Alpha-rooting and Correlation Method of Image Enhancement

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Abstract

• This paper analyzes the method of the Fourier transforms-based alpha-rooting in image enhancement and describes a new generalized method of alpha-rooting, by using the autocorrelation function of the image.

• The alpha-rooting can be described by the Taylor series in the frequency-domain, as well as in the spatial domain. In such a series, the alpha-rooting is the convolution of the image with the series of the autocorrelation functions.

• The application of the Taylor series in alpha-rooting allows us to use the parameterized filters even for alpha parameter in a much larger interval than [0,1].

• New correlation alpha-rooting filters are presented.

• Examples of application of these two filters for enhancement of the grayscale ‘jetplane’ image are given.
Alpha-rooting and approximation

In the $\alpha$-rooting method of image enhancement, the magnitude of the 2-D discrete Fourier transform (DFT) of the discrete image of $N \times M$ pixel is modified as follows:

$$|F_{p,s}| \to |F_{p,s}|^{\alpha}, \quad p = 0: (N - 1), \quad s = 0: (M - 1),$$  \hspace{1cm} (1)

where the parameter $\alpha$ is from the interval $(0,1)$. The phase $\varphi(p,s)$ of the transform does not change. The coefficients of the Fourier transform are multiplied by the coefficients $|F_{p,s}|^{\alpha - 1}$,

$$F_{p,s} = |F_{p,s}| e^{i\varphi(p,s)} \to Y_{p,s} = |F_{p,s}|^{\alpha} e^{i\varphi(p,s)} = |F_{p,s}|^{\alpha - 1} |F_{p,s}| e^{i\varphi(p,s)} = |F_{p,s}|^{\alpha - 1} F_{p,s}.$$  \hspace{1cm} (2)

Here, $\varphi(p,s)$ is the phase of the transform at frequency-point $(p,s)$.

The enhanced image is the inverse 2-D DFT of the modified transform

$$y_{n,m} = \frac{1}{NM} \sum_{p=0}^{N-1} \sum_{s=0}^{M-1} F_{p,s} |F_{p,s}|^{\alpha - 1} W_N^{-np} W_M^{-ms}, \quad n = 0: N - 1, m = 0: M - 1.$$  \hspace{1cm} (3)

where the exponential coefficients are $W_N^t = \exp \left( -\frac{i2\pi t}{N} \right)$ and $W_M^t = \exp \left( -\frac{i2\pi t}{M} \right)$.  

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For simplicity of calculation, we consider the 1-D case of alpha-rooting of a signal $f_n$, whose $N$-point discrete Fourier transform $F_p$ is modified as

$$F_p \rightarrow F_p |F_p|^{\alpha-1}, \quad p = 0: (N - 1).$$  \hfill (4)

The inverse discrete Fourier transform of $|F_p|^2$ is equal to the autocorrelation function

$$K(n) = \sum_{k=0}^{N-1} f_{n+k} f_k, \quad n = 0: (N - 1).$$  \hfill (5)

The signal is considered periodic and the subscripts in this sum are calculated by modulo $N$. For this autocorrelation function, we will use the notation $K = f \circ f$. Note that $K(-n) = K(n)$.

If we consider the inverse DFT of the spectrum $|F_p|^4$, we obtain the second order correlation, i.e., $K \circ K = f \circ f \circ f \circ f$, or simply $K^{(2)} = f^{(4)}$. 

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The Taylor series

\[
(1 + x)^a = 1 + \sum_{n=1}^{\infty} \binom{a}{n} x^n, \quad |x| < 1,
\]

can be written as

\[
x^a = \sum_{n=0}^{\infty} \binom{a}{n} (x - 1)^n = 1 + \sum_{n=1}^{\infty} \binom{a}{n} (x - 1)^n, \quad 0 < x < 2.
\]  \(6\)

Here, the binomial coefficients are

\[
\binom{a}{n} = \frac{a(a - 1) \cdots (a - n + 1)}{n!}.
\]

We consider the spectrum \(|F_p|^{\alpha-1}\) in the form

\[
|F_p|^{\alpha-1} = \left(|F_p|^2\right)^{\frac{\alpha-1}{2}}, \quad p = 0, 1, \ldots, (N - 1).
\]  \(7\)
To use the Taylor series for this power, we perform the normalization by the constant $A_0$, i.e.,

$$F_p = F_p/A, \quad p = 1, 2, ..., (N - 1),$$

such that $|F_p| < 1$. For instance, the normalization coefficient $A = N \times \max|F_p|$ can be used.

Denoting $b = (a - 1)/2$ and $G_p = |F_p|^2$, we write the following Taylor series:

$$|F_p|^\alpha - 1 = \sum_{n=0}^{\infty} \left(\frac{b}{n}\right)(G_p - 1)^n, \quad 0 < G_p < 1.$$  \hspace{1cm} (9)
Here, the cases when \( G_p = 0 \) and 1 are not considered, since the alpha-rooting does not change the values of such components. The inverse DFT of this series can be written as

\[
H(x) = \frac{1}{N} \sum_{p=0}^{N-1} |F_p|^\alpha W_N^{-px} = \sum_{n=0}^{\infty} \binom{b}{n} (K - \delta)^n(x), \quad x = 0, 1, \ldots, (N - 1). \tag{10}
\]

In this sum, \( \delta(x) \) is the discrete unit sample function, and the polynomials are

\[
(K - \delta)^n(x) = \sum_{m=0}^{n} \binom{n}{m} K^{(m)}(x) \ast (-\delta(x))^{n-m}, \tag{11}
\]

and \( K^{(m)}(x) = (K \circ K \circ \cdots \circ K)^m(x) \) and \( \delta^{(k)}(x) = (\delta \circ \delta \circ \cdots \circ \delta)^k(x) = \delta(x) \), when integer \( k > 0 \).

Thus, the function \( H(x) \), that represents the filter of the alpha-rooting in time domain, can be written in Eq. 10 as

\[
H(x) = \delta(x) + b(K(x) - \delta(x)) + \frac{b(b-1)}{2}(K^{(2)}(x) - 2K(x) + \delta(x)) + \\
+ \frac{b(b-1)(b-2)}{3!}(K^{(3)}(x) - 3K^{(2)}(x) + 3K(x) + \delta(x)) + \cdots. \tag{12}
\]
In the time-domain, the alpha-rooting $F_p \rightarrow F_p |F_p|^{\alpha-1}$ means the convolution of the signal $f_n$ with the obtained filter $H(x)$. This filter in the zeroth approximation is equal to

$$H_0(x) = \delta(x),$$

and in the first approximation, it is equal to

$$H_1(x) = \delta(x) + b(K(x) - \delta(x)) = (1 - b)\delta(x) + bK(x),$$

and in the 2\textsuperscript{nd} approximation, it is equal to

$$H_2(x) = \delta(x) + b(K(x) - \delta(x)) + \frac{b(b - 1)}{2}(K^{(2)}(x) - 2K(x) + \delta(x))$$

$$= \frac{(1 - b)(2 - b)}{2} \delta(x) + b(2 - b)K(x) + \frac{b(b - 1)}{2} K^{(2)}(x).$$
The results of the alpha-rooting of the signal $f_n$ when using these approximations are

\[ (y_0)_n = (f \ast H_0)_n = f_n, \]

\[ (y_1)_n = (f \ast H_1)_n = (1 - b)f_n + bf_n \ast K(n), \] \hspace{1cm} (15)

\[ (y_2)_n = (f \ast H_2)_n = \frac{(1 - b)(2 - b)}{2} f_n + b(2 - b)f_n \ast K(n) + \frac{b(b - 1)}{2} f_n \ast K(2)(n). \] \hspace{1cm} (16)

In Eq. 15, the coefficient $(1 - b) > 0$ when $0 < a < 3$. Using the coefficients

\[ c_1 = \frac{(1 - b)(2 - b)}{2}, \quad c_2 = b(2 - b), \quad \text{and} \quad c_3 = \frac{b(b - 1)}{2}, \]

The signal in the 2\textsuperscript{nd} approximation can be written as

\[ (y_2)_n = (f \ast H_2)_n = c_1 f_n + c_2 f_n \ast K(n) + c_3 f_n \ast K(2)(n). \]
In the first approximation, the convolution is the mean of the signal and its convolution with autocorrelation function.

It also can be written as the signal plus a gradient,

\[(y_1)_n = f_n - b[f_n - f_n * K(n)], \quad (-b > 0).\]

Thus, the correlation function is used to calculate a gradient of the signal (image).

It is interesting to note that the sum of the coefficients in Eq. 16 is equals to 1, too. Indeed,

\[c_1 + c_2 + c_3 = \frac{(1 - b)(2 - b)}{2} + b(2 - b) + \frac{b(b - 1)}{2} = \frac{2 - 3b + b^2 + 4b - 2b^2 + b^2 - b}{2} = 1.\]

The same is for the higher than the 2\textsuperscript{nd} order approximation of the alpha-rooting filter \(H(x)\) in Eq. 12.
**Example 1:**

Consider the $\alpha = \frac{3}{4}$ case, for which $b = \frac{\alpha - 1}{2} = -\frac{1}{8}$.

Then, the 1st and 2nd approximations of the alpha-rooting are the signals

$$ (y_1)_n = (f * H_1)_n = \frac{9}{8} f_n - \frac{1}{8} f_n * K(n) = (y_1)_n = f_n + \frac{1}{8}[f_n - f_n * K(n)], \quad (18) $$

and

$$ (y_2)_n = (f * H_2)_n = \frac{1}{64} [76.5 f_n - 17 f_n * K(n) + 4.5 f_n * K^{(2)}(n)]. $$

$$ = 1.1953 f_n - 0.2656 f_n * K(n) + 0.0703 f_n * K^{(2)}(n). \quad (19) $$

For the 0.9-rooting approximations, we obtain the signals

$$ (y_1)_n = (f * H_1)_n = \frac{21}{20} f_n - \frac{1}{20} f_n * K(n) = (y_1)_n = f_n + \frac{1}{20}[f_n - f_n * K(n)], \quad (20) $$

and

$$ (y_2)_n = (f * H_2)_n = 1.0762 f_n - 0.1025 f_n * K(n) + 0.0262 f_n * K^{(2)}(n). \quad (21) $$
One can see from these two examples that the coefficients $c_1, c_2,$ and $c_3$ for the 0.9–rooting became smaller, when comparing with the $\alpha = 0.75$ case.

The graphs of these coefficients as functions of $\alpha$, i.e., $c_1(\alpha), c_2(\alpha),$ and $c_3(\alpha)$ are shown in Fig. 1, when $\alpha$ runs the interval $[0.2,1]$. A significant contribution of the correlation function in alpha-rooting is when the parameter $\alpha$ takes small values. When $\alpha$ is close to 1, the coefficients $c_2(\alpha)$ and $c_3(\alpha)$ are very small.

![Graphs of coefficients $c_1(\alpha)$, $c_2(\alpha)$, and $c_3(\alpha)$ as functions of $\alpha$.](image)

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Figure 1. (a) The coefficients of the 2\textsuperscript{nd} approximation of the alpha-rooting.
The filter $H(x)$ in Eq. 12 was derived for the normalized signal $\tilde{f}_n = f_n/A$ and not for $f_n$. For instance, in the first approximation

$$(y_1)_n = \frac{f_n}{A} * H_1(n), \quad \text{or} \quad (y_1)_n A = f_n * H_1(n).$$

Here, the filter $H_1(n) = (1 - b)\delta(n) + bK(n)$ is defined by the autocorrelation function $K(n)$ of $\tilde{f}_n$, i.e., up to the constant $1/A^2$, it is the autocorrelation function $R(n)$ of the original signal. Thus,

$$A(y_1)_n = (1 - b)f_n + b \frac{1}{A^2} f_n * R(n). \quad (22)$$

Similarly, the 2nd approximation of the alpha-rooting can be written as

$$A(y_2)_n = c_1 f_n + c_2 \frac{1}{A^2} f_n * R(n) + c_3 \frac{1}{A^4} f_n * R^{(2)}(n). \quad (23)$$

The output of the alpha-rooting will be scaled to the original range, let say $[1,255]$. Therefore, the constant $A$ in the left parts of Eqs. 22 and 23, can be omitted. The alpha-rooting is not linear transformation.
1. The first modification of the alpha-rooting of the signal is defined as

\[(y_1)_n = (1 - b)f_n + (bA_1)f_n * K(n).\] (24)

2. The 2\textsuperscript{nd} modification of the alpha-rooting of the original signal is defined as

\[(y_2)_n = c_1f_n + (c_2A_2)f_n * K(n) + (c_3B_2)f_n * K^{(2)}(n).\] (25)

We will call the corresponding filters

\[H_1(n) = (1 - b)\delta(n) + (bA_1)K(n)\] (26)

and

\[H_2(n) = c_1\delta(n) + (c_2A_2)K(n) + (c_3B_2)K^{(2)}(n)\] (27)

the 1\textsuperscript{st} and 2\textsuperscript{nd} order correlation alpha-rooting filters, respectively.

The coefficients \(c_2, c_3\) at the autocorrelation function in the above approximations are small. Therefore, in these new filters, the parameters \(A_1, A_2, B_2\) will be chosen with large values.
Figure 2 show the “jetplane” image of 256×256 pixels in part (a), the first and second approximation of the 0.75-rooting in parts (b) and (c), respectively. The parameters of the filters are $A_1 = 10$, and $A_2 = 5, B_2 = 10$. In part d, the result of the traditional 0.75-rooting. The last image is shown with twice large amplitude (i.e., $2y_2$); otherwise, the displayed image will be very dark.

Figure 2. (a) The original image, (b) the 1st and (c) 2nd approximations, and (d) the alpha-rooting with $\alpha = 0.75$. 
Figure 3 shows the same image in part (a) and the first and second approximations of the 0.90-rooting in parts (b) and (c), respectively. The parameters of the filters are $A_1 = 10$ and $A_2 = 10$, $B_2 = 20$.

Figure 3. (a) The “jetplane” image, and (b) the 1st and (c) 2nd approximations of the alpha-rooting with $\alpha = 0.90$. 
The traditional alpha-rooting works for the parameter alpha from the interval (0,1). The above Eqs. 12-27 can be used for the values of alpha parameter outside this interval, too. Figure 4 shows the 1st and 2nd approximations, when \( \alpha = 1.5 \). The parameters of the filters are \( A_1 = -2 \) and \( A_2 = 1, B_2 = 20 \).

Figure 4 shows the 1st and 2nd approximations of the image when \( \alpha = 1.5 \).

Figure 5 shows the 1st approximation of the alpha-rooting, when \( \alpha = -0.75 \). The parameters of the filter \( A_1 = -2 \).

Figure 5. The 1st approximations of the image when \( \alpha = -0.75 \).

The best parameter of alpha for the proposed method can be found, by using a measure of enhancement. For instance, the EME enhancement measure can be used, as in the traditional method of alpha-rooting.
SUMMARY

This paper analyzes the traditional method of the Fourier transforms-based alpha-rooting in image enhancement and presents the approximations of the alpha-rooting, by using the autocorrelation function of the image. In the spatial domain, the alpha-rooting can be described by the Taylor series. For this, the inverse 2-D DFT is used. In such a series, the alpha-rooting is the convolution of the image with the series of the autocorrelation functions. The application of the Taylor series in the method of alpha-rooting allows to use the new correlation alpha-rooting filters even for values of alpha in an interval larger than [0,1]. Examples of application of correlation alpha-rooting filters for enhancement of the grayscale image ‘jetplane’ are given. We believe that the proposed methods can also be used to enhance color images.

REFERENCES