Quaternion Quantum Image Representation: New Models

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Abstract

- In this paper, we unite two approaches for processing color images, by proposing a quaternion representation in quantum imaging, which includes the color images in the RGB model together with the grayscale component or brightness.
- The concept of quaternion two-qubit is considered and applied for image representation in each quantum pixel. The colors at each pixel are processed as one unit in quaternion representation. Other new models for quaternion image representation are also described.
- It is shown that a quaternion image or four component image of $N \times M$ pixels, can be represented by $(r + s + 2)$ qubits, when $N = 2^r$ and $M = 2^s, r, s > 1$. The number of qubits for representing the image can be reduced to $(r + s)$, when using the quaternion 2-qubit concept.

Quantum Models of Color Images

We consider the discrete RGB color image $f = \{f_{n,m}\}$ of size $N \times N$, where $N = 2^r, r > 1$, as the quaternion image $q = \{q_{n,m}\}$,

$$q_{n,m} = a_{n,m} + ir_{n,m} + jg_{n,m} + kb_{n,m}. \tag{1}$$

Here, i, j, and k are imaginary units, $i^2 = j^2 = k^2 = -1$ and $a_{n,m}, r_{n,m}, g_{n,m}$, and $b_{n,m}$ are the gray, red, green, and blue components of the image, respectively. The image is in the RGB color model and is considered together with its grayscale image, which is calculated by $a_{n,m} = (r_{n,m} + g_{n,m} + b_{n,m})/3$.

The brightness $(0.30r_{n,m} + 0.59g_{n,m} + 0.11b_{n,m})$ can also be used $a_{n,m}$.

For simplicity of representation of images in quantum domain, we write the image as the 1-D vector constructed from image rows,

$$q_{n,m} \rightarrow q_k = a_k + ir_k + jg_k + kb_k, \qquad k = nN + m, k = 0{:}\,(N^2 - 1). \tag{2}$$

1st Quantum Model of Color Images

At each pixel k, the four colors can be used for amplitudes of the 2-qubit states, namely, they will be used as

$$a \to a|00\rangle, \quad r \to r|01\rangle, \quad g \to g|10\rangle, \quad b \to b|11\rangle, \tag{3}$$

to obtain two qubits $Q_2 = a|00\rangle + r|01\rangle + g|10\rangle + b|11\rangle$, which should be written with the normalized coefficient

$$Q_2 = \frac{a|00\rangle + r|01\rangle + g|10\rangle + b|11\rangle}{\sqrt{a^2 + r^2 + g^2 + b^2}}. \tag{4}$$

With such 2-qubit per 1-pixel presentation, the quantum representation of the quaternion image can be written as

$$|\check{q}\rangle = \frac{1}{N}\sum_{k=0}^{N^2-1}\left(\frac{a_k|00\rangle + r_k|01\rangle + g_k|10\rangle + b_k|11\rangle}{\sqrt{a_k^2 + r_k^2 + g_k^2 + b_k^2}}\right)|k\rangle. \tag{5}$$

and that requires $(2r+2)$ qubits.

In this model, four colors which are packed in two qubits.

The measurement of these $(2r+2)$ qubits in the state $|k\rangle$ results in the 2-qubit superposition $Q_2(k)$ which carries information of all colors at pixel k.

It should be noted that this model differs from the model representing the quaternion image as

$$|\check{q}\rangle = \frac{1}{A}\sum_{k=0}^{N^2-1}(a_k|00\rangle + r_k|01\rangle + g_k|10\rangle + b_k|11\rangle)|k\rangle, \qquad (6)$$

where the normalized coefficient is calculated by

$$A = \sqrt{\sum_{k=0}^{4^r-1}(a_k^2 + r_k^2 + g_k^2 + b_k^2)} = \sqrt{\sum_{n=0}^{2^r-1}\sum_{m=0}^{2^r-1}|q_{n,m}|^2} \qquad (7)$$

which is a large number for images.

2^{nd} **Quantum Model** (by color parts)

In quantum representation, the quaternion image can be written as the grayscale image composing by four parts, by using $(2r + 2)$ qubits,

$$|\breve{q}\rangle = \frac{1}{A}\left(\sum_{k=0}^{N^2-1} a_k|k\rangle + \sum_{k=0}^{N^2-1} r_k|k + N^2\rangle + \sum_{k=0}^{N^2-1} g_k|k + 2N^2\rangle + \sum_{k=0}^{N^2-1} b_k|k + 3N^2\rangle\right)$$

The normalized coefficient A is the norm of the quaternion image and is calculated as in Eq. 7 for Model 1.

This model of quantum representation of color images can be used in parallel color processing. The measurement of the qubits $|\breve{q}\rangle$ in a state $|n\rangle = |k\rangle, |k + N^2\rangle, |k + 2N^2\rangle$, or $|k + 3N^2\rangle$ gives us only one color at the pixel k. This fact can be considered a drawback of the model when compared with Model 1.

3^{rd} **Quantum Model** (with gray scale image)

The quaternion image can be map into the grayscale image of size $2N \times 2N$. For instance, the following 2×2 model can be used:

$$q_{n,m} \rightarrow [q]_{n,m} = \begin{bmatrix} a_{n,m} & r_{n,m} \\ g_{n,m} & b_{n,m} \end{bmatrix}.$$

In the 1-D representation of the image, this model also can be described as

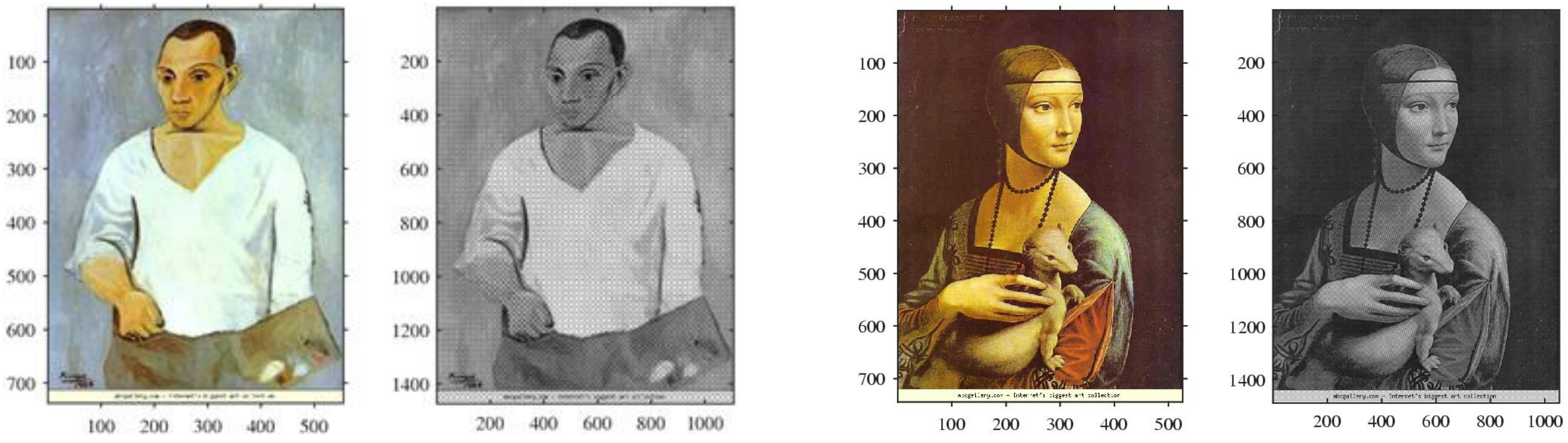
$$q_k \rightarrow \{[a_k \quad r_k \quad g_k \quad b_k], \qquad k = 0: (N^2 - 1)\}. \tag{9}$$

The quantum representation of the quaternion image can be written as

$$|\check{q}\rangle = \frac{1}{A} \sum_{k=0}^{4^r - 1} (a_k|4k\rangle + r_k|4k+1\rangle + g_k|4k+2\rangle + b_k|4k+3\rangle). \tag{10}$$

Unlike Model 2, each four colors (a_k, r_k, g_k, b_k) of the pixel k are recorded in the amplitudes of the base states next to each other.

Figure 1 shows the image of the Pablo Picasso's painting “Self-Portrait With A Palette” in part (a) and its quaternion-in-grayscale image twice the size in part (b). The image of Leonardo da Vinci’s painting “Portrait Of Cecilia Gallerani (Lady With An Ermine)” is shown in part (c) and its quaternion-in-grayscale image in part (d). The images were taken from Olga's Gallery, by address https://www.freeart.com/gallery/



(a) (b) (c) (d)

Figure 1. (a) Color Image “picasso171.jpg” and (b) its quaternion-in-grayscale image, (c) color image “leonardo9.jpg” and (d) its quaternion-in-grayscale image.

$$A = \sqrt{\sum_{m=0}^{2^r-1} \sum_{n=0}^{2^r-1} \left(a_{n,m}^2 + r_{n,m}^2 + g_{n,m}^2 + b_{n,m}^2\right)}.$$

If the range of all components of the quaternion image $q_{n,m}$ is the interval [0,1], then

$$A \leq \sqrt{\sum_{n=0}^{2^r-1} \sum_{m=0}^{2^r-1} 4} = 2 \times 2^r = 2N.$$

Such a representation requires also $(2r+2)$ qubits. The difference between this model and the model described by Eq. 8 is that four colors at each pixel are packed close to each other. In the model with Eq. 8, the placement of colors in neighbor pixels is very distant. For example, the red and green parts of the image are the amplitudes of the basic states, separated by at least N^2 states.

4^{th} **Quantum Model** (with quaternion amplitudes)

We consider the following quaternion-quantum representation of the image, when using the values of the quaternion image as the amplitudes of the basic states:

$$|\breve{q}\rangle = \frac{1}{A}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} q_{n,m}|n,m\rangle. \tag{11}$$

After mapping the image into the 1-D vector, such a representation can be written as

$$|\breve{q}\rangle = \frac{1}{A}\sum_{n=0}^{N^2-1} q_k|k\rangle. \tag{12}$$

This representation requires $2r$ qubits, that is, fewer qubits than in all other models described above.

For the image of size $N \times M = 2^r \times 2^s, r, s > 1,$ the similar quantum representation requites $(r + s)$ qubits.

5^{th} **Quantum Model** (with quaternion exponential function)

We consider the quaternion logarithm, which is calculated as follows. A quaternion number can be written as

$$q = a + q' = a + \frac{q'}{|q'|}|q'| = a + \mu\vartheta, \qquad (13)$$

where the pure unit quaternion $\mu = q'/|q'|$ and the real number ϑ=$|q'|$ will be considered as an angle.

Note that $\mu^2 = -1$, because $(q')^2 = -(r^2 + g^2 + b^2) = -|q'|^2$. If the colors of the image are in the range of [0,1], then $a \leq 1$ and $|q'| \leq \sqrt{3} < \pi$.

The quaternion exponent is

$$e^q = e^{a+q'} = e^{a+\mu\vartheta} = e^a e^{\mu\vartheta} = e^a(\cos\vartheta + \mu\sin\vartheta).$$

For only RGB colors, when $q = q'$, the exponent is $e^{q'} = e^{\mu\vartheta} = \cos\vartheta + \mu\sin\vartheta$.

At each pixel $k \sim (n, m)$, the image can be represented by the qubit

$$|\breve{q}_k\rangle = e^{a_k}(\cos\vartheta_k\,|0\rangle + \mu\sin\vartheta_k\,|1\rangle) = \cos\vartheta_k\,|0\rangle + \mu\sin\vartheta_k\,|1\rangle. \qquad (14)$$

This operation over the qubit is described by the diagonal matrix

$$R = R_{\vartheta_k} = \begin{bmatrix} \cos\vartheta_k & 0 \\ 0 & \mu\sin\vartheta_k \end{bmatrix}, \qquad \det R = \frac{1}{2}\mu\cdot\sin(2\vartheta_k)\,. \qquad (15)$$

The quantum representation of the image can be defined as

$$|\breve{q}\rangle = \frac{1}{A}\sum_{n=0}^{N^2-1} e^{a_k}(\cos\vartheta_k\,|0\rangle + \mu_k\sin\vartheta_k\,|1\rangle)|k\rangle. \qquad (16)$$

We also can consider the qubit in the form $e^{ia_k}(\cos\vartheta_k + \mu\sin\vartheta_k)|k\rangle$. In this case, the quantum representation of the image is written as

$$|\breve{q}\rangle = \frac{1}{A}\sum_{n=0}^{N^2-1} e^{ia_k}(\cos\vartheta_k\,|0\rangle + \mu_k\sin\vartheta_k\,|1\rangle)|k\rangle, \qquad A = N. \qquad (17)$$

6^{th} **Quantum Model** (with quaternion exponential function)

The quaternion exponent can be used also for the quantum representation of the image in the following way. We consider the quaternion q in polar form $q = |q|e^{\mu\vartheta}$. The quaternion can be written as

$$q = a + q' = |q|\left(\frac{a}{|q|} + \frac{q'}{|q'|} \cdot \frac{|q'|}{|q|}\right) = |q|(a + \mu\vartheta). \qquad (18)$$

If $\mu = q'/|q'|$, then $\mu^2 = -1$, and $\cos\vartheta = a/|q|$. Then, $\sin\vartheta = |q'|/|q|$ and

$$q = |q|(\cos\vartheta + \mu \cdot \sin\vartheta), \qquad \vartheta \in [0, \pi]. \qquad (19)$$

Thus, any pixel value of the image q_k can be written as

$$q_k = |q_k|(\cos\vartheta_k + \mu_k \cdot \sin\vartheta_k), \quad \vartheta_k \in [0, \pi],$$

or as the qubit

$$|\widetilde{q_k}\rangle = |q_k|(\cos\vartheta_k|0\rangle + \mu_k \cdot \sin\vartheta_k\,|1\rangle) = \cos\vartheta_k|0\rangle + \mu_k \cdot \sin\vartheta_k\,|1\rangle. \quad (20)$$

The quantum representation of the image can be given as

$$|\check{q}\rangle = \frac{1}{A}\sum_{n=0}^{N^2-1} |q_k|(\cos\vartheta_k\,|0\rangle + \mu_k \cdot \sin\vartheta_k\,|1\rangle)|k\rangle. \qquad (21)$$

with $(2r + 1)$ qubits. The normalized coefficient is calculated by

$$A = \sqrt{\sum_{k=0}^{2^{2r}-1} |q_k|^2} \leq 2\sqrt{2^{2r}} = 2^{r+1},$$

if all color components of the image are in the range of the unit interval [0,1].

SUMMARY

- Two concepts of the quaternion and quantum representations of color images can be united to obtain the quaternion-in-quantum representation of color images. Different quantum models for quaternion image representation are described. The images are considered in the RGB color models, but other color models, such as CMY(K) and XYZ can also be used.
- A quaternion image of $N \times M$ pixels, can be represented by $(r + s + 2)$ or $(r + s + 1)$ qubits, when $N = 2^r$ and $M = 2^s$, $r, s > 1$. The model of quantum quaternion image representation with $(r + s)$ is also described.

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