] and Sos S. Agaian[b]

[a]Department of Electrical and Computer Engineering

The University of Texas at San Antonio, San Antonio, Texas, USA,

and

[b]Computer Science Department, College of Staten Island and the Graduate Center, Staten Island, NY, USA

amgrigoryan@utsa.edu, sos.agaian@csi.cuny.edu

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Abstract

- This paper offers a new multiple signal restoration tool to solve the inverse problem, when signals are convoluted with a multiple impulse response and then degraded by an additive noise signal with multiple components.

  Inverse problems arise practically in all areas of science and engineering and refers to as methods of estimating data/parameters, in our case of multiple signals that cannot directly be observed.

- The presented tool is based on the mapping multiple signals into the quaternion domain, and then solving the inverse problem.

- Due to the non-commutativity of quaternion arithmetic, it is difficult to find the optimal filter in the frequency domain for degraded quaternion signals.

Presented Work

- As an alternative, we introduce an optimal filter by using special 4×4 matrices on the discrete Fourier transforms of signal components, at each frequency-point.

  The optimality of the solution is with respect to the mean-square-root error, as in the classical theory of the signal restoration by the Wiener filter.

- The Illustrative example of optimal filtration of multiple degraded signals in the quaternion domain is given. The computer simulations validate the effectiveness of the proposed method.

1. PROBLEM OF MULTIPLE SIGNAL RESTORATION

In the space of quaternion signals, in the model described the signal $q(t)$ convoluted with the function $h(t)$ plus a noise $n(t)$

$$i(t) = q(t) * h(t) + n(t), \qquad (1)$$

the signal $q(t)$ is restoring from the degraded signal $i(t)$.

The classic case: the inverse problem is solving by the optimal filter

$$Y(\omega) = \frac{H(\omega)}{|H(\omega)| + \phi_{N/Q}(\omega)}. \qquad (2)$$

Here, $H(\omega)$ is the Fourier transform of $h(t)$, and $\phi_{N/Q}(\omega)$ is the noise-signal ratio, and $\phi_N(\omega) =< |N(\omega)|^2 >$ and $\phi_Q(\omega) =< |Q(\omega)|^2 >$ are spatial spectral densities of the signal $q(t)$ and noise $n(t)$, respectively.

Inroduction to Quanterions

The quaternion number is composed by one real part and three-component imaginary part,

$$q = a + (bi + cj + dk) = a + bi + cj + dk,$$

where $a, b, c,$ and d are real numbers. Together with unit 1, three imaginary units i, j, and k are used with the multiplication laws, which are following:

$$i^2 = j^2 = k^2 = -1,\ ki = -ik = j,\ ij = -ji = k,\ jk = -kj = i.$$

The quaternion conjugate and modulus of q are defined as

$$\overline{q} = a - (bi + cj + dk) \text{ and } |q| = a^2 + b^2 + c^2 + d^2.$$

The multiplication of quaternions is not a commutative operation, i.e., $q_1 q_2 \neq q_2 q_1$ for many quaternions $q_2 \neq q_1$.

In the definition of the N-point quaternion DFT (QDFT), the exponential kernel is used the exponential kernel is used

$$W_\mu = \exp(-\mu 2\pi/N) = \cos(2\pi/N) - \mu \sin(2\pi/N),$$

where μ is a pure unit quaternion, $\mu = m_1 i + m_2 j + m_3 k$. For a such number, $|\mu| = 1$ and $\mu^2 = -1$.

The N-point right-side QDFT

$$Q_p = \sum_{n=0}^{N-1} q_n W_\mu^{np}, \qquad p = 0{:}\,(N-1).$$

The fast algorithms for the N-points QDFT exist [1].

Because of not commutativity of multiplication in quaternion arithmetic, the main operation of the cyclic convolution is not reduced to the multiplication of the QDFTs, as for the traditional N-point DFT.

1-D CONVOLUTION IN QUATERNIOIN ALGEBRA

In this section, we show a new technique for calculating the quaternion convolution by the Fourier transforms, which was developed by Grigoryan in 2015. The convolution will be transformed into the frequency domain and a new operation of multiplication of transforms by 4×4 matrices will be performed, instead of point-wise multiplication of the DFTs in the traditional method.

The quaternion signal $f_n, n = 0:(N-1)$, of length N

$$f_n = [(f_1)_n, (f_2)_n, (f_3)_n, (f_4)_n] = (f_1)_n + i(f_2)_n + j(f_3)_n + k(f_4)_n$$

can be considered in the following matrix representation.

We introduce the following 4 matrices with the multiplications shown in the table

$$E=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad I=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix},\quad J=\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix},\quad E=\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\tag{4}$$

.

Table 1. Table of multiplications of quaternion matrices E, I, J, K.

	E	I	J	K
E	E	I	J	K
I	I	$-E$	$-K$	J
J	J	K	$-E$	$-I$
K	K	$-J$	I	$-E$

(a) original table

	E	J	I	K
E	E	J	I	K
J	J	$-E$	K	$-I$
I	I	$-K$	$-E$	J
K	K	I	$-J$	$-E$

(b) new E-JIK-table

The quaternion signal at the point n can be written as a matrix, (we use the same notation f_n),

$$f_n=(f_1)_nE+(f_2)_nI+(f_3)_nJ+(f_4)_nK=\begin{bmatrix}(f_1)_n&-(f_2)_n&-(f_3)_n&-(f_4)_n\\(f_2)_n&(f_1)_n&(f_4)_n&-(f_3)_n\\(f_3)_n&-(f_4)_n&(f_1)_n&(f_2)_n\\(f_4)_n&(f_3)_n&-(f_2)_n&(f_1)_n\end{bmatrix},\tag{5}$$

Another quaternion sequence, which we call the impulse response characteristic of a linear system, is denoted by $h_n = ((h_1)_n, (h_2)_n, (h_3)_n, (h_4)_n)$ and can be presented in matrix form as

$$h_n = (h_1)_n E + (h_2)_n I + (h_3)_n J + (h_4)_n K. \quad (6)$$

We define the circular linear convolution as

$$g_n = (f * h)_n = \sum_{m=0}^{N-1} f_{n-m} h_m \triangleq \sum_{m=0}^{N-1} f_{(n-m)\mathrm{mod}N} h_m \,. \quad (7)$$

The following two sums are different: $(f * h)_n \neq (h * f)_n$.

To simplify calculations, we separate all E, I, J, K-components of the signals

$$f_E = \big((f_1)_0, (f_1)_1, (f_1)_2, \dots, (f_1)_{N-1}\big), \qquad f_I = \big((f_2)_0, (f_2)_1, (f_2)_2, \dots, (f_2)_{N-1}\big),$$
$$f_J = \big((f_3)_0, (f_3)_1, (f_3)_2, \dots, (f_3)_{N-1}\big), \qquad f_K = \big((f_4)_0, (f_4)_1, (f_4)_2, \dots, (f_4)_{N-1}\big),$$
$$h_E = \big((h_1)_0, (h_1)_1, (h_1)_2, \dots, (h_1)_{N-1}\big), \qquad h_I = \big((h_2)_0, (h_2)_1, (h_2)_2, \dots, (h_2)_{N-1}\big),$$
$$h_J = \big((h_3)_0, (h_3)_1, (h_3)_2, \dots, (h_3)_{N-1}\big), \qquad h_K = \big((h_4)_0, (h_4)_1, (h_4)_2, \dots, (h_4)_{N-1}\big).$$

The E, I, J-, and K-components of the quaternion convolution g_n are denoted by g_E, g_I, g_J, and g_K, respectively. By using the table of multiplications for the basic four quaternion matrices, we can open the convolution

$$g_n = [(f_1)_n E + (f_2)_n I + (f_3)_n J + (f_4)_n K] * [(h_1)_n E + (h_2)_n I + (h_3)_n J + (h_4)_n K]$$

and consider the following system of Eqs for the E, I, J, K-components of g_n:

$$\left.\begin{aligned} g_E &= f_E * h_E - f_I * h_I - f_J * h_J - f_K * h_K, \qquad & g_I &= f_E * h_I + f_I * h_E - f_J * h_K + f_K * h_J, \\ g_J &= f_E * h_J + f_I * h_K + f_J * h_E - f_K * h_I, \qquad & g_K &= f_E * h_K - f_I * h_J + f_J * h_I + f_K * h_E. \end{aligned}\right\}$$

For each real component of the signal f_n and the impulse response sequence h_n, we will use the corresponding capital letters for the corresponding DFTs. In the frequency domain, the system of Eq. 8 for frequency-point p can be written as

$$\begin{pmatrix} G_E \\ G_I \\ G_J \\ G_K \end{pmatrix} = \begin{pmatrix} H_E & -H_I & -H_J & -H_K \\ H_I & H_E & -H_K & H_J \\ H_J & H_K & H_E & -H_I \\ H_K & -H_J & H_I & H_E \end{pmatrix} \begin{pmatrix} F_E \\ F_I \\ F_J \\ F_K \end{pmatrix}.$$

This is a compact form of the equation $G = \boldsymbol{H}F$ at one frequency-point p, which is omitted from the notation. For this point, the matrix $\boldsymbol{H}$ is the 4×4 matrix

$$\boldsymbol{H} = \begin{pmatrix} H_E & -H_I & -H_J & -H_K \\ H_I & H_E & -H_K & H_J \\ H_J & H_K & H_E & -H_I \\ H_K & -H_J & H_I & H_E \end{pmatrix}.$$

This matrix is orthogonal.

The convoluted quaternion signal

$$g_n = (g_1)_n E + (g_2)_n I + (g_3)_n J + (g_4)_n K$$

can be calculated as

$$g_1 = \boldsymbol{F}^{-1} G_E,\quad g_2 = \boldsymbol{F}^{-1} G_I,\quad g_3 = \boldsymbol{F}^{-1} G_J,\quad g_4 = \boldsymbol{F}^{-1} G_K.$$

Here, $\boldsymbol{F}^{-1}$ is the matrix of the inverse N-point DFT. Together with the unite matrix E, we consider the following three new quaternion matrices:

$$I^* = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\qquad J^* = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\qquad K^* = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

TABLE II. MULTIPLICATION LAWS

	E	I*	J*	K*
E	E	I*	J*	K*
I*	I*	-E	K*	-J*
J*	J*	-K*	-E	I*
K*	K*	J*	-I*	-E

Statement 1. Given a frequency-point p, the inverse matrix $\boldsymbol{H}^{-1}$ at this point is

$$\boldsymbol{H}^{-1} = \frac{1}{\sqrt{\det \boldsymbol{H}}} \boldsymbol{H}^T = \frac{1}{\sqrt{\det \boldsymbol{H}}} \left(EH_E - I^* H_I - J^* H_J - K^* H_K\right), \tag{13}$$

where the square root of the determinant of the matrix is calculated by $\sqrt{\det \boldsymbol{H}} = H_E^2 + H_I^2 + H_J^2 + H_K^2$. The operation $\boldsymbol{H}^T$ denotes the transposition of the complex matrix $\boldsymbol{H}$.

The reconstruction of the frequency-vector F at the frequency-point p is the inverse transform

$$\begin{pmatrix} F_E \\ F_I \\ F_J \\ F_K \end{pmatrix} = \frac{1}{\sqrt{\det \boldsymbol{H}}} \begin{pmatrix} H_E & H_I & H_J & H_K \\ -H_I & H_E & H_K & -H_J \\ -H_J & -H_K & H_E & H_I \\ -H_K & H_J & -H_I & H_E \end{pmatrix} \begin{pmatrix} G_E \\ G_I \\ G_J \\ G_K \end{pmatrix}. \tag{14}$$

The original quaternion signal $f_n = (f_1)_n E + (f_2)_n I + (f_3)_n J + (f_4)_n K$ is reconstructed from the components of the inverse transform by

$$f_1 = \boldsymbol{F}^{-1} F_E, \qquad f_2 = \boldsymbol{F}^{-1} F_I, \quad f_3 = \boldsymbol{F}^{-1} F_J, \quad f_4 = \boldsymbol{F}^{-1} F_K. \tag{15}$$

Algorithm 1. Inverse problem: 1-D Quaternion convolution

1. Input signal $f_n = ((f_1)_n, (f_2)_n, (f_3)_n, (f_4)_n)$, and the impulse response $h_n = ((h_1)_n, (h_2)_n, (h_3)_n, (h_4)_n)$.
2. Calculate four 1-D DFTs F_E, F_I, F_J, and F_K of $(f_1)_n$, $(f_2)_n$, $(f_3)_n$, and $(f_4)_n$, respectively.
3. Calculate 1-D DFTs H_E, H_I, H_J, and H_K of $(h_1)_n$, $(h_2)_n$, $(h_3)_n$, and $(h_4)_n$, respectively.
4. Compose the convoluted quaternion signal g_n from the inverse N-point DFTs by Eq. 11.
   a. For each frequency-point $p \in \{0,1,2,\ldots,(N-1)\}$, calculate the 4×4 matrix $\boldsymbol{H}$ by Eq. 10.
   b. Calculate the data G_E, G_I, G_J, and G_K by Eq. 9.
   c. Calculate four inverse N-point DFTs of G_E, G_I, G_J, and G_K.
5. Compose the quaternion signal f_n from the inverse N-point DFTs by Eq. 15.
   a. Calculate the inverse 4×4 matrix $\boldsymbol{H}^{-1}$ by Eq. 13.
   b. Apply the inverse matrix on the vector (G_E, G_I, G_J, G_K).
   c. Calculate four inverse N-point DFTs of F_E, F_I, F_J, and F_K.

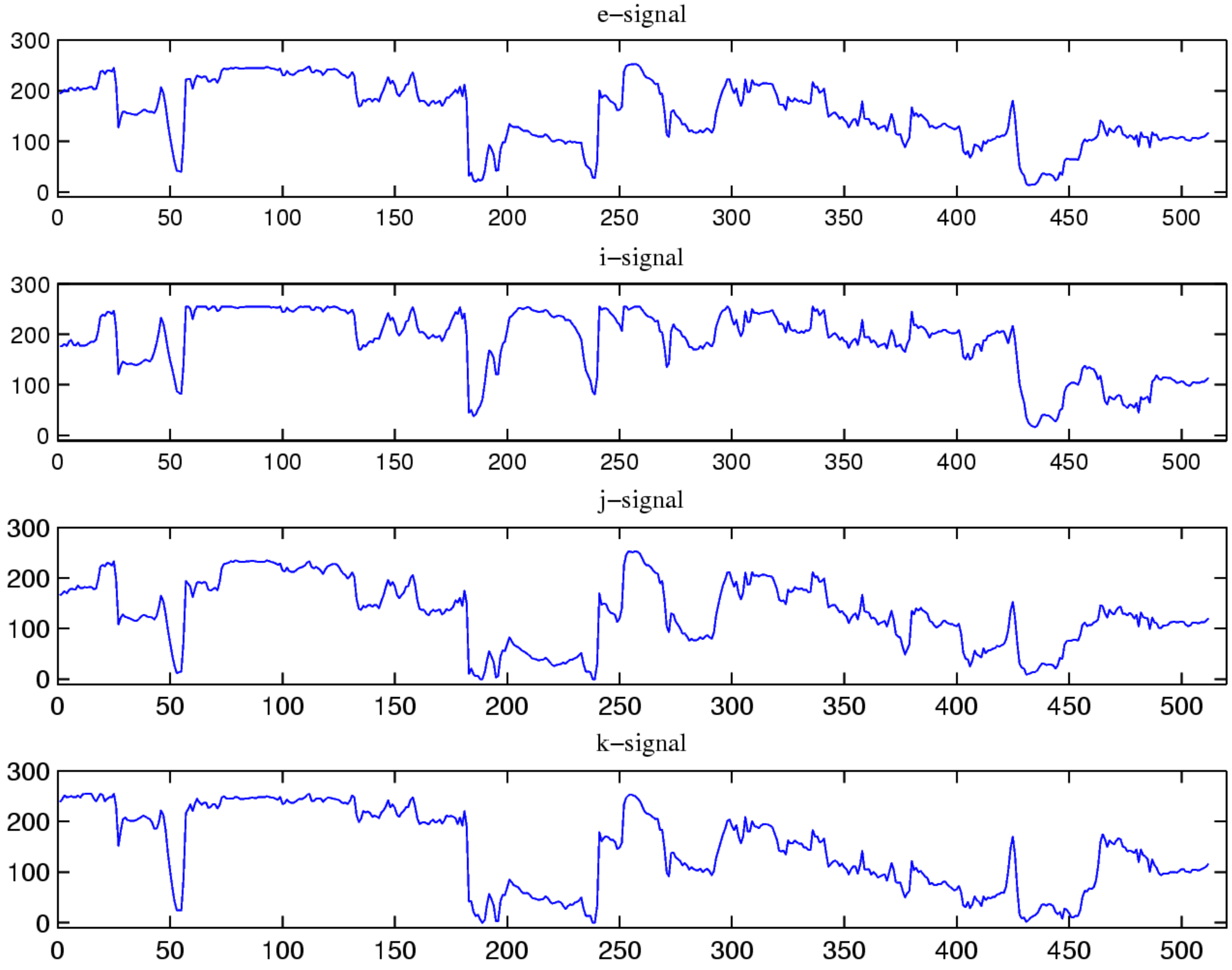

Figure 2: Four components of the original quaternion signal f_n.

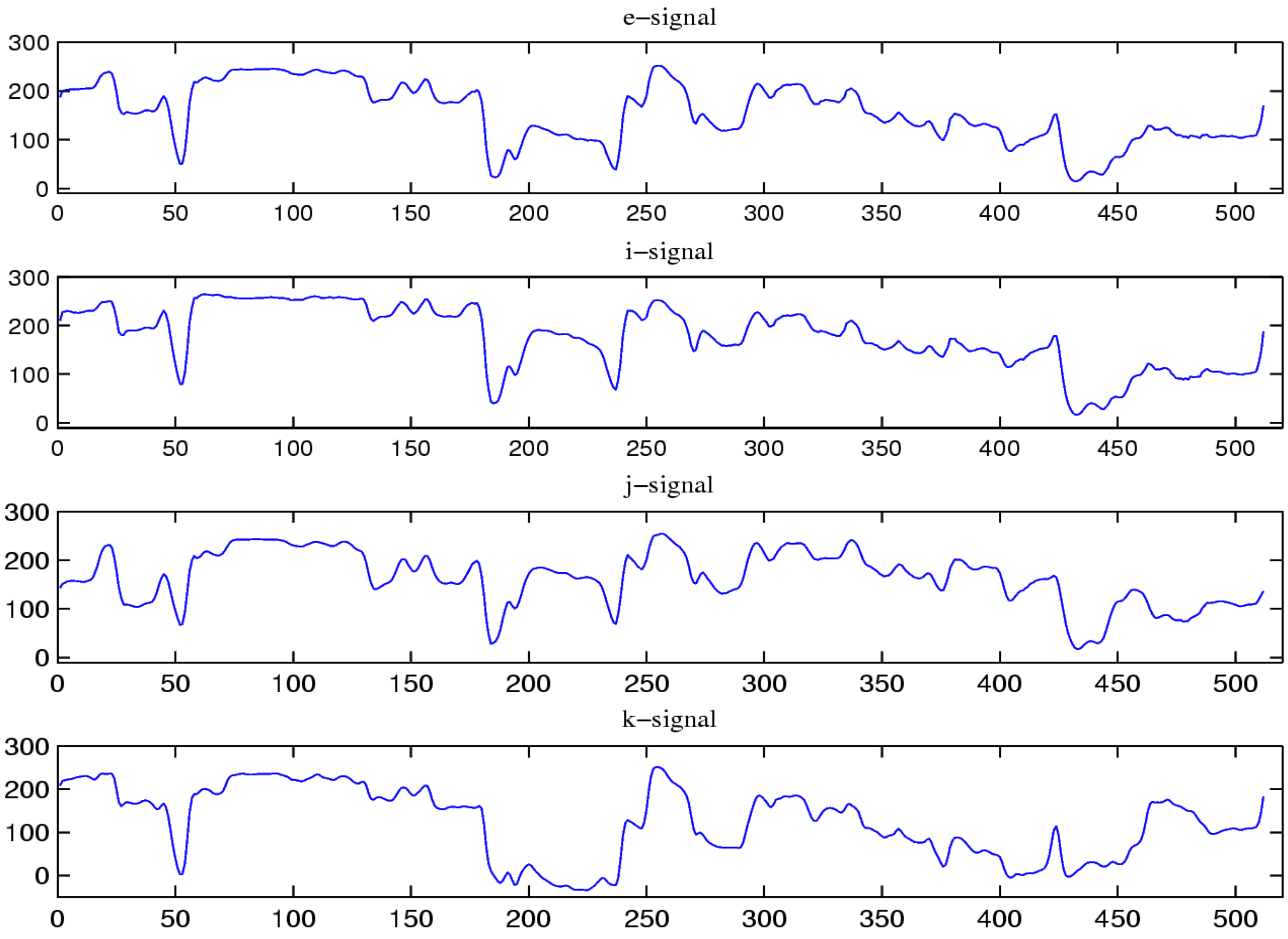

Figure 3: Four components of the convoluted quaternion signal g_n.

We consider the corresponding binary matrix $\boldsymbol{A}$ of signs of elements of $\boldsymbol{H}$,

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 2 \end{pmatrix}. \tag{16}$$

Here, the vector $\boldsymbol{K}$ is the sum of elements of the matrix $\boldsymbol{A}$ by rows. Therefore, the components of the convolution are plotted after the following amplitude transformation:

$$\left((g_1)_n, (g_2)_n, (g_3)_n, (g_4)_n\right) \rightarrow (-(g_1)_n/2, (g_2)_n/2, (g_3)_n/2, (g_4)_n/2)$$

The means-square-root error therefore is calculated as

$$e(f,g) = \frac{1}{N}\sqrt{\sum_{n=0}^{N-1}\left[\left((f_1)_n + \frac{1}{2}(g_1)_n\right)^2 + \sum_{k=2}^{4}\left((f_k)_n - \frac{1}{2}(g_k)_n\right)^2\right]}. \tag{17}$$

Quaternion Convolution and Noise

The frequency characteristics of the input and output signals are considered in forms of quaternion matrices in the (E, I^*, J^*, K^*)-basis, i.e., at each frequency-points p, we consider that $F = EF_E + I^*F_I + J^*F_J + K^*F_K$ and $G = EG_E + I^*G_I + J^*G_J + K^*G_K$.

It is important to note and not difficult to verify that equation (9) can be written as

$$\begin{pmatrix} G_E & -G_I & -G_J & -G_K \\ G_I & G_E & -G_K & G_J \\ G_J & G_K & G_E & -G_I \\ G_K & -G_J & G_I & G_E \end{pmatrix} = \begin{pmatrix} H_E & -H_I & -H_J & -H_K \\ H_I & H_E & -H_K & H_J \\ H_J & H_K & H_E & -H_I \\ H_K & -H_J & H_I & H_E \end{pmatrix} \begin{pmatrix} F_E & -F_I & -F_J & -F_K \\ F_I & F_E & -F_K & F_J \\ F_J & F_K & F_E & -F_I \\ F_K & -F_J & F_I & F_E \end{pmatrix}.$$

Thus, we obtain the equation $G = \boldsymbol{H}F$, which means that at the frequency-point p, $G_p = \boldsymbol{H}_p F_p$ where all components of this equation are 4×4 matrices.

We consider the model with the noise, when $g_n = f_n * h_n + n_n$ and n_n is a noise. The transformation of this equation into the frequency-domain results in the similar equation

$$G_p = \boldsymbol{H}_p F_p + N_p, \qquad p = 0:(N-1). \tag{18}$$

Here $N = EN_E + I^*N_I + J^*N_J + K^*N_K$ and $N = (N)_p$.

Consider a quaternion signal $O = EO_E + I^*O_I + J^*O_J + K^*O_K$ which is a filtered signal $O = YG$, i.e., at the frequency p, we have $O_p = Y_pG_p = Y_p(\boldsymbol{H}_pF_p + N_p)$. The filter, or the frequency characteristics Y is defined as the filter that minimizes the RMS error between O and F. To derive such an optimal filter, we consider the Lagrangian

$$\mathcal{L}(Y_p) = \langle \left| F_p - Y_p(\boldsymbol{H}_pF_p \ + \ N_p) \right|^2 \rangle = \text{min}.$$

Therefore, the optimal filter is

$$Y = \frac{\boldsymbol{H}'\phi_F}{\|\boldsymbol{H}\|^2\phi_F + \phi_N} = \frac{\overline{\boldsymbol{H}}^T\phi_F}{\|\boldsymbol{H}\|^2\phi_F + \phi_N} = \frac{\overline{\boldsymbol{H}}^T}{\|\boldsymbol{H}\|^2 + \phi_{N/F}}. \qquad (21)$$

Here, the noise to signal ratio $\phi_{N/F} = \phi_N/\phi_F$. In the given (E, I^*, J^*, K^*)-basis, the matrix-filter $Y = Y_p$ can be written as

$$Y = \frac{\phi_F}{\|\boldsymbol{H}\|^2\phi_F + \phi_N}\left(\overline{\boldsymbol{H}}_E - I^*\overline{\boldsymbol{H}}_I - J^*\overline{\boldsymbol{H}}_J - K^*\overline{\boldsymbol{H}}_K\right). \qquad (22)$$

Thus, the inverse problem of restoration has been solved. The result of optimal filtration is $O = YG$ at the frequency-point p. The filtered quaternion signal $\hat{f} = (\hat{f}_E, \hat{f}_I, \hat{f}_J, \hat{f}_K)$ is calculated as

$$\hat{f}_E = \boldsymbol{F}^{-1} O_E, \qquad \hat{f}_I = \boldsymbol{F}^{-1} O_I, \qquad \hat{f}_J = \boldsymbol{F}^{-1} O_J, \qquad \hat{f}_K = \boldsymbol{F}^{-1} O_K.$$

The optimality is with respect to the MSR error. In the case when there is no noise, $\phi_N = 0$, we obtain the inverse quaternion filter described above.

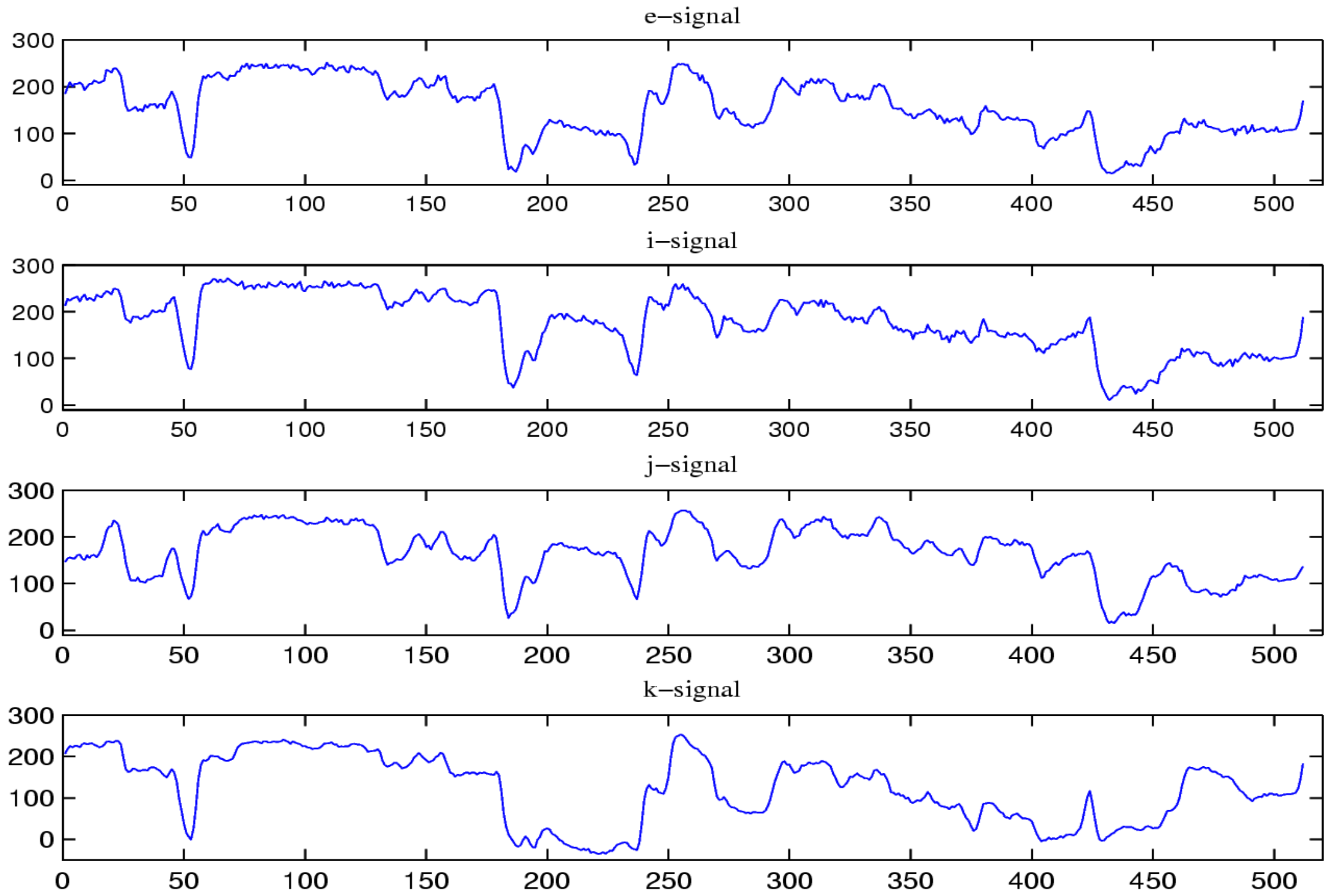

Figure 4: The e-, i-, j-, and k components of the noisy quaternion signal.

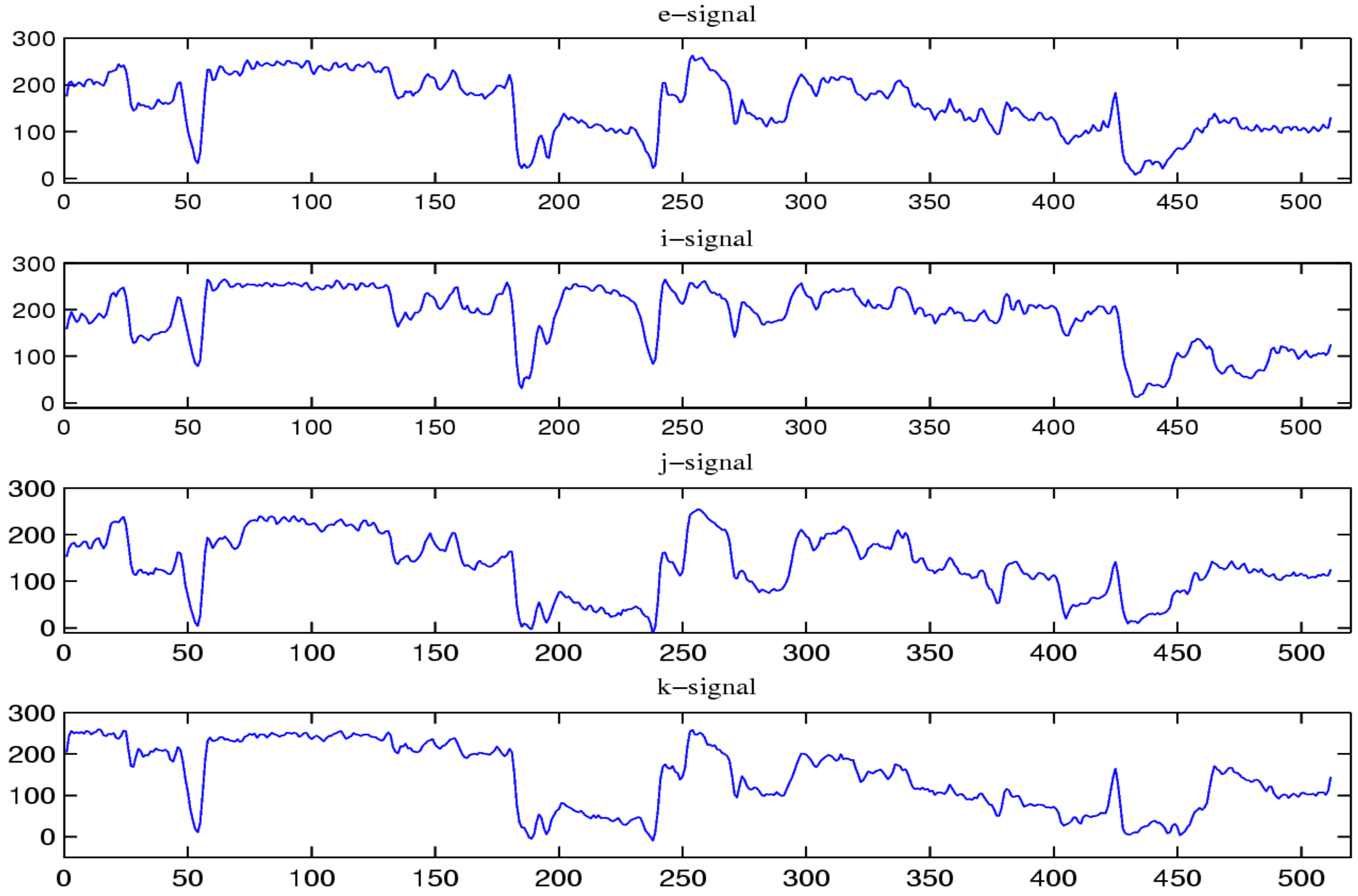

Figure 5: The e-, i-, j-, and k components of the filtered quaternion signal o_n.

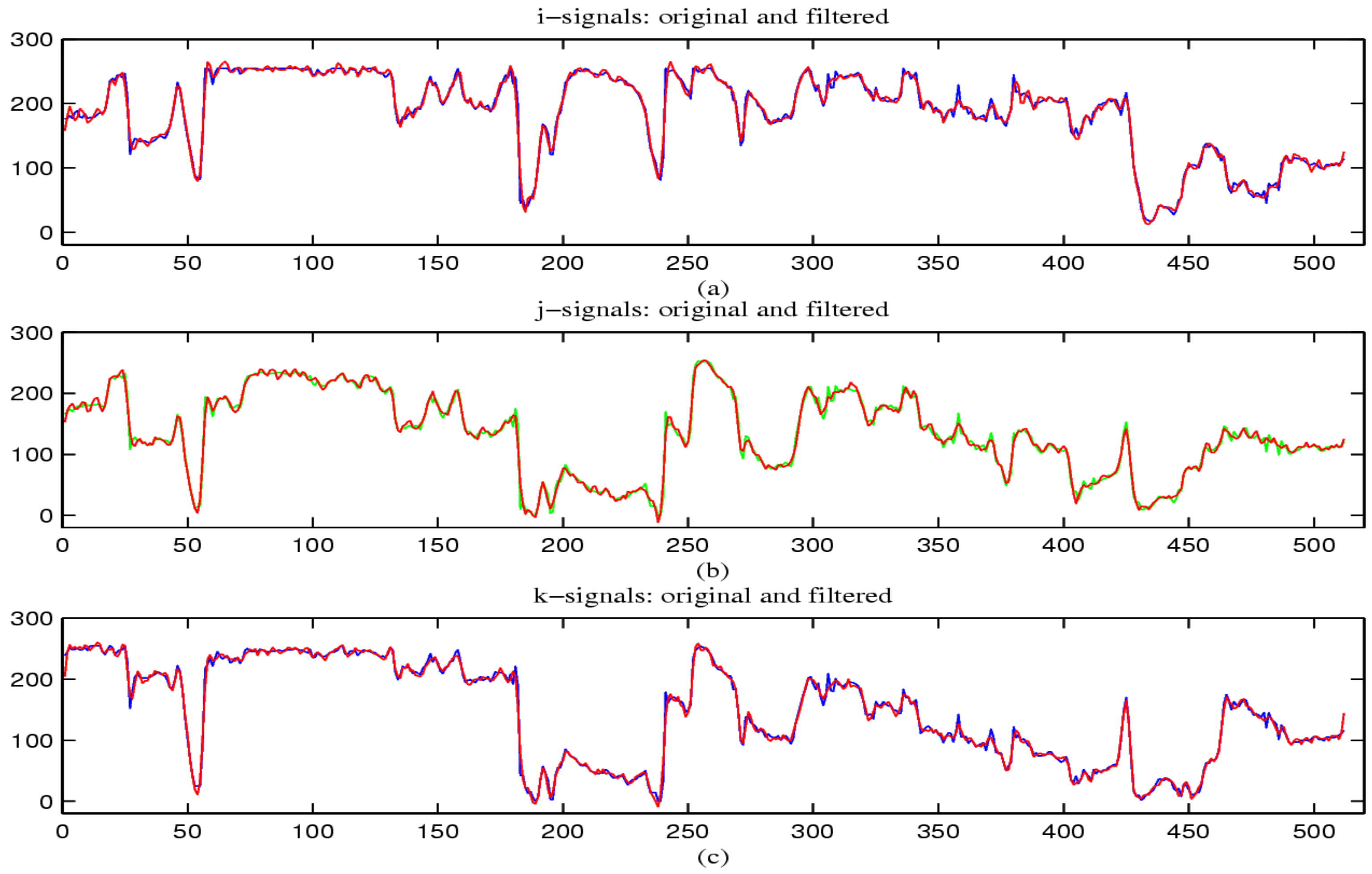

Figure 6: The e-, i-, j-, and k components of the filtered quaternion signal o_n are plotted together with the corresponding components of the original signal f_n.

SUMMARY

- The multiple signals were considered in the quaternion space and the cyclic quaternion convolution was described and calculated in the frequency domain, by using the right-side quaternion discrete Fourier transform (QDFT).
- The traditional approach of multiplication QDFTs does not work in quaternion algebra, because of non-commutativity of quaternion multiplication.
- The components of the convolution were processed in the frequency domain by special 4×4 matrices that allows us to solve the inverse problem, namely, to restore the degraded quaternion signal in a linear system with the convolution plus a noise.
- The characteristic of the quaternion optimal filter was found.

REFERENCES

1. A.M. Grigoryan, S.S. Agaian, Quaternion and Octonion Color Image Processing with MATLAB, SPIE PRESS, vol. PM279 (2018), (https://doi.org/10.1117/3.2278810).
2. A.M. Grigoryan, E.R. Dougherty, “Bayesian robust optimal linear filters,” Signal Processing, vol. 81, no. 12, pp. 2503–2521 (2001).
3. A.M. Grigoryan, E.R. Dougherty, “Robustness of optimal binary filters,” Journal of Electronic Imaging, vol. 7, no. 1, pp. 117–126 (1998), (https://doi.org/10.1117/1.482633)