Alpha-rooting and Correlation Method of Image Enhancement

Artyom M. Grigoryan[a] and Sos S. Agaian[b]

[a]Department of Electrical and Computer Engineering

The University of Texas at San Antonio, San Antonio, Texas, USA

and

[b]Computer Science Department, College of Staten Island and the Graduate Center, Staten Island, NY, USA

amgrigoryan@utsa.edu, sos.agaian@csi.cuny.edu

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Abstract

- This paper analyzes the method of the Fourier transforms-based alpha-rooting in image enhancement and describes a new generalized method of alpha-rooting, by using the autocorrelation function of the image.
- The alpha-rooting can be described by the Taylor series in the frequency-domain, as well as in the spatial domain. In such a series, the alpha-rooting is the convolution of the image with the series of the autocorrelation functions.
- The application of the Taylor series in alpha-rooting allows us to use the parameterized filters even for alpha parameter in a much larger interval than [0,1].
- New correlation alpha-rooting filters are presented.
- Examples of application of these two filters for enhancement of the grayscale 'jetplane' image are given.

Alpha-rooting and approximation

In the α-rooting method of image enhancement, the magnitude of the 2-D discrete Fourier transform (DFT) of the discrete image of $N \times M$ pixel is modified as follows:

$$\left|F_{p,s}\right| \to \left|F_{p,s}\right|^{\alpha}, \qquad p = 0{:}\,(N-1), \qquad s = 0{:}\,(M-1), \tag{1}$$

where the parameter α is from the interval (0,1). The phase $\varphi(p,s)$ of the transform does not change. The coefficients of the Fourier transform are multiplied by the coefficients $\left|F_{p,s}\right|^{\alpha-1}$,

$$F_{p,s} = \left|F_{p,s}\right| e^{i\varphi(p,s)} \to Y_{p,s} = \left|F_{p,s}\right|^{\alpha} e^{i\varphi(p,s)} = \left|F_{p,s}\right|^{\alpha-1} \left|F_{p,s}\right| e^{i\varphi(p,s)} = \left|F_{p,s}\right|^{\alpha-1} F_{p,s}. \tag{2}$$

Here, $\varphi(p,s)$ is the phase of the transform at frequency-point (p,s).
The enhanced image is the inverse 2-D DFT of the modified transform

$$y_{n,m} = \frac{1}{NM} \sum_{p=0}^{N-1} \sum_{s=0}^{M-1} F_{p,s} \left|F_{p,s}\right|^{\alpha-1} W_N^{-np} W_M^{-ms}, \qquad n = 0{:}\,N-1), m = 0{:}\,(M-1). \tag{3}$$

where the exponential coefficients are $W_N^t = \exp\left(-\frac{i2\pi t}{N}\right)$ and $W_M^t = \exp\left(-\frac{i2\pi t}{M}\right)$.

For simplicity of calculation, we consider the 1-D case of alpha-rooting of a signal f_n, whose N-point discrete Fourier transform F_p is modified as

$$F_p \rightarrow F_p \left|F_p\right|^{\alpha-1}, \qquad p = 0{:}\,(N-1). \tag{4}$$

The inverse discrete Fourier transform of $\left|F_p\right|^2$ is equal to the autocorrelation function

$$K(n) = \sum_{k=0}^{N-1} f_{n+k}\, f_k, \qquad n = 0{:}\,(N-1). \tag{5}$$

The signal is considered periodic and the subscripts in this sum are calculated by modulo N.
For this autocorrelation function, we will use the notation $K = f \circ f$. Note that $K(-n) = K(n)$.

If we consider the inverse DFT of the spectrum $\left|F_p\right|^4$, we obtain the second order correlation, i.e., $K \circ K = f \circ f \circ f \circ f$, or simply $K^{(2)} = f^{(4)}$.

The Taylor series

$$(1+x)^a = 1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n, \qquad |x| < 1,$$

can be written as

$$x^a = \sum_{n=0}^{\infty} \binom{a}{n} (x-1)^n = 1 + \sum_{n=1}^{\infty} \binom{a}{n} (x-1)^n, \qquad 0 < x < 2. \tag{6}$$

Here, the binomial coefficients are

$$\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}.$$

We consider the spectrum $\left|F_p\right|^{\alpha-1}$ in the form

$$\left|F_p\right|^{\alpha-1} = \left(\left|F_p\right|^2\right)^{\frac{\alpha-1}{2}}, \qquad p = 0,1,\ldots,(N-1). \tag{7}$$

To use the Taylor series for this power, we perform the normalization by the constant A_0, i.e.,

$$F_p = F_p/A, \quad p = 1, 2, \dots, (N-1), \tag{8}$$

such that $|F_p| < 1$. For instance, the normalization coefficient $\mathrm{A} = N \times \max|F_p|$ can be used.

Denoting $b = (a-1)/2$ and $G_p = |F_p|^2$, we write the following Taylor series:

$$|F_p|^{\alpha-1} = \sum_{n=0}^{\infty} \binom{b}{n} (G_p - 1)^n, \qquad 0 < G_p < 1. \tag{9}$$

Here, the cases when $G_p = 0$ and 1 are not considered, since the alpha-rooting does not change the values of such components. The inverse DFT of this series can be written as

$$H(x) = \frac{1}{N}\sum_{p=0}^{N-1} \left|F_p\right|^{\alpha-1} W_N^{-px} = \sum_{n=0}^{\infty} \binom{b}{n} (K-\delta)^n(x), \qquad x = 0,1,\dots,(N-1). \tag{10}$$

In this sum, $\delta(x)$ is the discrete unit sample function, and the polynomials are

$$(K-\delta)^n(x) = \sum_{m=0}^{n} \binom{n}{m} K^{(m)}(x) * \left(-\delta(x)\right)^{n-m}, \tag{11}$$

and $K^{(m)}(x) = \big(\underbrace{K \circ K \circ \cdots \circ K}_{m \text{ times}}\big)(x)$ and $\delta^{(k)}(x) = \big(\underbrace{\delta \circ \delta \circ \cdots \circ \delta}_{k \text{ times}}\big)(x) = \delta(x)$, when integer $k > 0$.

Thus, the function $H(x)$, that represents the filter of the alpha-rooting in time domain, can be written in Eq. 10 as

$$H(x) = \delta(x) + b\big(K(x) - \delta(x)\big) + \frac{b(b-1)}{2}\Big(K^{(2)}(x) - 2K(x) + \delta(x)\Big) +$$
$$+ \frac{b(b-1)(b-2)}{3!}\Big(K^{(3)}(x) - 3K^{(2)}(x) + 3K(x) + \delta(x)\Big) + \cdots . \tag{12}$$

In the time-domain, the alpha-rooting $F_p \to F_p\left|F_p\right|^{\alpha-1}$ means the convolution of the signal f_n with the obtained filter $H(x)$. This filter in the zeroth approximation is equal to

$$H_0(x) = \delta(x),$$

and in the first approximation, it is equal to

$$H_1(x) = \delta(x) + b\big(K(x) - \delta(x)\big) = (1-b)\delta(x) + bK(x), \tag{13}$$

and in the 2nd approximation, it is equal to

$$H_2(x) = \delta(x) + b\big(K(x) - \delta(x)\big) + \frac{b(b-1)}{2}\Big(K^{(2)}(x) - 2K(x) + \delta(x)\Big)$$

$$= \frac{(1-b)(2-b)}{2}\delta(x) + b(2-b)K(x) + \frac{b(b-1)}{2}K^{(2)}(x). \tag{14}$$

The results of the alpha-rooting of the signal f_n when using these approximations are

$$(y_0)_n = (f * H_0)_n = f_n,$$

$$(y_1)_n = (f * H_1)_n = (1-b)f_n + bf_n * K(n), \tag{15}$$

$$(y_2)_n = (f * H_2)_n = \frac{(1-b)(2-b)}{2} f_n + b(2-b)f_n * K(n) + \frac{b(b-1)}{2} f_n * K^{(2)}(n). \tag{16}$$

In Eq. 15, the coefficient $(1-b) > 0$ when $0 < a < 3$. Using the coefficients

$$c_1 = \frac{(1-b)(2-b)}{2}, \qquad c_2 = b(2-b), \qquad \text{and} \quad c_3 = \frac{b(b-1)}{2},$$

The signal in the 2nd approximation can be written as

$$(y_2)_n = (f * H_2)_n = c_1 f_n + c_2 f_n * K(n) + c_3 f_n * K^{(2)}(n).$$

In the first approximation, the convolution is the mean of the signal and its convolution with autocorrelation function.

It also can be written as the signal plus a gradient,

$$(y_1)_n = f_n - b[f_n - f_n * K(n)], \qquad (-b > 0).$$

Thus, the correlation function is used to calculate a gradient of the signal (image).

It is interesting to note that the sum of the coefficients in Eq. 16 is equals to 1, too. Indeed,

$$c_1 + c_2 + c_3 = \frac{(1-b)(2-b)}{2} + b(2-b) + \frac{b(b-1)}{2} = \frac{2 - 3b + b^2 + 4b - 2b^2 + b^2 - b}{2} = 1.$$

The same is for the higher than the 2nd order approximation of the alpha-rooting filter $H(x)$ in Eq. 12.

Example 1:

Consider the $\alpha = \frac{3}{4}$ case, for which $b = \frac{\alpha - 1}{2} = -\frac{1}{8}$.
Then, the 1st and 2nd approximations of the alpha-rooting are the signals

$$(y_1)_n = (f * H_1)_n = \frac{9}{8} f_n - \frac{1}{8} f_n * K(n) = (y_1)_n = f_n + \frac{1}{8}[f_n - f_n * K(n)], \quad (18)$$

and

$$(y_2)_n = (f * H_2)_n = \frac{1}{64}\left[76.5 f_n - 17 f_n * K(n) + 4.5 f_n * K^{(2)}(n)\right]. \quad (19)$$

$$= 1.1953 f_n - 0.2656 f_n * K(n) + 0.0703 f_n * K^{(2)}(n).$$

For the 0.9-rooting approximations, we obtain the signals

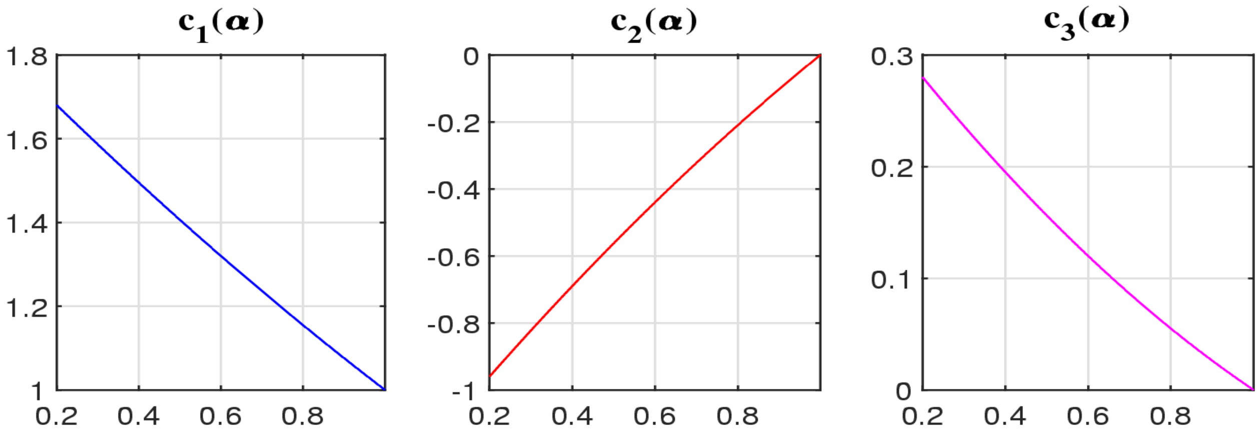
$$(y_1)_n = (f * H_1)_n = \frac{21}{20} f_n - \frac{1}{20} f_n * K(n) = (y_1)_n = f_n + \frac{1}{20}[f_n - f_n * K(n)], \quad (20)$$

and

$$(y_2)_n = (f * H_2)_n = 1.0762 f_n - 0.1025 f_n * K(n) + 0.0262 f_n * K^{(2)}(n). \quad (21)$$

One can see form these two examples that the coefficients c_1, c_2, and c_3 for the 0.9–rooting became smaller, when comparing with the $\alpha = 0.75$ case.

The graphs of these coefficients as functions of α, i.e., $c_1(\alpha), c_2(\alpha)$, and $c_3(\alpha)$ are shown in Fig. 1, when α runs the interval [0.2,1]. A significant contribution of the correlation function in alpha-rooting is when the parameter α takes small values. When α is close to 1, the coefficients $c_2(\alpha)$ and $c_3(\alpha)$ are very small.



(a) (b) (c)

Figure 1. (a) The coefficients of the 2nd approximation of the alpha-rooting.

The filter $H(x)$ in Eq. 12 was derived for the normalized signal $\check{f}_n = f_n/A$ and not for f_n. For instance, in the first approximation

$$(y_1)_n = \frac{f_n}{A} * H_1(n), \qquad \text{or} \quad (y_1)_n A = f_n * H_1(n).$$

Here, the filter $H_1(n) = (1-b)\delta(n) + bK(n)$ is defined by the autocorrelation function $K(n)$ of $\check{f}_n$, i.e., up to the constant $1/A^2$, it is the autocorrelation function $R(n)$ of the original signal. Thus,

$$A(y_1)_n = (1-b)f_n + b\frac{1}{A^2}f_n * R(n). \tag{22}$$

Similarly, the 2nd approximation of the alpha-rooting can be written as

$$A(y_2)_n = c_1 f_n + c_2\frac{1}{A^2}f_n * R(n) + c_3\frac{1}{A^4}f_n * R^{(2)}(n). \tag{23}$$

The output of the alpha-rooting will be scaled to the original range, let say [1,255]. Therefore, the constant A in the left parts of Eqs. 22 and 23, can be omitted. The alpha-rooting is not linear transformation.

1. The first modification of the alpha-rooting of the signal is defined as

$$(y_1)_n = (1-b)f_n + (bA_1)f_n * K(n). \tag{24}$$

2. The 2nd modification of the alpha-rooting of the original signal is defined as

$$(y_2)_n = c_1 f_n + (c_2 A_2)f_n * K(n) + (c_3 B_2)f_n * K^{(2)}(n). \tag{25}$$

We will call the corresponding filters

$$H_1(n) = (1-b)\delta(n) + (bA_1)K(n) \tag{26}$$

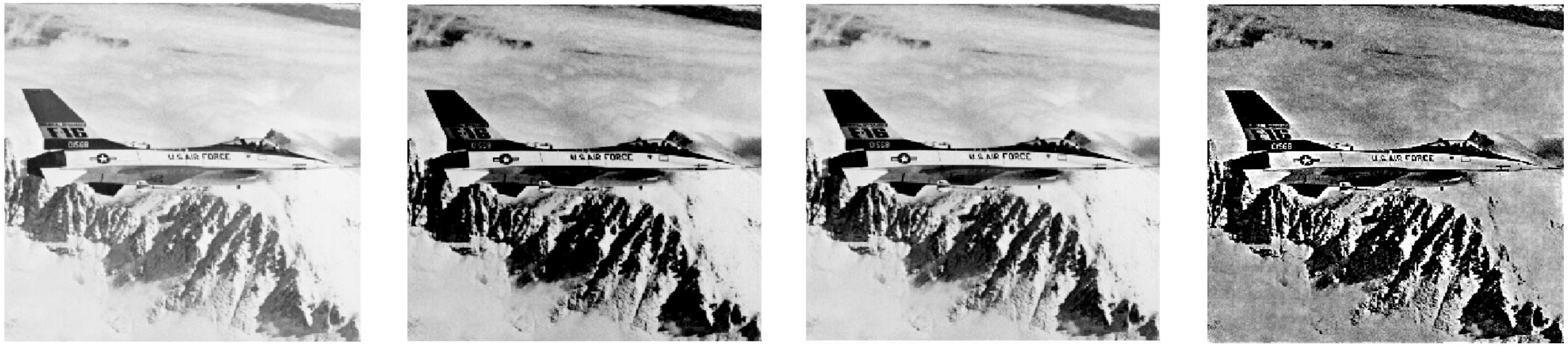
and

$$H_2(n) = c_1\delta(n) + (c_2 A_2)K(n) + (c_3 B_2)K^{(2)}(n) \tag{27}$$

the 1st and 2nd order correlation alpha-rooting filters, respectively.

The coefficients c_2, and c_3 at the autocorrelation function in the above approximations are small. Therefore, in these new filters, the parameters A_1, A_2, and B_2 will be chosen with large values.

Figure 2 show the “jetplane” image of 256×256 pixels in part (a), the first and second approximation of the 0.75-rooting in parts (b) and (c), respectively. The parameters of the filters are $A_1 = 10$, and $A_2 = 5, B_2 = 10$. In part d, the result of the traditional 0.75-rooting. The last image is shown with twice large amplitude (i.e., $2y_2$); otherwise, the displayed image will be very dark.



(a) (b) (c) (d)

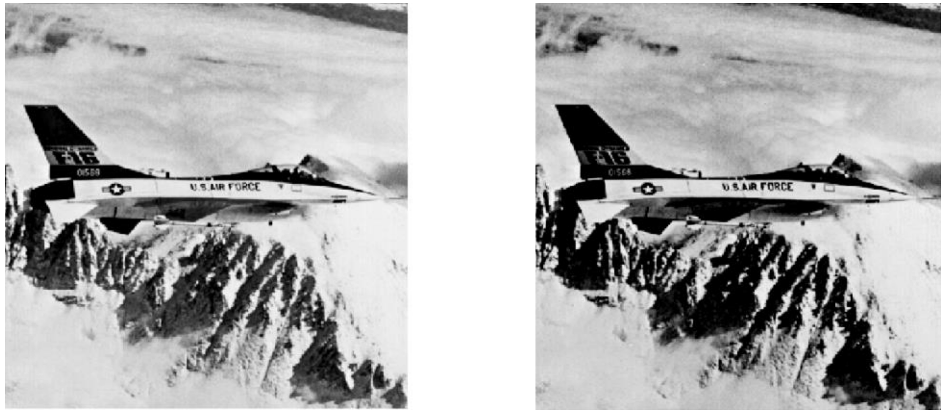
Figure 2. (a) The original image, (b) the 1st and (c) 2nd approximations, and (d) the alpha-rooting with $\alpha = 0.75$.

Figure 3 shows the same image in part (a) and the first and second approximations of the 0.90-rooting in parts (b) and (c), respectively. The parameters of the filters are $A_1 = 10$ and $A_2 = 10,\ B_2 = 20$.

(a) (b) (c)

Figure 3. (a) The “jetplane” image, and (b) the 1st and (c) 2nd approximations of the alpha-rooting with $\alpha = 0.90$.

The traditional alpha-rooting works for the parameter alpha from the interval (0,1). The above Eqs. 12-27 can be used for the values of alpha parameter outside this interval, too. Figure 4 shows the 1st and 2nd approximations, when α =1.5. The parameters of the filters are $A_1 = -2$ and $A_2 = 1,\ B_2 = 20$.



(a) (b)

Figure 4. (a) The 1st and (b) 2nd approximations of the image when $\alpha = 1.5$.

Figure 5 shows the 1st approximation of the alpha-rooting, when $\alpha = -0.75$. The parameters of the filter $A_1 = -2$.

Figure 5. The 1st approximations of the image when $\alpha = -0.75$.

The best parameter of alpha for the proposed method can be found, by using a measure of enhancement. For instance, the EME enhancement measure can be used, as in the traditional method of alpha-rooting

SUMMARY

This paper analyzes the traditional method of the Fourier transforms-based alpha-rooting in image enhancement and presents the approximations of the alpha-rooting, by using the autocorrelation function of the image. In the spatial domain, the alpha-rooting can be described by the Taylor series. For this, the inverse 2-D DFT is used. In such a series, the alpha-rooting is the convolution of the image with the series of the autocorrelation functions. The application of the Taylor series in the method of alpha-rooting allows to use the new correlation alpha-rooting filters even for values of alpha in an interval larger than [0,1]. Examples of application of correlation alpha-rooting filters for enhancement of the grayscale image 'jetplane' are given. We believe that the proposed methods can also be used to enhance color images.

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