# Brief Lectures in EE-3523: Signals and Systems II 

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## PREFACE

These notes were used in my lectures during the teaching the undergraduate-level class of EE 3523 signals and systems in the Department of Electrical Engineering at the University of Texas at San Antonio. These lectures are based on the material of many text-books and my notes, and among the books I should mention the following books (as it mentioned in the syllabus):

Signals, Systems, and Transforms, (3rd Ed., 2003), by Charles L. Philips, John
M. Parr, and Eve A. Riskin, Chapters 7, 8, 10-13.

Text:

1. Signals and Systems: Analysis using Transform methods and MATLAB (2nd Ed., 2004) by M.J. Roberts.
2. Fundamentals of Signals and Systems: with MATLAB Examples (2000) by

Edward Kamen and Bonnie Heck.
3. Signals and Systems, (2nd Ed., 2002) by Simon Haykin and Barry Van Veen.

I decided to post these briefs because of many requests from our students, specially the students who took my class EE-3423: Signals and Systems I. I hope these lectures will be useful for the graduate students as well, to remember many important concepts from signals and systems and use them in other DSP classes. It is not a final version of the lectures, it should be improved, and I will work on the text later and will add many problem-examples. If you have comments, please let me know (email: amgrigoryan@utsa.edu).

I hope these briefs will help students to understand better many basic topics in S\&S.

Thank you.

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## PART I

## LECTURES

## I. Characteristics of continuous-time systems

We first consider the RC circuit that is defined by the following differential equation of the 1st order

$$
\begin{equation*}
C \frac{d v(t)}{d t}+\frac{1}{R} v(t)=0 . \tag{1}
\end{equation*}
$$

The solution of this equation is

$$
v(t)=v_{0} e^{-\frac{t}{R C}}, \quad v_{0}=v(0)
$$

where $\tau=R C$ is the time constant for this exponentially decaying function. The voltage across the capacitor decays exponentially with time at a rate determined by the time constant.

Let us consider the case when the time constant is 1 s . Our goal is to determine the voltage across the capacitor $y(t)=v(t)$ from the following input voltage

$$
\begin{equation*}
x(t)=e^{-4 t}[u(t)-u(t-3)] . \tag{2}
\end{equation*}
$$

The impulse response of the RC system is $h(t)=e^{-t} u(t)$, therefore

$$
\begin{aligned}
y(t) & =x(t) * h(t)=\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d \tau=\int_{0}^{3 \mid t} e^{-4 \tau} e^{-(t-\tau)} d \tau \\
& =\int_{0}^{3 \mid t} e^{-(t+3 \tau)} d \tau=e^{-t} \int_{0}^{3 \mid t} e^{-3 \tau} d \tau=\left.e^{-t} \frac{1}{-3} e^{-3 \tau}\right|_{0} ^{3 \mid t}
\end{aligned}
$$

and finally

$$
y(t)= \begin{cases}0, & t<0  \tag{3}\\ \frac{1}{3} e^{-t}\left(1-e^{-3 t}\right), & t \in[0,3] \\ \frac{1}{3} e^{-t}\left(1-e^{-9}\right), & t>3\end{cases}
$$

Example 1: For a given input $x(t)$ and impulse function $h(t)$ of the LTI system, let us determine the response of the system,

$$
\begin{aligned}
& x(t)=2 u(t-1)-2 u(t-3) \\
& h(t)=u(t+1)-2 u(t-1)+u(t-3) .
\end{aligned}
$$

We recall here the following properties of the convolution

$$
\begin{aligned}
u\left(t-t_{0}\right) * u(t) & =\left(t-t_{0}\right) u\left(t-t_{0}\right) \\
u\left(t-\left[t_{1}+t_{2}\right]\right) * u(t) & =u\left(t-t_{1}\right) * u\left(t-t_{2}\right)
\end{aligned}
$$

for ant $t_{0}, t_{1}$, and $t_{2}$. Therefore, the following calculations hold

$$
\begin{aligned}
y(t)= & {[2 u(t-1)-2 u(t-3)] *[u(t+1)-2 u(t-1)+u(t-3)] } \\
= & 2 t u(t)-4(t-2) u(t-2)+2(t-4) u(t-4) \\
& -2(t-2) u(t-2)+4(t-4) u(t-4)-2(t-6) u(t-6) \\
= & 2 t u(t)-6(t-2) u(t-2)+6(t-4) u(t-4)-2(t-6) u(t-6)
\end{aligned}
$$

Thus

$$
y(t)= \begin{cases}0, & t<0  \tag{4}\\ 2 t, & t \in[0,2] \\ 12-4 t, & t \in[2,4] \\ 2 t-12, & t \in[4,6] \\ 0, & t>6\end{cases}
$$

. 1 Impulse response
We consider the concept of linear time-invariant system in the continuous-time case. The continuous function (continuous-time signal) $f(t)$ can be written as

$$
\begin{equation*}
f(t)=\int_{-\infty}^{+\infty} \delta(t-s) f(s) d s, \quad t \in(-\infty,+\infty) \tag{5}
\end{equation*}
$$

Therefore, the response of the linear time-invariant system $L[f]$ at point $t$ can be written by formal calculations as

$$
\begin{align*}
(L[f])(t) & =L\left[\int_{-\infty}^{+\infty} \delta(t-s) f(s) d s\right]=\int_{-\infty}^{+\infty} L[\delta(t-s) f(s)] d s \\
& =\int_{-\infty}^{+\infty} f(s)(L[\delta])(t-s) d s \\
& =\int_{-\infty}^{+\infty} f(t-s)(L[\delta](s)) d s . \tag{6}
\end{align*}
$$

The function $h(s)=L[\delta](s)$ is called the impulse response function of the system $L$, which is the output (response) of the system when the impulse delta function $x(t)=\delta(t)$ is input,

$$
\begin{equation*}
\int_{-\infty}^{\infty} h(t-s) \delta(s) d s=h(t-0)=h(t) \tag{7}
\end{equation*}
$$

. 2 Step response
We define the unit step function $u(t)$ as follows

$$
u(t)= \begin{cases}1, & t>0  \tag{8}\\ 1 / 2, & t=0 \\ 0, & t<0\end{cases}
$$

The response of the system on the input which is the unit step function $u(t)$ is called a step response

$$
x(t)=u(t) \rightarrow s(t)=y(t)
$$

Example 2: Consider the step response of the RC circuit described by the differential equation of order one. The impulse response function of the RC circuit is

$$
\begin{equation*}
h(t)=\frac{1}{R C} e^{-\frac{t}{R C}} u(t) . \tag{9}
\end{equation*}
$$

Then

$$
\begin{aligned}
s(t) & =h(t) * u(t)=\int_{-\infty}^{\infty} \frac{1}{R C} e^{-\frac{\tau}{R C}} u(\tau) u(t-\tau) d \tau \\
& =u(t) \int_{0}^{t>0} \frac{1}{R C} e^{-\frac{\tau}{R C}} d \tau= \begin{cases}0, & t \leq 0 \\
\left(1-e^{-\frac{t}{R C}}\right), & t>0 .\end{cases}
\end{aligned}
$$

The step response as the impulse response function describes uniquely the LTI system. Moreover, a simple relation exists between these two functions. Indeed, defining for positive $T$ the functions

$$
u_{T}(t)= \begin{cases}1, & t>T / 2 \\ \frac{t}{T}+\frac{1}{2}, & t \in[-T / 2, T / 2]\end{cases}
$$

we can see that

$$
\frac{d u_{T}(t)}{d t} \rightarrow \delta(t), \quad T \rightarrow 0
$$

Assuming $u_{T}(t) \rightarrow u(t)$ [for that we need define $u(0)=1 / 2$ ], we have

$$
\begin{gather*}
\frac{d u(t)}{d t}=\delta(t), \quad \rightarrow \quad u(t)=\int_{-\infty}^{t} \delta(\tau) d \tau \\
s(t)=h(t) * u(t)=\int_{-\infty}^{t} h(\tau) d \tau \quad \rightarrow \quad h(t)=\frac{d s(t)}{d t} . \tag{10}
\end{gather*}
$$

For instance, the impulse response function of the RC circuit can be calculated as

$$
h(t)=\frac{d s(t)}{d t}=\left(1-e^{-\frac{t}{R C}}\right)^{\prime} u(t)=\frac{1}{R C} e^{-\frac{t}{R C}} u(t) .
$$

. 3 LTI systems
We now write the linear time-invariant system $L$ as

$$
\begin{equation*}
y(t)=\int_{-\infty}^{\infty} h(\tau) x(t-\tau) d \tau \tag{11}
\end{equation*}
$$

Equation 11 is the linear convolution of the input signal, $x(t)$, with the function $h(t)$

$$
y(t)=(h * x)(t)=h(t) * x(t)=x(t) * h(t) .
$$

The response of the LTI system to a complex exponential input is the same exponential input with modified amplitude. Indeed, consider the exponential function with parameter $s, x(t)=$ $x_{s}(t)=\exp (s t)$. The output signal for this system is defined as follows

$$
\begin{aligned}
y(t) & =\int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d \tau=e^{s t} \int_{-\infty}^{+\infty} h(\tau) e^{-s \tau} d \tau \\
& =e^{s t} H(s)=x_{s}(t) H(s)
\end{aligned}
$$

where $H(s)$ is the transfer function of the system. The exponential input is modified by the amount $H(s)$ at the output of the linear system.

It directly follows from the above equation, that if an input is composed by linear combination of two sinusoidal signals, then the output is also the linear combination of two such signals

$$
\begin{align*}
x(t) & =k_{1} x_{s_{1}}(t)+k_{2} x_{s_{2}}(t)  \tag{12}\\
y(t) & =k_{1} H\left(s_{1}\right) x_{s_{1}}(t)+k_{2} H\left(s_{2}\right) x_{s_{2}}(t) . \tag{13}
\end{align*}
$$

We can see that only amplitudes of inputs signals are changed, when they pass the linear system.
In general, for a linear combination of exponential inputs, the response of the system is defined as

$$
x(t)=\sum_{k} c_{k} e^{s_{k} t} \rightarrow y(t)=\sum_{k} c_{k} H\left(s_{k}\right) e^{s_{k} t} .
$$

Example 3: Consider the system that is described as

$$
x(t) \rightarrow y(t)=x(t-5)+x(t+2)
$$

Then, the response on the exponential signal is defined as

$$
x_{s}(t)=e^{s t} \rightarrow e^{s(t-5)}+e^{s(t+2)}=e^{s t}\left(e^{-5 s}+e^{2 s}\right)
$$

and the corresponding frequency response is

$$
\begin{equation*}
H(s)=e^{-5 s}+e^{2 s} . \tag{14}
\end{equation*}
$$

There is a simple relation between the impulse and frequency response functions

$$
h(t) * x(t)=H(s) x(t) .
$$

We have

$$
h(t)=\delta(t-5)+\delta(t+2)
$$

and using the convolution operation, we obtain

$$
\begin{aligned}
e^{s t} & \rightarrow e^{s t} * \delta(t-5)+e^{s t} * \delta(t+2)=e^{s(t-5)} * \delta(t)+e^{s(t+2)} * \delta(t) \\
& =e^{s(t-5)}+e^{s(t+2)}=e^{s t}\left[e^{-5 s}+e^{2 s}\right]=H(s) x(t) .
\end{aligned}
$$

We now consider the sinusoidal signal $x(t)=\cos (2 t)$. Then, the following holds

$$
\begin{aligned}
x(t) & =\frac{1}{2}\left[e^{j 2 t}+e^{-j 2 t}\right] \rightarrow y(t)=\frac{1}{2}\left[H(2 j) e^{j 2 t}+H(-2 j) e^{-j 2 t}\right] \\
& =\operatorname{Re} H(2 j) e^{j 2 t}=\operatorname{Re}\left[\left(e^{-10 j}+e^{4 j}\right) e^{j 2 t}\right]=\cos (2 t-10)+\cos (2 t+4) \\
\cos (2 t) & \rightarrow \cos (2 t-10)+\cos (2 t+4) .
\end{aligned}
$$

In general, for a complex exponential signal $x_{\omega}(t)=e^{j \omega t}$, the output signal for the LTI system is defined as follows

$$
\begin{equation*}
y(t)=\int_{-\infty}^{+\infty} h(\tau) e^{j \omega(t-\tau)} d \tau=e^{j \omega t} \int_{-\infty}^{+\infty} h(\tau) e^{-j \omega \tau} d \tau=e^{j \omega t} H(\omega)=x_{\omega}(t) H(\omega) \tag{15}
\end{equation*}
$$

where $H(\omega)$ is the Fourier transform of $h(t)$. The amplitude of the input sinusoid of frequency $\omega$ is only changed by the amount $H(\omega)$ at the output of the system.

Because of linearity of the system, for any real inputs $x_{1}(t)$ and $x_{2}(t)$, the response of a linear combination of inputs is the linear combination of their outputs. Therefore

$$
\begin{array}{lll}
x_{1}(t) & +j \cdot & x_{2}(t)=x(t)=e^{j \omega_{0} t} \\
\downarrow L & & \downarrow L \\
& \downarrow L \\
y_{1}(t) & +j \cdot & y_{2}(t)=y(t)=H\left(\omega_{0}\right) e^{j \omega_{0} t}=\left|H\left(\omega_{0}\right)\right| e^{j\left(\omega_{0} t+\theta\right)}
\end{array}
$$

and

$$
\begin{aligned}
\cos \left(\omega_{0} t\right) & =x_{1}(t)=\operatorname{Re} x(t) \rightarrow y_{1}(t)=\operatorname{Re} y(t)=\operatorname{Re} H\left(\omega_{0}\right) e^{j \omega_{0} t} \\
& =\operatorname{Re}\left[\left|H\left(\omega_{0}\right)\right| e^{j \theta}\right] e^{j \omega_{0} t}=\left|H\left(\omega_{0}\right)\right| \operatorname{Re} e^{j\left(\omega_{0} t+\theta\right)} \\
& =\left|H\left(\omega_{0}\right)\right| \cos \left(\omega_{0} t+\theta\right)
\end{aligned}
$$

where $\theta=\theta\left(\omega_{0}\right)=\arg H\left(\omega_{0}\right)$ is the phase of the frequency response.
Thus, for a given frequency $\omega_{0}$,

$$
\begin{align*}
\cos \left(\omega_{0} t\right) & \rightarrow\left|H\left(\omega_{0}\right)\right| \cos \left(\omega_{0} t+\theta\right), \quad H=(|H|, \theta)  \tag{16}\\
\sin \left(\omega_{0} t\right) & \rightarrow\left|H\left(\omega_{0}\right)\right| \sin \left(\omega_{0} t+\theta\right) .
\end{align*}
$$

Applying this property for Example 3 when the transfer response defined in (14), we first assume that

$$
H(s)=\left[H_{1}(s)=e^{-5 s}\right]+\left[H_{2}(s)=e^{2 s}\right]
$$

and, then, applying (16) for $\omega_{0}=2$, we obtain

$$
\begin{aligned}
\cos (2 t) & \rightarrow\left[\left|e^{-j 5 \cdot 2}\right| \cos (2 t+(-5) \cdot 2)\right]+\left[\left|e^{j 2 \cdot 2}\right| \cos (2 t+2 \cdot 2)\right] \\
& =\cos (2 t-10)+\cos (2 t+4) \\
& =[2 \cos (7)] \cos (2 t-3)=|H(2)| \cos (2 t+\theta) .
\end{aligned}
$$

Figure 1 shows the input and response of the system.
In the complex case of parameter $s=j \omega$, the frequency response $H(\omega)$ of a LTI system is defined as follows

$$
\begin{gather*}
x(t)=e^{j \omega t} \rightarrow y(t)=e^{j \omega t} \int_{-\infty}^{+\infty} h(\tau) e^{-j \omega \tau} d \tau=H(\omega) e^{j \omega t} \\
H(\omega)=\int_{-\infty}^{+\infty} h(\tau) e^{-j \omega \tau} d \tau \tag{17}
\end{gather*}
$$



Fig. 1. (a) Input signal $x(t)=\cos (2 t)$. (b) Response of the system on the signal.

Absolute value of frequency response is called the magnitude response and the phase of the frequency response is called the phase response.

Example 4: Consider the system with the following impulse response function

$$
h(t)=\frac{1}{2}[\delta(t-1)+\delta(t)+\delta(t+1)] .
$$

The frequency response of the system is calculated as follows

$$
\begin{aligned}
H(\omega) & =\frac{1}{2} \int_{-\infty}^{+\infty}[\delta(\tau-1)+\delta(\tau)+\delta(\tau+1)] e^{-j \omega \tau} d \tau \\
& =\frac{1}{2}\left[e^{-j \omega}+1+e^{j \omega}\right]=\frac{1}{2}+\cos (\omega) .
\end{aligned}
$$

The impulse response is the periodic, symmetric (even) function and the frequency response is real function, and $\arg H(\omega)=0$ for $\omega \in[-2 \pi / 3,2 \pi / 3]$, and $\arg H(\omega)= \pm \pi$ for $|\omega| \in(2 \pi / 3, \pi)$.

If we consider the impulse response

$$
h(t)=\frac{1}{2}[\delta(t-1)+\delta(t)]
$$

then the frequency response

$$
H(\omega)=\frac{1}{2}\left[e^{-j \omega}+1\right]=\frac{1}{2} e^{-j \frac{\omega}{2}}\left[e^{j \frac{\omega}{2}}+e^{-j \frac{\omega}{2}}\right]=e^{-j \frac{\omega}{2}} \cos \left(\frac{\omega}{2}\right) .
$$

Therefore, $|H(\omega)|=|\cos (\omega / 2)|$ and $\arg H(\omega)=\omega / 2$ or $\omega / 2-\pi$, depending on the sign of $\cos (\omega / 2)$.
Example 5: Consider the RC circuit with the impulse response

$$
h(t)=\frac{1}{R C} e^{-\frac{t}{R C}} u(t) .
$$

Then

$$
\begin{aligned}
H(\omega) & =\int_{0}^{+\infty} \frac{1}{R C} e^{-\frac{t}{R C}} e^{-j \omega t} d t=\frac{1}{R C} \int_{0}^{+\infty} e^{-\left[j \omega+\frac{1}{R C}\right] t} d t \\
& =\frac{1}{R C} \cdot \frac{1}{j \omega+\frac{1}{R C}}(1-0)=\frac{\frac{1}{R C}}{j \omega+\frac{1}{R C}}=\frac{1}{j(R C) \omega+1}
\end{aligned}
$$

and magnitude and phase frequency responses are

$$
\begin{aligned}
|H(\omega)| & =\sqrt{\frac{\left(\frac{1}{R C}\right)^{2}}{\omega^{2}+\left(\frac{1}{R C}\right)^{2}}}=\sqrt{\frac{1}{(R C \omega)^{2}+1}} \\
\arg H(\omega) & =-\tan ^{-1}[R C \omega] .
\end{aligned}
$$

Figure 2 shows the magnitude and phase responses of the RC circuit, for the cases when $R C=1$ and $R C=4$.

Example 6 (Integrator) Consider a causal system $H$ which is described as

$$
H: x(t) \rightarrow y(t)=\int_{-\infty}^{t} x(\tau) d \tau, \quad t \in(-\infty,+\infty)
$$

$H$ is linear time invariant system, since for a given $t_{0}$, we have the following

$$
H: x\left(t-t_{0}\right) \rightarrow \int_{-\infty}^{t} x\left(\tau-t_{0}\right) d \tau=\int_{-\infty}^{t-t_{0}} x\left(\tau^{\prime}\right) d \tau^{\prime}=y\left(t-t_{0}\right) .
$$

The impulse response function of the system is $h(t)=u(t)($ if $t \neq 0)$.
If $x(t)=C e^{j \omega t}, \omega \neq 0$, then the response of the system on $x(t)$ at point $t \neq 0$ is defined as

$$
\begin{aligned}
y(t) & =\int_{-\infty}^{t} C e^{j \omega \tau} d \tau=C \int_{-\infty}^{t} d e^{j \omega \tau} \cdot \frac{1}{j \omega} \\
& =C \frac{1}{j \omega} e^{j \omega t}=\frac{1}{j \omega} C e^{j \omega t}=\frac{1}{j \omega} x(t) .
\end{aligned}
$$



Fig. 2. (a) Magnitude frequency responses. (b) Phase responses of the system.
So, we can see that response function $H(\omega)=1 /(j \omega)$ (but not the Fourier transform of the $u(t)$ function). Therefore, the response of the system to the linear combination of exponential signals is defined as

$$
x(t)=c_{1} e^{j \omega_{1} t}+c_{2} e^{j \omega_{2} t} \rightarrow y(t)=\frac{1}{j \omega_{1}} c_{1} e^{j \omega_{1} t}+\frac{1}{j \omega_{2}} c_{2} e^{j \omega_{2} t}
$$

and in the general case

$$
y(t)=\frac{1}{j \omega_{1}} c_{1} e^{j \omega_{1} t}+\frac{1}{j \omega_{2}} c_{2} e^{j \omega_{2} t}+\pi\left[c_{1} \delta\left(\omega_{1}\right)+c_{2} \delta\left(\omega_{2}\right)\right] .
$$

Example 7 (Time invariance) In many practical applications, a real physical system (circuit) can be modeled as a LTI system under a specific conditions. We can consider as an example, the simple resistor network consisting of two resistors that vary with time. The resistors are connected cascade-wise, and the voltage across the second resistor can be defined as

$$
y(t)=\frac{r_{2}(t)}{r_{1}(t)+r_{2}(t)} x(t)
$$

where $x(t)$ is an applied voltage.
This network can be considered to be a LTI system $x(t) \rightarrow y(t)$ only if

$$
\frac{r_{1}(t)}{r_{2}(t)}=\text { cons. }
$$

## . 4 n-order LTI systems

We now consider a general $n$ th-order linear system $H: x(t) \rightarrow y(t)$ that is described by the $n$-order linear differential equation with constant coefficients,

$$
\begin{align*}
a_{0} y(t) & +a_{1} y^{(1)}(t)+a_{2} y^{(2)}(t)+\ldots+a_{n} y^{(n)}(t) \\
& =b_{0} x(t)+b_{1} x^{(1)}(t)+b_{2} x^{(2)}(t)+\ldots+b_{m} x^{(m)}(t) \tag{18}
\end{align*}
$$

with coefficients $a_{k}$ and $b_{l},(k=0: n, l=0: m) . y^{(k)}$ and $x^{(l)}$ denote respectively the derivatives $d^{k} y(t) / d t^{k}$ and $d^{l} x(t) / d t^{l}$. We consider the exponential input function with amplitude $X$,

$$
x(t)=X e^{s t}, \quad t \in(-\infty,+\infty)
$$

and output in the form

$$
y(t)=H(s) x(t), \quad t \in(-\infty,+\infty)
$$

where $H(s)$ is a function of $s$.
Since

$$
x^{(l)}(t)=s^{l} x(t)=s^{l} e^{s t}, \quad l=0: m
$$

and

$$
y^{(k)}(t)=H(s) x^{(k)}(t)=H(s) s^{k} x(t)=s^{k} y(t)
$$

for $\mathrm{k}=0, \ldots, \mathrm{n}$, we can write Eq. 18 as

$$
\begin{equation*}
\left[a_{0}+a_{1} s+a_{2} s^{2}+\ldots+a_{n} s^{n}\right] y(t)=\left[b_{0}+b_{1} s+b_{2} s^{2}+\ldots+b_{m} s^{m}\right] x(t) \tag{19}
\end{equation*}
$$

And finally

$$
\begin{equation*}
y(t)=\frac{b_{0}+b_{1} s+b_{2} s^{2}+\ldots+b_{m} s^{m}}{a_{0}+a_{1} s+a_{2} s^{2}+\ldots+a_{n} s^{n}} x(t) . \tag{20}
\end{equation*}
$$

Therefore the $n$ th-order linear system $H: x(t) \rightarrow y(t)$, that has been defined by the $n$-order linear differential equation with constant coefficients, is described by the transform function

$$
\begin{equation*}
H(s)=\frac{b_{0}+b_{1} s+b_{2} s^{2}+\ldots+b_{m} s^{m}}{a_{0}+a_{1} s+a_{2} s^{2}+\ldots+a_{n} s^{n}} \tag{21}
\end{equation*}
$$

when inputs are exponential functions.
The following formula for computing the response function of the system by the impulse function:

$$
\begin{equation*}
H(s)=\int_{-\infty}^{+\infty} e^{-s \tau} h(\tau) d \tau \tag{22}
\end{equation*}
$$

Thus, the linear convolution system can be represented by two ways:

$$
\begin{array}{ll}
x(t) \longrightarrow y(t)=x(t) * h(t) \quad \text { (integral representation) } \\
x(t) \longrightarrow y(t)=H(s) x(t) \quad \text { (polynomial representation). }
\end{array}
$$

$$
\begin{equation*}
C \frac{d y(t)}{d t}+\frac{1}{R} y(t)=x(t) \tag{23}
\end{equation*}
$$

where we denote $y(t)=v(t)$ and $x(t)=i(t)$. The impulse response of the RC system is

$$
\begin{equation*}
h(t)=\frac{1}{C} e^{-\frac{t}{R C}} u(t) \tag{24}
\end{equation*}
$$

1. (Laplace transform) The transfer function is

$$
H(s)=\frac{1}{C s+\frac{1}{R}}=\frac{\frac{1}{C}}{s+\frac{1}{C R}} \rightarrow h(t)=\frac{1}{C} e^{-\frac{t}{R C}} u(t) .
$$

2. (Integrator) By multiplying both sides of (23) by $\exp (t / R C)$, we obtain

$$
\begin{aligned}
e^{\frac{t}{R C}} \frac{d y(t)}{d t}+e^{\frac{t}{R C}} \frac{1}{C R} y(t) & =\frac{1}{C} e^{\frac{t}{R C}} x(t) \\
\frac{d}{d t}\left[e^{\frac{t}{R C}} y(t)\right] & =\frac{1}{C} e^{\frac{t}{R C}} x(t)
\end{aligned}
$$

So

$$
\begin{aligned}
e^{\frac{t}{R C}} y(t) & =\int_{-\infty}^{t} \frac{1}{C} e^{\frac{\tau}{R C}} x(\tau) d \tau= \\
y(t) & =\int_{-\infty}^{\infty} \frac{1}{C} e^{-\frac{(t-\tau)}{R C}} u(t-\tau) x(\tau) d \tau=h(t) * x(t)
\end{aligned}
$$

3. (Delta function) The impulse response is the output of the system when the input is $\delta(t)$

$$
x(t)=\delta(t) \rightarrow y(t)=h(t) .
$$

We show that $h(t)$ in (24) is the solution of

$$
C \frac{d y(t)}{d t}+\frac{1}{R} y(t)=\delta(t)
$$

Indeed, for any continuous at point $t=0$ function $f(t)$, the following calculations hold

$$
\begin{gathered}
\int_{-\infty}^{\infty}\left[C \frac{d h(t)}{d t}+\frac{1}{R} h(t)\right] f(t) d t \\
\int_{-\infty}^{\infty}\left[\begin{array}{c}
\left.C\left[\frac{1}{C}\left(-\frac{1}{R C}\right) e^{-\frac{t}{R C}} u(t)+\frac{1}{C} e^{-\frac{t}{R C}} \frac{d u(t)}{d t}\right]\right] f(t) d t \\
\\
+\frac{1}{R} \frac{1}{C} e^{-\frac{t}{R C}} u(t)
\end{array}\right. \\
\int_{-\infty}^{\infty} C\left[\frac{1}{C} e^{-\frac{t}{R C}} \frac{d u(t)}{d t}\right] f(t) d t=\int_{-\infty}^{\infty}\left[e^{\left.-\frac{t}{R C} \delta(t)\right] f(t) d t}\right. \\
\quad=\int_{-\infty}^{\infty}\left[e^{-\frac{t}{R C}} f(t)\right] \delta(t) d t=f(0) .
\end{gathered}
$$

II. State-variable description for continuous LTI systems

In this section, we consider state-variable description (SVD) of a LTI system, namely a model which is specified in terms of a set of variables that describe the internal state of the system. We derive such model in terms of matrix equations that correspond to a system of coupled first order differential equations. That system will also describe the state of the system to the current input and output. A state can be defined as a minimal number of signals that represent the system's memory of the past.

We consider a LTI system, $L: v(t) \rightarrow y(t)$, that is described by a differential equation of order $n$. The following notations are used for SVD of the system

$$
x(t)=\left[\begin{array}{c}
x_{1}(t)  \tag{25}\\
x_{2}(t) \\
\vdots \\
x_{n}(t)
\end{array}\right], \quad \dot{x}(t)=\left[\begin{array}{c}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t) \\
\vdots \\
\dot{x}_{n}(t)
\end{array}\right] .
$$

The components $x_{k}(t)$ are called state variables of the systems. The $n$-dimensional vector function $x(t)$ is called $a$ state trajectory (or state response) of the system. The component of the second vector are derivatives of $x_{k}(t)$, i.e. $\dot{x}_{k}(t)=x_{k}^{\prime}(t), k=1: n$. The input of the system will be denoted by $v(t)$.

The general state equations of the SISO system with state $x(t)$, relative to the input $v(t)$ and output $y(t)$ at time $t$, can be written as

$$
\begin{align*}
\dot{x}(t) & =F(x(t), v(t), t)  \tag{26}\\
y(t) & =G(x(t), v(t), t) \tag{27}
\end{align*}
$$

where $F$ and $G$ are functions to be found. These equations define the state model of the system. This is a time-domain model with two parts connected in cascade form as shown in Fig. 3.


Fig. 3. Structure of the state model of a LTI system.
The first equation describes the state response $x(t)$ resulting from application of an input signal $v(t)$. The second equation describes the response of the system with the given state and input.

We focus only on the simple form of the above equations, namely linear form

$$
\begin{align*}
\dot{x}(t) & =A(t) x(t)+B(t) v(t)  \tag{28}\\
y(t) & =C(t) x(t)+D(t) v(t) \tag{29}
\end{align*}
$$

Example 1: Consider the system with the differential equation of order one

$$
\dot{y}(t)+3 y(t)=4 v(t) .
$$

Introducing the variable $x(t)=y(t)$, this equation can be written as the following system

$$
\left\{\begin{array}{l}
\dot{x}(t)=-3 x(t)+4 v(t) \\
y(t)=x(t)
\end{array}\right.
$$

In this case $A(t)=-3, B(t)=4, C(t)=1$, and $D(t)=0$.
Example 2: Consider the system with the differential equation of order 2

$$
\ddot{y}(t)-2 \dot{y}(t)+3 y(t)=4 v(t) .
$$

Introducing the variable

$$
x_{1}(t)=y(t), \quad x_{2}(t)=\dot{x}_{1}(t)
$$

the above differential equation can be written as the following system

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=x_{2}(t) \\
\dot{x}_{2}(t)=-3 x_{1}(t)+2 x_{2}(t)+4 v(t) \\
y(t)=x_{1}(t) .
\end{array}\right.
$$

The first two equations define the state response $x(t)$ of the system, and the last equation describes the system response $y(t)$. In the terms of matrix

$$
\begin{aligned}
\dot{x}(t) & =\left[\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t)
\end{array}\right]=\left[\begin{array}{rr}
0 & 1 \\
-3 & 2
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{l}
0 \\
4
\end{array}\right] v(t) \\
y(t) & =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]
\end{aligned}
$$

In this case

$$
A(t)=\left[\begin{array}{rr}
0 & 1 \\
-3 & 2
\end{array}\right], \quad B(t)=\left[\begin{array}{l}
0 \\
4
\end{array}\right], \quad C(t)=\left[\begin{array}{ll}
1 & 0
\end{array}\right], \quad D(t)=0 .
$$

Thus, the following substitution have been done for the given differential equation

| $\ddot{y}(t)$ | $-2 \cdot$ | $\dot{y}(t)$ | $+3 \cdot$ | $y(t)$ | $=$ | $4 v(t)$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :--- | :--- |
|  |  | $\downarrow$ |  | $\downarrow$ |  |  |  |
|  |  | $x_{2}(t)$ |  | $x_{1}(t)$ | $\rightarrow$ |  | $\dot{x}_{1}(t)=x_{2}(t)$ |
| $\downarrow$ | $\downarrow$ |  | $\downarrow$ |  |  |  |  |
| $\dot{x}_{2}(t)$ | $-2 \cdot$ | $x_{2}(t)$ | $+3 \cdot$ | $x_{1}(t)$ | $=$ | $4 v(t)$ |  |

It should be noted that the state-variables for the state-model of the system can be defined in different way. In other words, there is no unique state-variable description of the system. For instance, consider the following state-model

$$
\begin{array}{|cccccccl|}
\hline \ddot{y}(t) & -2 \cdot & \dot{y}(t) & +3 \cdot & y(t) & = & 4 v(t) & \\
& & \downarrow & & \downarrow & & & \\
& & x_{1}(t) & & x_{2}(t) & \rightarrow & & \dot{x}_{2}(t)=x_{1}(t) \\
\downarrow & \downarrow & & \downarrow & & & & \\
\dot{x}_{1}(t) & -2 . & x_{1}(t) & +3 \cdot & x_{2}(t) & = & 4 v(t) & \\
\hline
\end{array}
$$

Then, in the matrix form the state-model can be written as

$$
\begin{aligned}
& \dot{x}(t)=\left[\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t)
\end{array}\right]=\left[\begin{array}{rr}
2 & -3 \\
1 & 0
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{l}
4 \\
0
\end{array}\right] v(t) \\
& y(t)=\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]
\end{aligned}
$$

One can note the change in matrices

$$
A(t)=\left[\begin{array}{rr}
0 & 1 \\
-3 & 2
\end{array}\right] \rightarrow\left[\begin{array}{rr}
2 & -3 \\
1 & 0
\end{array}\right], \quad b(t)=\left[\begin{array}{l}
0 \\
4
\end{array}\right] \rightarrow\left[\begin{array}{l}
4 \\
0
\end{array}\right], \quad C(t)=\left[\begin{array}{ll}
1 & 0
\end{array}\right] \rightarrow\left[\begin{array}{ll}
0 & 1
\end{array}\right]
$$

by rules $a_{n, k} \rightarrow a_{2-n, 2-k}, b_{k} \rightarrow b_{2-k}$, and $c_{n} \rightarrow c_{2-n}$, respectively.
Example 3: Consider the RLC circuit that is described by the differential equation of order 2

$$
\begin{aligned}
L \frac{d i(t)}{d t}+R i(t)+v_{c}(t) & =v_{i}(t) \\
v_{c}(t) & =\frac{1}{C} \int_{-\infty}^{t} i(\tau) d \tau
\end{aligned}
$$

where $v_{i}(t)$ is the input voltage and $v_{c}(t)$ is the capacitor voltage (output).
The substitutions are the following for this system

$$
\begin{array}{|cccccllll|}
\hline L \frac{d i(t)}{d t} & +R & i(t) & + & v_{c}(t) & = & v_{i}(t) & \\
& & \downarrow & & \downarrow & & & \\
& & x_{2}(t) & & x_{1}(t) & \rightarrow & & \dot{x}_{1}(t)=\frac{1}{C} x_{2}(t) \\
\downarrow & \downarrow & & & & & & \\
L \dot{x}_{2}(t) & +R & & x_{2}(t) & + & x_{1}(t) & & & v_{i}(t) \\
\hline
\end{array}
$$

Thus, we write the above system of equations as

$$
\begin{align*}
\dot{x}_{1}(t) & =\frac{1}{C} x_{2}(t)  \tag{30}\\
\dot{x}_{2}(t) & =-\frac{1}{L} x_{1}(t)-\frac{R}{L} x_{2}(t)+\frac{1}{L} v_{i}(t)
\end{align*}
$$

where $x_{1}(t)=v_{c}(t)$ and $x_{2}(t)=i(t)$. In the matrix form the above equations can be written as

$$
\begin{align*}
\dot{x}(t) & =\left[\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t)
\end{array}\right]=\left[\begin{array}{rr}
0 & \frac{1}{C} \\
-\frac{1}{L} & -\frac{R}{C}
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{c}
0 \\
\frac{1}{L}
\end{array}\right] v_{i}(t)  \tag{31}\\
y(t) & =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right] \tag{32}
\end{align*}
$$

## State response (rule):

In the general case of the differential equation of order $n$

$$
L_{y}\left(a_{n}, a_{n-1}, \ldots, a_{1}, a_{0}\right)(t)=L_{v}\left(b_{n}, b_{n-1}, \ldots, b_{1}, b_{0}\right)(t)
$$



Fig. 4. Diagram of realization of the system with state variables.
where (we assume $a_{n}=1$ )

$$
\begin{aligned}
& L_{y}\left(1, a_{n-1}, \ldots, a_{1}, a_{0}\right)=y^{(n)}+a_{n-1} y^{(n-1)}+\ldots+a_{1} y^{\prime}+a_{0} y \\
& L_{v}\left(b_{n}, b_{n-1}, \ldots, b_{1}, b_{0}\right)=b_{n} v^{(n)}+b_{n-1} v^{(n-1)}+\ldots+b_{1} v^{\prime}+b_{0} v
\end{aligned}
$$

the following substitutions can be performed

| $y^{(n)}$ | $+a_{n-1}$. | $\begin{gathered} y^{(n-1)} \\ \downarrow \\ x_{n} \end{gathered}$ | $+a_{n-2}$. | $\begin{gathered} y^{(n-2)} \\ \downarrow \\ x_{n-1} \end{gathered}$ | $+\ldots+a_{2}$ | $\begin{gathered} y^{(2)} \\ \downarrow \\ x_{3} \end{gathered}$ | $+a_{1}$. | $\begin{gathered} y^{\prime} \\ \downarrow \\ x_{2} \end{gathered}$ | $+a_{0}$. | $\begin{gathered} y \\ \downarrow \\ x_{1} \end{gathered}$ | $=v(t)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\downarrow$ | $\downarrow$ |  | $\downarrow$ |  | $\downarrow$ |  | $\downarrow$ |  | $\downarrow$ |  |  |
| $\dot{x}_{n}(t)$ | $+a_{n-1}$. | $x_{n}(t)$ | $+a_{n-2}$. | $x_{n-1}(t)$ | $+\ldots+a_{2}$. | $x_{3}(t)$ | $+a_{1}$. | $x_{2}(t)$ | $+a_{0}$. | $x_{1}(t)$ | $=v(t)$ |

Therefore, we have the following system of equations for state variables

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=x_{2}(t) \\
\dot{x}_{2}(t)=x_{3}(t) \\
\dot{x}_{3}(t)=x_{4}(t) \\
\quad \ldots \\
\dot{x}_{n-2}(t)=x_{n-1}(t) \\
\dot{x}_{n-1}(t)=x_{n}(t) \\
\dot{x}_{n}(t)=-a_{0} x_{1}(t)-a_{1} x_{2}(t)-a_{2} x_{3}(t)-\ldots-a_{n-2} x_{n-1}(t)-a_{n-1} x_{n}(t)+v(t)
\end{array}\right.
$$

In the matrix form, the state-variable description of the system is defined by

$$
\dot{x}(t)=\left[\begin{array}{ccccc}
0 & 1 & & &  \tag{33}\\
& 0 & 1 & & \\
& & 0 & \ldots & \\
& & & 0 & 1 \\
-a_{0} & -a_{1} & -a_{2} & \ldots & -a_{n-1}
\end{array}\right] x(t)+\left[\begin{array}{c}
0 \\
0 \\
\ldots \\
0 \\
1
\end{array}\right] v(t)
$$

and

$$
y(t)=\left[\begin{array}{lllll}
b_{0}-a_{0} b_{n} & b_{1}-a_{1} b_{n} & b_{2}-a_{2} b_{n} & \ldots & b_{n-1}-a_{n-1} b_{n}
\end{array}\right]\left[\begin{array}{c}
x_{1}(t) \\
x_{2}(t) \\
x_{3}(t) \\
\ldots \\
x_{n}(t)
\end{array}\right]+b_{n} v(t)
$$

Note that another state model is described by the state-variables defined as follows:


Therefore, we have the following system of equations for new state variables

$$
\left\{\begin{array}{l}
\dot{x}_{1}(t)=-a_{n-1} x_{1}(t)-a_{n-2} x_{2}(t)-a_{n-3} x_{3}(t)-\ldots-a_{1} x_{n-1}(t)-a_{0} x_{n}(t)+v(t) \\
\dot{x}_{2}(t)=x_{1}(t) \\
\dot{x}_{3}(t)=x_{2}(t) \\
\quad \ldots \\
\dot{x}_{n-2}(t)=x_{n-3}(t) \\
\dot{x}_{n-1}(t)=x_{n-2}(t) \\
\dot{x}_{n}(t)=x_{n-1}(t) \quad \quad\left[x_{n}(t)=y(t)\right] .
\end{array}\right.
$$

In the matrix form, the corresponding state-variable description of the system is defined by

$$
\dot{x}(t)=\left[\begin{array}{ccccc}
-a_{n-1} & -a_{n-2} & \ldots & -a_{1} & -a_{0}  \tag{34}\\
1 & 0 & & & \\
& 1 & 0 & \ldots & \\
& & 1 & 0 & \\
& & & 1 & 0
\end{array}\right] x(t)+\left[\begin{array}{c}
1 \\
0 \\
\ldots \\
0 \\
0
\end{array}\right] v(t)
$$

and

$$
y(t)=\left[\begin{array}{lllll}
b_{n-1}-a_{n-1} & b_{n-2}-a_{n-2} b_{n} & \ldots & b_{1}-a_{1} b_{n} & b_{0}-a_{0} b_{n}
\end{array}\right]\left[\begin{array}{c}
x_{1}(t) \\
x_{2}(t) \\
x_{3}(t) \\
\ldots \\
x_{n}(t)
\end{array}\right]+b_{n} v(t) .
$$

Example 4: Consider the LTI system described by the differential equation of order two

$$
L_{y}(1,3,2)=L_{v}(5,4)
$$

The order of equation is 2 , therefore the state variable consists of two components, and we will use the matrix $(2 \times 2)$ for $A$. The SVD model for this system is defined by

$$
\dot{x}(t)=\left[\begin{array}{rr}
0 & 1 \\
-a_{0} & -a_{1}
\end{array}\right] x(t)+\left[\begin{array}{l}
0 \\
1
\end{array}\right] v(t)=\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right] x(t)+\left[\begin{array}{l}
0 \\
1
\end{array}\right] v(t)
$$

and $\left(b_{2}=0\right)$

$$
y(t)=\left[\begin{array}{ll}
b_{0}-a_{0} b_{2} & b_{1}-a_{1} b_{2}
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+b_{2} v(t)=\left[\begin{array}{ll}
4 & 5
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]
$$

Example 5: Consider the diagram of realization, which is given in Fig. 5. The diagram describes a LTI system described by the differential equation of order two

$$
\begin{equation*}
y^{(2)}(t)=3 y^{\prime}(t)-2 y(t)-v^{\prime}(t)+4 v(t) \tag{35}
\end{equation*}
$$


$y^{(2)}(t)-3 y^{\prime}(t)+2 y(t)=-v^{\prime}(t)+4 v(t)$
Fig. 5. Diagram of realization of the system.
The state-model is defined as follows

$$
\begin{align*}
\dot{x}(t) & =\left[\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t)
\end{array}\right]=\left[\begin{array}{rr}
0 & 1 \\
-2 & 3
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] v(t)  \tag{36}\\
y(t) & =\left[\begin{array}{ll}
4 & -1
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right] \tag{37}
\end{align*}
$$

Example 6: We now consider the system described by differential equation

$$
\begin{equation*}
y^{(2)}(t)=3 y^{\prime}(t)-2 y(t)+2 v^{(2)}(t)-v^{\prime}(t)+4 v(t) \tag{38}
\end{equation*}
$$

The state-response is the same as in (36), one need to change only the formula for response of the system

$$
y(t)=\left[\begin{array}{ll}
4-2 \cdot 2 & -1+3 \cdot 2
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]=\left[\begin{array}{ll}
0 & 5
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]=5 x_{2}(t)
$$

In this case we can simplify the state-model as

$$
\dot{x}_{1}(t)=\left[\begin{array}{ll}
-2 & 3
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]=-2 x_{2}(t)+3 x_{2}(t), \quad y(t)=5 x_{2}(t) .
$$

III. Discrete-time representation of differential equations

In this section, we describe the method of discretization (in time)

$$
t \rightarrow n T, \quad y(t) \rightarrow y(n)=y(n T), \quad x(t) \rightarrow x(n)=x(n T)
$$

of a differential equation with constant coefficients. The goal is to obtain the approximation of the differential equation in the form of the difference equation. The time sampling $T$ is assumed to be small for correct approximation.

## A. The 1st order differential equation

We first consider the differential equation of order 1

$$
\begin{equation*}
y^{\prime}(t)+a y(t)=b x(t) \tag{39}
\end{equation*}
$$

which results in the following evaluation at point $t=n T$

$$
\begin{equation*}
y^{\prime}\left(\left.t\right|_{t=n T}\right)+a y(n)=b x(n), \quad n \in Z . \tag{40}
\end{equation*}
$$

By definition of the derivative, we obtain the approximation

$$
y^{\prime}\left(\left.t\right|_{t=n T}\right)=\lim _{\Delta \rightarrow 0} \frac{y(n T+\Delta)-y(n T)}{\Delta} \approx \frac{y(n T+T)-y(n T)}{T} .
$$

Therefore Eq. 40 can be approximated as

$$
\begin{align*}
\frac{y(n T+T)-y(n T)}{T}+a y(n) & =b x(n) \\
y(n+1)-y(n)+a T y(n) & =b T x(n) \\
y(n+1) & =y(n)[1-a T]+b T x(n)  \tag{41}\\
y(n) & =y(n-1)[1-a T]+b T x(n-1) . \tag{42}
\end{align*}
$$

Let us assume that the initial condition for this equation to be $y(0)$ and let $x(t)=0$ for all $n$. Then,

$$
\begin{aligned}
& y(1)=(1-a T) y(0)+b T x(0)=(1-a T) y(0) \\
& y(2)=(1-a T) y(1)+b T x(1)=(1-a T)^{2} y(0), \ldots
\end{aligned}
$$

and the solution is

$$
\begin{equation*}
y(n)=(1-a T)^{n} y(0), \quad n \geq 0 . \tag{43}
\end{equation*}
$$

The exact solution of (39) is

$$
\begin{equation*}
y_{c}(t)=e^{-a t} y(0) u(t) \tag{44}
\end{equation*}
$$

whose approximation is

$$
\begin{equation*}
y_{c}(n)=y_{c}\left(\left.t\right|_{t=n T}\right)=e^{-a n T} y(0) u(n)=\left(e^{-a T}\right)^{n} y(0) u(n) . \tag{45}
\end{equation*}
$$

For small $T$, the solution in (43) can thus be considered as the accurate approximation of the solution (45). Indeed, according to Taylor's expansion (series) of the exponential function, we have

$$
\begin{aligned}
e^{-a T} & =1-a T+\frac{(a T)^{2}}{2!}-\frac{(a T)^{3}}{3!}+\ldots+(-1)^{n} \frac{(a T)^{n}}{n!}+\ldots \\
& \approx 1-a T, \quad \text { when } T \ll \varepsilon>0 .
\end{aligned}
$$

Example 1: Consider the RC circuit with the input-output differential equation

$$
C \frac{d y(t)}{d t}+\frac{1}{R} y(t)=x(t) \rightarrow \frac{d y(t)}{d t}+\frac{1}{R C} y(t)=\frac{1}{C} x(t)
$$

where $x(t)$ is the current applied to the circuit, $y(t)$ is the voltage across the capacitor. In this case, $\alpha=1 / R C$ and $b=1 / C$, and the solution in (42) is

$$
\begin{equation*}
y(n)=\left(1-\frac{T}{R C}\right) y(n-1)+\frac{T}{C} x(n-1)=A y(n-1)+B x(n-1) \tag{46}
\end{equation*}
$$

where

$$
\begin{equation*}
A=1-\frac{T}{R C}, \quad B=\frac{T}{C} . \tag{47}
\end{equation*}
$$

Example 2: Consider the RC circuit with parameters $C=1 \mathrm{~F}, R=1 \Omega$, and sampling period $T=0.2 \mathrm{~s}$. The difference equation 46 takes the form

$$
y(n)=\frac{4}{5} y(n-1)+\frac{1}{5} x(n-1) .
$$

If the initial conditions are conditions $y(0)=0$ and $x(0)=1$, then response of the system to the input $x(t)=1, t>0$, is calculated by the simple recursion as

$$
\begin{aligned}
y(1) & =\frac{1}{5} \\
y(2) & =\frac{4}{5} \cdot \frac{1}{5}+\frac{1}{5} \\
y(3) & =\frac{4^{2}}{5^{2}} \cdot \frac{1}{5}+\frac{4}{5} \cdot \frac{1}{5}+\frac{1}{5} \\
\cdots(n) & =\frac{4^{n-1}}{5^{n-1}} \cdot \frac{1}{5}+\ldots+\frac{4^{2}}{5^{2}} \cdot \frac{1}{5}+\frac{4}{5} \cdot \frac{1}{5}+\frac{1}{5} .
\end{aligned}
$$

Therefore

$$
y(n)=1-\left(\frac{4}{5}\right)^{n}, \quad n=1,2, \ldots
$$

Figure 6 shows the solution of this difference equation for $n=1: 40$. The curve of the exact solution

$$
\begin{equation*}
y_{c}(t)=\left(1-e^{-\frac{t}{R C}}\right) u(t) \tag{48}
\end{equation*}
$$

is also shown.


Fig. 6. Approximation of the solution for the $\mathrm{RC}=1$ circuit when the sampling time is 0.2 s .

## B. 2nd order equation

We now consider the differential equation of order 2

$$
\begin{equation*}
y^{\prime \prime}(t)+a_{1} y^{\prime}(t)+a_{0} y(t)=b_{1} x^{\prime}(t)+b_{0} x(t) \tag{49}
\end{equation*}
$$

which results in the following evaluation at point $t=n T$

$$
\begin{equation*}
y^{\prime \prime}\left(\left.t\right|_{t=n T}\right)+a_{1} y^{\prime}\left(\left.t\right|_{t=n T}\right)+a_{0} y(n)=b_{1} x^{\prime}\left(\left.\right|_{t=n T}\right)(n)+b_{0} x(n), \quad n \in Z . \tag{50}
\end{equation*}
$$

The approximation of the second derivative will be calculated as follows

$$
\begin{aligned}
y^{\prime \prime}\left(t_{t=n T}\right) & =\lim _{\Delta \rightarrow 0} \frac{y^{\prime}(t+\Delta)-y^{\prime}(t)}{\Delta}=\lim _{\Delta \rightarrow 0} \lim _{T \rightarrow 0} \frac{\frac{y(t+\Delta+T)-y(t+\Delta)}{T}-\frac{y(t+T)-y(t)}{T}}{\Delta} \\
& =\lim _{T \rightarrow 0}\left[\frac{y(t+T+T)-y(t+T)}{T^{2}}-\frac{y(t+T)-y(t)}{T^{2}}\right] \\
& \approx \frac{y(n T+2 T)-2 y(n T+T)+y(n T)}{T^{2}}=\frac{y(n+2)-2 y(n+1)+y(n)}{T^{2}} .
\end{aligned}
$$

Therefore Eq. 49 can be approximated as

$$
\begin{aligned}
\frac{y(n+2)-2 y(n+1)+y(n)}{T^{2}}+a_{1} \frac{y(n+1)-y(n)}{T}+a_{0} y(n) & =b_{1} \frac{x(n+1)-x(n)}{T}+b_{0} x(n) \\
y(n+2)-\left[2-a_{1} T\right] y(n+1)+\left[1-a_{1} T+a_{0} T^{2}\right] y(n) & =b_{1} T x(n+1)-\left[b_{1}-b_{0} T\right] T x(n)
\end{aligned}
$$

$$
\begin{aligned}
y(n+2) & =\left[2-a_{1} T\right] y(n+1)-\left[1-a_{1} T+a_{0} T^{2}\right] y(n)+b_{1} T x(n+1)-\left[b_{1}-b_{0} T\right] T x(n) \\
y(n) & =\left[2-a_{1} T\right] y(n-1)-\left[1-a_{1} T+a_{0} T^{2}\right] y(n-2)+b_{1} T x(n-1)-\left[b_{1}-b_{0} T\right] T x(n-2) .
\end{aligned}
$$

We obtain thus the difference equation of order two

$$
\begin{equation*}
y(n)=A_{1} y(n-1)+A_{2} y(n-2)+B_{1} x(n-1)+B_{2} x(n-2) \tag{51}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{1}=2-a_{1} T, \quad A_{2}=-\left(1-a_{1} T+a_{0} T^{2}\right), \quad B_{1}=b_{1} T, \quad B_{2}=-\left(b_{1}-b_{0} T\right) T . \tag{52}
\end{equation*}
$$

We assume that two initial conditions are given, $y^{\prime}(0)$ and $y(0)$. For difference equation the first initial condition takes the form

$$
y^{\prime}(0)=\frac{y(T)-y(0)}{T}=\frac{y(1)-y(0)}{T}
$$

and therefore

$$
\begin{equation*}
y(1)=T y^{\prime}(0)+y(0) . \tag{53}
\end{equation*}
$$

Example 3: Consider the RLC circuit with the input-output differential equation

$$
L \frac{d i(t)}{d t}+R i(t)+\frac{1}{C} \int_{-\infty}^{t} i(\tau) d \tau=v(t)
$$

which we write in the form

$$
L \frac{d^{2} y(t)}{d t^{2}}+R y^{\prime}(t)+\frac{1}{C} y(t)=x^{\prime}(t)
$$

or

$$
\begin{equation*}
\frac{d^{2} y(t)}{d t^{2}}+\frac{R}{L} y^{\prime}(t)+\frac{1}{L C} y(t)=\frac{1}{L} x^{\prime}(t) \tag{54}
\end{equation*}
$$

We denote

$$
a_{1}=\frac{R}{L}, \quad a_{0}=\frac{1}{L C}, \quad b_{1}=\frac{1}{L}, \quad b_{0}=0 .
$$

and according to (52) obtain the following approximation of the differential equation

$$
y(n)=\left[2-\frac{R}{L} T\right] y(n-1)-\left[1-\frac{R}{L} T+\frac{1}{L C} T^{2}\right] y(n-2)+\frac{1}{L} T x(n-1)-\frac{1}{L} T x(n-2) .
$$

In the case, $R=2, L=C=1$, we obtain

$$
\begin{aligned}
y(n) & =2(1-T) y(n-1)-(1-T)^{2} y(n-2)+T x(n-1)-T x(n-2) \\
& =(1-T)[2 y(n-1)-(1-T) y(n-2)]+T[x(n-1)-x(n-2)]
\end{aligned}
$$

## C. MATLAB source code

The program shown below can be used for calculating the response of the LTI causal system defined by

$$
\begin{equation*}
y(n)=\sum_{k=1}^{N} a_{k} y(n-k)+\sum_{m=0}^{M} b_{m} x(n-m) \tag{55}
\end{equation*}
$$

where we assume $M \leq N$. In the code, the matrix form of this equation is used

$$
y(n)=\left[a_{N}, a_{N-1}, \ldots, a_{2}, a_{1}\right]\left[\begin{array}{l}
y(n-N) \\
y(n-N-1) \\
\ldots \\
y(n-2) \\
y(n-1)
\end{array}\right]+\left[b_{M}, b_{M-1}, \ldots, b_{2}, b_{1}\right]\left[\begin{array}{l}
x(n-M) \\
x(n-M-1) \\
\ldots \\
x(n-2) \\
x(n-1)
\end{array}\right]+b_{0} x(n) .
$$

One can see that each new value of $y(n)$ is calculated from the $N$ previous values $y(n-1), y(n-$ $2), \ldots, y(n-N)$ and $M$ previous input values $x(n-1), x(n-2), \ldots, x(n-M)$ plus the input $x(n)$. This is an $N$ th-order recursion.

```
Call: recur.m
Calculate response of the system x(t)-> y(t)
    L_y(1,a)=L_x(b)
    a=(a1,a2,\ldots,an) and b=(b1,b2,\ldots,bn)
    with initial conditions x0 and yO.
    function y=recur(a,b,t,x,x0,y0)
        n=length(t);
        N=length(a);
        M=length(b);
        x=[x0 x];
        y=[y0 zeros(1,n)];
        a_reverse=a(N:-1:1);
        b_reverse=b(M:-1:1);
        for k=N+1:N+n
            y(k)=-a_reverse*y(k-N:k-1)' + b_reverse*x(k-N:k-1-N+M)';
        end;
        y=y(N+1:N+n);
    a=[l-3/2 1]; b=[llll}
    y0=[2 1];
    x0=[0 0];
    t=0:20;
    n=length(t);
    x=ones(1,n);
    y=recur(a,b,t,x,x0,y0);
    figure;
    plot(t,y,'r--'); % plot the response
    stem(t,y,'b'); % plot the response as a stem plot
```

IV. Discrete-Time Fourier Transform EE-3523

In this section, we consider the concept of the discrete-time Fourier transform (DTFT). For a given sequence $x(n), n=0, \pm 1, \pm 2, \ldots$, the pair of the discrete-time Fourier transforms is defined by

$$
\begin{align*}
X\left(e^{j \omega}\right) & =\sum_{n=-\infty}^{\infty} x(n) e^{-j \omega n}, \quad \omega \in(-\pi, \pi),  \tag{49}\\
x(n) & =\frac{1}{2 \pi} \int_{-\pi}^{\pi} X\left(e^{j \omega}\right) e^{j \omega n} d \omega, \quad n \in Z . \tag{50}
\end{align*}
$$

We assume here, that the infinite series in (49) converges (uniformly), in order to the inverse Fourier transform (50) to be defined. The discrete-time Fourier transform of a sequence $x(n)$ exists if, for example, the sequence is absolute summable, i.e. the following (sufficient) condition holds

$$
\sum_{n=-\infty}^{\infty}|x(n)|<\infty
$$

It is clear, that the DTFT for a finite length sequence $x(n)$ exists.
We should note that the DTFT $X\left(e^{j \omega}\right)$ is a periodic function on the unit circle and

$$
\begin{aligned}
X\left(e^{j \omega+2 k \pi}\right) & =\sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2 k \pi) n}=\sum_{n=-\infty}^{\infty} x(n) e^{-j \omega n} e^{-j 2 k n \pi} \\
& =\sum_{n=-\infty}^{\infty} x(n) e^{-j \omega n}=X\left(e^{j \omega}\right), \quad k=0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

The frequencies $\omega$ and $\omega+2 k \pi$ are indistinguishable. Therefore, in the inverse formula of the DTFT, we need define only integral over the interval $(-\pi, \pi)$ (or any other interval of length $2 \pi$ ). Therefore, we consider that "hight frequencies" are those that close to the boundary of the interval, $\pm \pi$, (and odd multiple of $\pi$ ), "low frequencies" are those that close to the original 0 (and even multiple of $\pi)$.

The DTFT is the complex transformation. Writing components of the DTFT in the form

$$
X\left(e^{j \omega}\right)=\left|X\left(e^{j \omega}\right)\right| e^{j \vartheta(\omega)}
$$

we call $\left|X\left(e^{j \omega}\right)\right|$ to be the amplitude function and $\vartheta(\omega)$ to be the phase function of the DTFT.
Example 1: If the sequence $x(n)=2^{-n} u(n)$, then the DTFT exists and the following holds

$$
\begin{aligned}
X\left(e^{j \omega}\right) & =\frac{1}{1-2^{-1} e^{-j \omega}}=\frac{1}{1-2^{-1} e^{-j \omega}} \cdot \frac{1-2^{-1} e^{+j \omega}}{1-2^{-1} e^{+j \omega}} \\
& =\frac{1-2^{-1} e^{j \omega}}{1+4^{-1}-\cos \omega}=\frac{1-2^{-1}(\cos \omega+j \sin \omega)}{1+4^{-1}-\cos \omega} \\
& =\frac{1-2^{-1} \cos \omega}{1+4^{-1}-\cos (\omega)}+j \frac{-2^{-1} \sin \omega}{1+4^{-1}-\cos \omega} .
\end{aligned}
$$



Fig. 7. (a) Amplitude spectrum of the sequence $x(n)=2^{-n} u(n)$, (b) phase of the spectrum, (c) real part and (d) imaginary part of the DTFT.

The DTFT characteristics of $x(n)$ are (see Fig. 7)

$$
\begin{align*}
\left|X\left(e^{j \omega}\right)\right|^{2} & =\frac{1+4^{-1}-\cos \omega}{\left(1+4^{-1}-\cos \omega\right)^{2}}  \tag{51}\\
\vartheta(\omega) & =\arg X\left(e^{j \omega}\right)=-\arctan \left(\frac{\sin \omega}{2-\cos \omega}\right) \tag{52}
\end{align*}
$$

Example 2: We calculate the DTFTs of the following sequences

$$
x_{1}(n)=u(n), \quad x_{2}(n)=a^{n} u(n), \quad x_{3}(n)=a^{n} u(n-1)
$$

where $a$ is a real number. Then, the first DTFT

$$
\begin{equation*}
X_{1}\left(e^{j \omega}\right)=?=\sum_{n=-\infty}^{\infty} x_{1}(n) e^{-j \omega n}=\sum_{n=0}^{\infty} e^{-j \omega n} \tag{53}
\end{equation*}
$$

cannot be defined directly, since the series does not converge ( $u(n)$ is not absolute summable). The next DTFT

$$
X_{2}\left(e^{j \omega}\right)=\sum_{n=-\infty}^{\infty} x_{2}(n) e^{-j \omega n}=\sum_{n=0}^{\infty} a^{n} e^{-j \omega n}=\sum_{n=0}^{\infty}\left(a e^{-j \omega}\right)^{n}=\frac{1}{1-a e^{-j \omega}}
$$

if only $\left|a e^{-j \omega}\right|=|a|<1$. So, the DTFT of $x_{2}(n)$ exists for $|a|<1$.

Similarly, we obtain the following calculations for the third DTFT

$$
\begin{align*}
X_{3}\left(e^{j \omega}\right) & =\sum_{n=-\infty}^{\infty} x_{3}(n) e^{-j \omega n}=\sum_{n=1}^{\infty} a^{n} e^{-j \omega n} \\
& =\sum_{n=1}^{\infty}\left(a e^{-j \omega}\right)^{n}=\frac{a e^{-j \omega}}{1-a e^{-j \omega}}=\frac{a}{e^{j \omega}-a} \tag{54}
\end{align*}
$$

if $|a|<1$ (otherwise, the DTFT cannot be defined).
A. Properties of the discrete-time Fourier transform

$$
\mathcal{F}: x(n) \rightarrow X\left(e^{j \omega}\right) .
$$

2. (Time reversal transformation)

$$
\begin{array}{ccc}
x(n) & \rightarrow & x(-n) \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
X\left(e^{j \omega}\right) & \rightarrow & X\left(e^{-j \omega}\right)
\end{array}
$$

3. (Time shift)

$$
\begin{array}{ccc}
x(n) & \rightarrow & x\left(n-n_{0}\right) \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
X\left(e^{j \omega}\right) & \rightarrow & e^{-j \omega n_{0}} X\left(e^{j \omega}\right)
\end{array}
$$

4. (Rotation)

$$
\begin{array}{ccc}
x(n) & \rightarrow & e^{j \omega_{0} n} x(n) \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
X\left(e^{j \omega}\right) & \rightarrow & X\left(e^{j\left(\omega-\omega_{0}\right)}\right)
\end{array}
$$

5. (Derivative of the DTFT)

$$
\begin{array}{cccc}
x(n) & \rightarrow & n x(n) \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
X\left(e^{j \omega}\right) & \rightarrow & j\left[X\left(e^{j \omega}\right)\right]^{\prime} .
\end{array}
$$

Indeed

$$
\begin{aligned}
X^{\prime}\left(e^{j \omega}\right) & =\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j \omega n}\right]^{\prime}=\sum_{n=-\infty}^{\infty}-j n x(n) e^{-j \omega n} \\
& =-j \sum_{n=-\infty}^{\infty}[n x(n)] e^{-j \omega n}=-j \mathcal{F}[n x] .
\end{aligned}
$$

6. (Convolution)

$$
\begin{array}{ccccc}
x(n) & y(n) & \rightarrow & x(n) * y(n) \\
\downarrow \mathcal{F} & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
X\left(e^{j \omega}\right) & Y\left(e^{j \omega}\right) & \rightarrow & X\left(e^{j \omega}\right) Y\left(e^{j \omega}\right)
\end{array}
$$

7. (Convolution in frequency domain)

$$
\begin{array}{cccc}
x(n) & y(n) & \rightarrow & x(n) \cdot y(n) \\
\downarrow \mathcal{F} & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
X\left(e^{j \omega}\right) & Y\left(e^{j \omega}\right) & \rightarrow & \frac{1}{2 \pi} \int_{-\pi}^{\pi} X\left(e^{j(\omega-\tau)}\right) Y\left(e^{j \tau}\right) d \tau
\end{array}
$$

8. (Parseval's equality) The energy of the signal can be expressed in the frequency domain as

$$
\begin{equation*}
\|x\|^{2}=\sum_{n=-\infty}^{\infty}|x(n)|^{2}=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left|X\left(e^{j \omega}\right)\right|^{2} d \tau=\|X\|^{2} \tag{55}
\end{equation*}
$$

For any two sequences $x(n)$ and $y(n)$ with the discrete-time Fourier transforms $X\left(e^{j \omega}\right)$ and $Y\left(e^{j \omega}\right)$, the distance between signals in the time domain coincides with the distance in the frequency domain

$$
\|x-y\|^{2}=\sum_{n=-\infty}^{\infty}|x(n)-y(n)|^{2}=\frac{1}{2 \pi} \int_{-\infty}^{\infty}\left|X\left(e^{j \omega}\right)-Y\left(e^{j \omega}\right)\right|^{2} d \tau=\|X-Y\|^{2}
$$

Therefore, if two sequences have the same DTFT, they are equal.

## B. Frequency response of the system

In the discrete case of the sequences

$$
x(n), \quad n=0, \pm 1, \pm 2, \ldots
$$

we consider the linear system with impulse characteristics $h(n)$ that is described as

$$
H: x(n) \rightarrow y(n)=x(n) * h(n)=\sum_{m=-\infty}^{\infty} h(m) x(n-m)=\sum_{m=-\infty}^{\infty} x(m) h(n-m)
$$

Given $\omega$, let an input $x(n)$ be the exponential sequence

$$
x(n)=e^{j \omega n}=\left(e^{j \omega}\right)^{n} \quad n=0, \pm 1, \pm 2, \ldots
$$

Then, the corresponding output of the system is

$$
y(n)=\sum_{m=-\infty}^{\infty} h(m) e^{j \omega(n-m)}=e^{j \omega n} \sum_{m=-\infty}^{\infty} h(m) e^{-j \omega m}=e^{j \omega n} H\left(e^{j \omega}\right)=x(n) H\left(e^{j \omega}\right)
$$

where we denote by

$$
H\left(e^{j \omega}\right)=\sum_{m=-\infty}^{\infty} h(m) e^{-j \omega m}=\sum_{m=-\infty}^{\infty} h(m)\left(e^{j \omega}\right)^{-n}
$$

the frequency characteristics of the system $H$ (or, the discrete-time Fourier transform of $h(t)$ ).

Writing components of the DTFT in the polar form

$$
H\left(e^{j \omega}\right)=\left|H\left(e^{j \omega}\right)\right| e^{j \vartheta(\omega)}
$$

we call $\left|H\left(e^{j \omega}\right)\right|$ to be the amplitude response function and $\vartheta(\omega)$ to be the phase response function of the system $H$.

The inverse formula holds

$$
h(m)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} H\left(e^{j \omega}\right) e^{j \omega m} d \omega
$$

We recall that the discrete-time Fourier transform of a sequence $h(n)$ exists if the sequence is absolute summable. And in this case, we call the system $H$ to be stable. But, not for any absolute square integrable sequence $h(n)$, the DTFT can be defined. It means that the condition

$$
\sum_{n=-\infty}^{\infty}|h(n)|^{2}<\infty
$$

is not sufficient for the DTFT existence. As an example, we consider the following function in the frequency domain

$$
H\left(e^{j \omega}\right)=\operatorname{rect}(\omega), \quad \omega \in(-\pi, \pi) .
$$

The inverse Fourier transform for this function take values of the sinc sequence

$$
h(n)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} H\left(e^{j \omega}\right) e^{j \omega n} d \omega=\frac{1}{2 \pi} \int_{-1 / 2}^{1 / 2} e^{j \omega n} d \omega=\frac{1}{2 \pi} \operatorname{sinc}\left(\frac{n}{2}\right) .
$$

The sequence $h(n)$ is not absolute summable, but the sums of finite number of terms

$$
H_{N}\left(e^{j \omega}\right)=\sum_{m=-N}^{N} h(m) e^{-j \omega m}
$$

converges to the function $H\left(e^{j \omega}\right)$ as $N \rightarrow \infty$ in the sense

$$
\int_{-\infty}^{\infty}\left|H_{N}\left(e^{j \omega}\right)-H\left(e^{j \omega}\right)\right|^{2} d \omega \rightarrow 0, \quad \text { as } N \rightarrow \infty
$$

Figure 8 shows the approximations of the discrete-time Fourier transform $H\left(e^{j \omega}\right)$ by finite series $H_{7}\left(e^{j \omega}\right), H_{11}\left(e^{j \omega}\right), H_{21}\left(e^{j \omega}\right)$, and $H_{100}\left(e^{j \omega}\right)$.

Example 3: Let the impulse characteristics function be the following exponential sequence

$$
h(n)=a^{n} u_{1}(n)=\left\{\begin{aligned}
a^{n}, & n=0,1,2, \ldots \\
0, & n=-1,-2, \ldots
\end{aligned}\right.
$$

for a given number $a, 0<|a|<1$.


Fig. 8. Approximation by $N=7,11,21$, and 100 .

Then, the frequency characteristics is

$$
H\left(e^{j \omega}\right)=\sum_{m=0}^{\infty} h(m) e^{-j \omega m}=\sum_{m=0}^{\infty} a^{m} e^{-j \omega m}=\sum_{m=0}^{\infty}\left(a e^{-j \omega}\right)^{m}=\frac{1}{1-a e^{-j \omega}}
$$

and the absolute response function is defined as

$$
\left|H\left(e^{j \omega}\right)\right|^{2}=\frac{1}{1-a e^{-j \omega}} \cdot \frac{1}{1-a e^{j \omega}}=\frac{1}{1-2 a \cos \omega+a^{2}}
$$

The phase response function $\varphi(\omega)$ can be calculated as follows:

$$
\begin{gathered}
H\left(e^{j \omega}\right)=\frac{1}{1-a e^{-j \omega}}=\frac{1-a \cos (\omega)}{1-2 a \cos (\omega)+a^{2}}+j \frac{-a \sin (\omega)}{1-2 a \cos (\omega)+a^{2}} \\
\varphi(\omega)=\arg \left(H\left(e^{j \omega}\right)\right)=-\tan ^{-1}\left(\frac{a \sin (\omega)}{1-a \cos (\omega)}\right)
\end{gathered}
$$

So,

$$
h(n)=a^{n} u_{1}(n) \rightarrow H\left(e^{j \omega}\right)=\frac{1}{1-a e^{-j \omega}}
$$

For this system, the output is defined as follows:

$$
\begin{aligned}
y(n)=\sum_{m=0}^{\infty} a^{m} x(n-m) & =x(n)+a x(n-1)+a^{2} x(n-2)+a^{3} x(n-3)+\ldots \\
& =x(n)+a\left[x(n-1)+a x(n-2)+a^{2} x(n-3)+\ldots\right] \\
& =x(n)+a y(n-1) \\
& =x(n)+a x(n-1)+a^{2} y(n-2)
\end{aligned}
$$



Fig. 9. The diagram of realization of system $H$.
which represents the deference equation with the diagram of realization of Fig. 9.
Example 4: Consider the simplest case when the impulse response is the unit impulse

$$
h(n)=u_{0}(n)= \begin{cases}1, & n=0 \\ 0, & n \neq 0 .\end{cases}
$$

Then, $H\left(e^{j \omega}\right)=1$, and

$$
e^{j \omega n} \rightarrow e^{j \omega n} * u_{0}(n)=1 \cdot e^{j \omega n} .
$$

In the general case

$$
x(n) * u_{0}(n)=\sum_{k} x(n-k) u_{0}(k)=x(n) .
$$

If the impulse response $h(n)=u_{0}\left(n-n_{0}\right)$, then the shift by $n_{0}$ will yield the transfer response to be equal

$$
H\left(e^{j \omega}\right)=e^{-j \omega n_{0}} \cdot 1
$$

Example 5: Consider the following impulse response and input

$$
h(n)=a^{n} u(n), \quad x(n)=b^{n} u(n)
$$

where $|a|<1$ and $|b|<1$. The convolution $y(n)$ of these two sequences is calculated in the spectral domain as follows

$$
\begin{aligned}
h(n) & \rightarrow H\left(e^{j \omega}\right)=\frac{1}{1-a e^{-j \omega}} \\
x(n) & \rightarrow X\left(e^{j \omega}\right)=\frac{1}{1-b e^{-j \omega}} \\
x(n) * h(n) & \rightarrow X\left(e^{j \omega}\right) H\left(e^{j \omega}\right)=\frac{1}{1-a e^{-j \omega}} \cdot \frac{1}{1-b e^{-j \omega}} \\
Y\left(e^{j \omega}\right) & =\frac{A}{1-a e^{-j \omega}}+\frac{B}{1-b e^{-j \omega}}
\end{aligned}
$$

where $A=a /(a-b)$ and $B=b /(b-a)$, if $a \neq b$. Therefore

$$
y(n)=A h(n)+B x(n) .
$$

In the $b=a$ case, the discrete-time Fourier transform is

$$
Y\left(e^{j \omega}\right)=\left[\frac{1}{1-a e^{-j \omega}}\right]^{2} .
$$

Note that

$$
\begin{aligned}
{\left[\frac{1}{1-a e^{-j \omega}}\right]^{2} } & =\left[\frac{1}{1-a e^{-j \omega}}\right]^{\prime} \cdot\left(\frac{1}{-j a e^{-j \omega}}\right) \\
& =j\left[\frac{1}{1-a e^{-j \omega}}\right]^{\prime} \cdot \frac{1}{a} e^{j \omega}
\end{aligned}
$$

therefore, the following table holds

$$
\begin{array}{ccccc}
{\left[\frac{1}{1-a e^{-j \omega}}\right]^{2}} & : & j\left[\frac{1}{1-a e^{-j \omega}}\right]^{\prime} & \rightarrow & j\left[\frac{1}{1-a e^{-j \omega}}\right]^{\prime} \times e^{j \omega \cdot 1} \frac{1}{a} \\
\uparrow \mathcal{F} & & \uparrow \mathcal{F} & & \\
y(n) & : & n\left[a^{n} u(n)\right] & \rightarrow & (n+1)\left[a^{n+1} u(n+1)\right] \frac{1}{a}
\end{array}
$$

Thus,

$$
y(n)=\frac{1}{a}(n+1) a^{n+1} u(n+1)=\frac{1}{a}(n+1) x(n+1)
$$

and we can note that $y(-1)=0$ and $y(0)=a$.
B. 1 Linear Filter

Consider a causal LTI system that is described by a difference equation of order $N$

$$
\begin{equation*}
y(n)+\sum_{k=1}^{N} a_{k} y(n-k)=\sum_{m=0}^{M} b_{m} x(n-m) \quad(M \leq N) . \tag{56}
\end{equation*}
$$

According to the property of the convolution, we obtain

$$
H\left(e^{j \omega}\right)=\frac{Y\left(e^{j \omega}\right)}{X\left(e^{j \omega}\right)}
$$

and from (56)

$$
\begin{gathered}
Y\left(e^{j \omega}\right)+\sum_{k=1}^{N} a_{k} Y\left(e^{j \omega}\right) e^{-j k \omega}=\sum_{m=0}^{M} b_{m} X\left(e^{j \omega}\right) e^{-j m \omega} \\
H\left(e^{j \omega}\right)=\frac{\sum_{m=0}^{M} b_{m} e^{-j m \omega}}{1+\sum_{k=1}^{N} a_{k} e^{-j k \omega}} .
\end{gathered}
$$

Example 6: Consider the difference equation of the first order

$$
y(n)-a y(n-1)=2 x(n) .
$$

Then

$$
H\left(e^{j \omega}\right)=\frac{2}{1-a e^{-j \omega}} \leftarrow 2 a^{n} u(n) .
$$

Example 7: Consider a LTI system $L$ defined by the following difference equation of order two

$$
\begin{equation*}
y(n)-\frac{5}{6} y(n-1)+\frac{1}{6} y(n-2)=3 x(n) \tag{57}
\end{equation*}
$$

Then

$$
H\left(e^{j \omega}\right)=\frac{3}{1-\frac{5}{6} e^{-j \omega}-\frac{1}{6} e^{-j 2 \omega}}=\frac{3}{\left[1-\frac{1}{3} e^{-j \omega}\right]\left[1-\frac{1}{2} e^{-j \omega}\right]}
$$

or

$$
H\left(e^{j \omega}\right)=\frac{A}{1-\frac{1}{3} e^{-j \omega}}+\frac{B}{1-\frac{1}{2} e^{-j \omega}}
$$

where $A=-2$ and $B=3$. Therefore, the impulse response of the system is

$$
\begin{equation*}
h(n)=-2\left(\frac{1}{3}\right)^{n} u(n)+3\left(\frac{1}{2}\right)^{n} u(n) \tag{58}
\end{equation*}
$$

Let us rewrite difference equation 57 for the impulse response as

$$
\begin{gathered}
h(n)-\frac{5}{6} h(n-1)+\frac{1}{6} h(n-2)=3 u_{0}(n) \\
h(n)-\frac{1}{3} h(n-1)-\frac{1}{2} h(n-1)+\frac{1}{6} h(n-2)=\left[h(n)-\frac{1}{3} h(n-1)\right]-\frac{1}{2}\left[h(n-1)-\frac{1}{3} h(n-2)\right]
\end{gathered}
$$

Therefore, defining the signal

$$
\begin{equation*}
w(n)=h(n)-\frac{1}{3} h(n-1) \tag{59}
\end{equation*}
$$

we obtain the following difference equation in (57)

$$
\begin{equation*}
w(n)-\frac{1}{2} w(n-1)=3 u_{0}(n) \tag{60}
\end{equation*}
$$

Thus, the system represents the cascade connection of two systems $L=L_{1} \circ L_{2}$, which are described respectively by difference equations

$$
\begin{array}{ll}
L_{1}: x(n) \rightarrow y(n), & y(n)-\frac{1}{3} y(n-1)=x(n) \\
L_{2}: & x(n) \rightarrow y(n),
\end{array} \quad y(n)-\frac{1}{2} y(n-1)=3 x(n) .
$$

In other words, the system $L$ works as

$$
\begin{equation*}
x(t) \rightarrow\left[L_{2}\right] \rightarrow w(n) \rightarrow\left[L_{1}\right] \rightarrow y(n) \tag{61}
\end{equation*}
$$

But the system $L_{1}$ has impulse response $h_{1}(n)=3^{-n} u(n)$, and the system $L_{2}$ has impulse response $h_{2}(n)=2^{-n} u(n)$.

The cascade connection of two systems $L=L_{1} \circ L_{2}$ can be represented as the parallel connection

$$
L=L_{1} \circ L_{2}=A L_{1}+B L_{2}=-6 L_{1}+9 L_{2}
$$

which yields the impulse response for the system $L$ to be equal as shown (58).
Example 8: Consider now the response $y(n)$ of a LTI system $L$ to input $x(n)=b^{n} u(n), a<1$. In the frequency domain, we have

$$
Y\left(e^{j \omega}\right)=H\left(e^{j \omega}\right) X\left(e^{j \omega}\right)=H\left(e^{j \omega}\right) \cdot \frac{1}{1-b e^{-j \omega}}
$$

where $H\left(e^{j \omega}\right)$ is the frequency response of $L$. If the system $L$ is defined by a difference equation of order $N$, then the system can be represented as cascade-wise (or parallel) connection of $N$ simple system $L_{k}$ with frequency responses $H_{k}\left(e^{j \omega}\right)$ and impulse responses $h_{k}(n)=a_{k}^{n} u(n), k=1: N$.

Then we can write

$$
Y\left(e^{j \omega}\right)=\sum_{k=1}^{N} \frac{A_{k}}{1-a_{k} e^{-j \omega}}+\frac{B}{1-b e^{-j \omega}}=\sum_{k=1}^{N} A_{k} H_{k}\left(e^{j \omega}\right)+B X\left(e^{j \omega}\right)
$$

where $A_{k}$ and $B$ are constants. Therefore, the response of the system can be written as

$$
y(n)=\sum_{k=1}^{N} A_{k} h_{k}(n)+B x(n)=\sum_{k=1}^{N} A_{k} a_{k}^{n} u(n)+B b^{n} u(n) .
$$

For example, consider the LTI system of Example 7, for which the frequency response is

$$
H\left(e^{j \omega}\right)=\frac{3}{1-\frac{5}{6} e^{-j \omega}-\frac{1}{6} e^{-j 2 \omega}} .
$$

Let the input discrete-time signal be

$$
\begin{equation*}
x(n)=\left(\frac{1}{4}\right)^{n} u(n) . \tag{62}
\end{equation*}
$$

Then

$$
\begin{aligned}
Y\left(e^{j \omega}\right) & =H\left(e^{j \omega}\right) X\left(e^{j \omega}\right)=\frac{3}{\left[1-\frac{1}{3} e^{-j \omega}\right]\left[1-\frac{1}{2} e^{-j 2 \omega}\right]} \cdot \frac{1}{1-\frac{1}{4} e^{-j \omega}} \\
& =\frac{A}{1-\frac{1}{3} e^{-j \omega}}+\frac{B}{1-\frac{1}{2} e^{-j 2 \omega}}+\frac{C}{1-\frac{1}{4} e^{-j \omega}}
\end{aligned}
$$

where $A=-24, B=18$, and $C=-9$. The impulse response therefore equals

$$
h(n)=-24\left(\frac{1}{3}\right)^{n} u(n)+18\left(\frac{1}{2}\right)^{n} u(n)-9\left(\frac{1}{4}\right)^{n} u(n) .
$$

As in Eq. 57, the impulse response is composed as a linear combination of exponential sequences with bases $1 / 3,1 / 2$, and $1 / 4$. We recall here, that the input signal $x(n)$ can be considered as the impulse response of system $L_{3}$, and the system $L$ as parallel connection of three systems.
V. Sampling Theorem
I. Digital signal processing provides an alternative method for processing the analog signals. To perform processing digitally, there is a need for an interface between the analog signal and digital processor. This interface is called an analog-to-digital (A-D) converter. The output of A-D is a digital signal to be processed in the digital processor.


Fig. 10. Diagram of digital signal processing.
We consider the sampling process, where the analog signal $x(t)$ (or continuous-time signal) is measured periodically every $T$ seconds, as $x(n T) . T$ is the sampling time, or period, and time is counted at points multiple to $T$, i.e. $t=0, T, 2 T, 3 T, \ldots,(t=n T, \mathrm{n}=0, \ldots)$, or $t=0, \pm T, \pm 2 T, \pm 3 T, \ldots$.

The following problems are considered:

1. How does the sampling effect on the Fourier spectrum of the original signal?
2. What is the maximal value of sampling interval $T$ ?
3. Is it possible to reconstruct the original continuous-time signal $x(t)$ from its sampled version, by interpolating between the samples $x(n T)$.

We consider an absolute (or square) integrable function $x(t)$ defined on the real line $R$ for which the Fourier transform exists

$$
\begin{equation*}
X(\omega)=(\mathcal{F} \circ x)(\omega)=\int_{-\infty}^{+\infty} x(t) e^{-j \omega t} d t, \quad \omega \in(-\infty,+\infty) \tag{70}
\end{equation*}
$$

The inverse Fourier transform is described as

$$
\begin{equation*}
x(t)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} X(\omega) e^{j \omega t} d \omega, \quad t \in(-\infty,+\infty) \tag{71}
\end{equation*}
$$

A.

Let $T$ be an interval of sampling the function $x(t)$, that results in the sequence

$$
\begin{gathered}
x(t) \rightarrow \text { Analog/Digital Converter } \rightarrow x(n T) \\
x(n T)=x(t)_{\mid t=n T}, \quad n=0, \pm 1, \pm 2, \ldots .
\end{gathered}
$$

We take the pair of the discrete-time Fourier transforms for the sampled signal $x(t)$

$$
\begin{align*}
X\left(e^{j \omega T}\right) & =\sum_{n=-\infty}^{\infty} x(n T) e^{-j \omega n T}, \quad(\operatorname{period} \text { is } 2 \pi / T)  \tag{72}\\
x(n T) & =\frac{T}{2 \pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X\left(e^{j \omega T}\right) e^{j \omega n T} d \omega \tag{73}
\end{align*}
$$

We consider the relation between the spectrum of the sequence $x(n T)$ with the Fourier transform of the continuous-time function $x(t)$. For that, in (71), the integral on the line $(-\infty,+\infty)$ will be calculated by sum of integrals on segments of length $2 \pi / T$, i.e. $\ldots,[-3 \pi / T,-\pi / T),[-\pi / T, \pi / T)$, $[\pi / T, 3 \pi / T), \ldots$. We use the fact that $2 \pi / T$ is the period for both the DFT $X\left(e^{j \omega T}\right)$ and basis functions $e^{j \omega n T}$.

It follows directly from (71), that

$$
\begin{aligned}
x(t)_{\mid t=n T} & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} X(\omega) e^{j \omega n T} d \omega \\
& =\frac{1}{2 \pi} \sum_{m=-\infty}^{\infty} \int_{-\frac{\pi}{T}+2 m \frac{\pi}{T}}^{\frac{\pi}{T}+2 m \frac{\pi}{T}} X(\omega) e^{j \omega n T} d \omega \\
& =\frac{1}{2 \pi} \sum_{m=-\infty}^{\infty} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X\left(\omega+m \frac{2 \pi}{T}\right) e^{j\left(\omega+m \frac{2 \pi}{T}\right) n T} d \omega \\
& =\frac{T}{2 \pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\left[\frac{1}{T} \sum_{m=-\infty}^{\infty} X\left(\omega+m \frac{2 \pi}{T}\right)\right] e^{j \omega n T} d \omega .
\end{aligned}
$$

From (73), we therefore obtain the following expression for the DFT

$$
\begin{align*}
X\left(e^{j \omega T}\right) & =\frac{1}{T} \sum_{m=-\infty}^{\infty} X\left(\omega+m \frac{2 \pi}{T}\right)  \tag{74}\\
& =\frac{1}{T} X(\omega)+\frac{1}{T} \sum_{m= \pm 1, \pm 2, \ldots} X\left(\omega+m \frac{2 \pi}{T}\right)
\end{align*}
$$

It means that the periodic spectral function $X\left(e^{j \omega}\right)$ consists of infinite number of spectral components of the original spectra $X(\omega)$. Namely, the DFT itself represents the scaled original Fourier transform plus an infinite number of the shifted versions (copies, or replicas) of the Fourier transforms. The shifts are by periods $m(2 \pi / T), m= \pm 1, \pm 2, \ldots$.

The sampling process generates high frequency components and every frequency component of the original signal is periodically replicated ${ }^{1}$ over entire frequency axis with period

$$
f_{s}=\frac{1}{T} \quad(\text { sampling rate (frequency) })
$$

in units of samples per second.
B. Let us assume that the signal has bounded spectrum, i.e. $X(\omega)=0$ for all $|\omega|>\Omega$, and $\pi / T>\Omega$ for a positive number $\Omega$. In this case, for all integers $m \neq 0$, we have

$$
\left.X\left(\omega+m \frac{2 \pi}{T}\right)\right|_{(-\Omega, \Omega)}=0
$$

${ }^{1}$ The periodicity is in term of Hz and means that $X\left(e^{j \omega T}\right)=X\left(e^{j 2 \pi f T}\right)$ is periodic when shifting $f \rightarrow f+f_{s}$.

This means that if a frequency $\omega \in(-\Omega, \Omega)$, then $X\left(\omega+m \frac{2 \pi}{T}\right)=0$ and the replicas of $X(\omega)$ do not overlap (see Fig. 11(a)).


Fig. 11. (a) Effect of sampling when $\pi / T>\Omega$, and (b) effect of sampling when $\pi / T<\Omega$.

Then, we can state that the Fourier transform of the continuous-time signal can be defined from the discrete Fourier transform of the sampled signal as

$$
\begin{equation*}
\frac{1}{T} X(\omega)=X\left(e^{j \omega T}\right), \quad \text { or } \quad X(\omega)=T X\left(e^{j \omega T}\right) \tag{75}
\end{equation*}
$$

for all frequencies $|\omega| \leq \Omega$.
And opposite, if period of sampling $T$ such that $\pi / T<\Omega$, then at least two additional terms $X\left(\omega+\frac{2 \pi}{T}\right)$ and $X\left(\omega-\frac{2 \pi}{T}\right)$ will contribute to form the spectrum of the sampled signal in the interval $[-\Omega, \Omega]$ as shown in Fig. 11(b), and the Fourier transform $X(\omega)$ in the interval $[-\Omega, \Omega]$ cannot be defined as

$$
X(\omega)=T X\left(e^{j \omega T}\right), \quad \omega \in[-\Omega, \Omega]
$$

from the spectrum of the sampled signal. The reason is the small sampling rate $f_{s}$ (or, not small
sampling time $T$ ), which results in the overlapping (aliasing) condition

$$
\frac{\pi}{T}<\Omega
$$

To determine the Fourier transform of the function $x(t)$, the period of sampling should be taken as a period $T_{1}$ which provides the condition

$$
-\frac{\pi}{T_{1}}<-\Omega \quad \text { and } \quad \Omega<\frac{\pi}{T_{1}}
$$

and, therefore $T_{1} \leq T$.
C. From the first example of Fig. 11 (part a), one can see that it is possible to reconstruct the original spectrum from the continuous-discrete Fourier transform, taking

$$
\begin{equation*}
X(\omega)=T X\left(e^{j \omega T}\right), \quad|\omega| \leq \frac{\pi}{T}>\Omega \tag{76}
\end{equation*}
$$

since $X(\omega) \equiv 0$ when $\omega$ lies outside the interval $(-\pi / T, \pi / T)$. From the second example, we observe that one can never reconstruct the original spectrum from $X\left(e^{j \omega T}\right)$. Namely, the original Fourier transform can be reconstructed partially (reconstruction for low frequencies).

It should be noted that the sampling condition in terms of frequencies in Hz is derived as follows

$$
\frac{\pi}{T} \geq \Omega \quad \Leftrightarrow \quad \frac{1}{2 T}=\underline{\frac{f_{s}}{2} \geq f_{b}}=\frac{\Omega}{2 \pi}
$$

(see illustration of this condition in Figure 12). In other words, the sampling rate $f_{s} \geq 2 f_{b}$.


Fig. 12. Condition of the proper sampling the signal bounded by $\Omega$.
D. The following statement holds.

Theorem 1: Let $x(t)$ be a function with the bounded spectrum, i.e.

$$
\begin{equation*}
X(\omega)=0, \quad|\omega|>\Omega \tag{77}
\end{equation*}
$$

and let such value $\Omega>0$ exists. Then $x(t)$ can be described uniquely by its samples taken at discrete time with the sampling interval

$$
T<\frac{\pi}{\omega_{s}}
$$

for a frequency $\omega_{s}$ such that $\Omega \leq \omega_{s} \leq \pi / T$.
This statement has been proved first by Kotelnikov [V.A. Kotelnikov, Theory of potential noise stability, Moscow, Nauka, 1956] but is often called Uttekker and Shannon's sampling theorem.

Example 1: Consider the signal $x(t)$ that is bounded in the spectral domain of $50 \cdot 10^{3} \mathrm{rad} / \mathrm{sec}$. Then, $\Omega=25 \cdot 10^{3} \mathrm{rad} / \mathrm{sec}$. Let us assume that $T=10^{-4} \mathrm{sec}$. Checking the condition

$$
\omega_{\text {period } / 2}=\frac{\pi}{T}=3.14 \cdot 10^{4}>\Omega
$$

we see that $T$ can be considered as a good sampling period.
If we take $T=1.5 \cdot 10^{-4} \mathrm{sec}$, then after checking the condition

$$
\omega_{\text {period } / 2}=\frac{\pi}{T}=3.14 / 1.5 \cdot 10^{4} \approx 2.28 \cdot 10^{4}<\Omega
$$

we can state that such $T$ cannot be used for sampling a signal with bounded spectrum of 50 . $10^{3} \mathrm{rad} / \mathrm{sec}$. We need reduce by $\Delta T$ the sampling period for signal reconstruction. The minimum change in sampling time is defined as

$$
\Delta T=T-\frac{\pi}{\Omega}=1.5 \cdot 10^{-4}-\frac{\pi}{2.5 \cdot 10^{4}}=10^{-4}\left(1.5-\frac{\pi}{2.5}\right)=2.4336 \cdot 10^{-5} \mathrm{sec}
$$

In terms of hertz, the spectrum is bounded by the frequency

$$
f_{b}=\frac{\Omega}{2 \pi}=\frac{25 \cdot 10^{3}}{2 \pi} \mathrm{~Hz}=3.9789 \cdot 10^{3} \mathrm{~Hz}
$$

the sampling rate $f_{s}$ therefore should be greater or equal $2 f_{b}=7.9578 \cdot 10^{3} \mathrm{~Hz}$.
Owing to the Kotelnikov theorem (given bellow), each function with the bounded spectrum can be restored from its discrete values chosen in the defined way. So, the one-dimensional signal $x(t)$, that is given in the finite interval $[0, L]$ and satisfies the condition of spectrum to be band limited: $X(\omega)=0$ for all $|\omega|>\Omega$, can be represented one-to-one by the discrete sequence of its values

$$
\begin{equation*}
\left\{x_{n}=x(n T), \quad n=0:(N-1)\right\} \tag{78}
\end{equation*}
$$

taken with a sampling interval $T \leq \pi / \Omega=1 /\left(2 f_{b}\right)$ such that $N=L / T$ is an integer. The restoration of the assumed continuous-time function from its discrete values (78) is described by the expansion formula of the function:

$$
\begin{equation*}
x(t)=\sum_{n=0}^{N-1} x_{n} \operatorname{sinc}\{\Omega(t-n T)\}, \quad t \in[0, L] \tag{79}
\end{equation*}
$$

by the basis functions

$$
\begin{equation*}
\operatorname{sinc}\{\Omega(t-n T)\}=\frac{\sin \{\Omega(t-n T)\}}{\Omega(t-n T)} \tag{80}
\end{equation*}
$$

that are the shifted by $n T$ and time-scaled by $\Omega$ versions of $\operatorname{sinc}(t)$ functions

$$
\operatorname{sinc}(t) \rightarrow \sin c(\underline{\Omega} t) \rightarrow \operatorname{sinc}(\Omega(t-\underline{n T})) .
$$

Thus, a "signal" with the bounded spectrum (77) is defined completely by the finite number of its values at points situated uniformity on the segment $[0, L]$.


Fig. 13. Elements of the decomposition of the reconstructed signal by basis $\operatorname{sinc}(t)$ functions.

To proof the statement of the Kotelnikov theorem in (79) uses the condition of not aliasing

$$
\begin{equation*}
X(\omega)=T X\left(e^{j \omega T}\right) \quad|\omega|<\Omega<\frac{\pi}{T} . \tag{81}
\end{equation*}
$$

Eq. 79 in the general case has the form

$$
\begin{aligned}
x(t) & =\frac{T \Omega}{\pi} \sum_{n \in Z} x(n T) \operatorname{sinc}[\Omega(t-n T)] \\
& =\sum_{n \in Z} x(n T) \operatorname{sinc}[\Omega(t-n T)] \quad\left(\text { if } \frac{\pi}{T}=\Omega\right) .
\end{aligned}
$$

E. In the real word, the signals are not bound limited. They are limited in time but not in frequency domain.

Example 2: Consider the rectangle function on the interval $\left[-T_{0}, T_{0}\right]$

$$
x(t)=\operatorname{rect}\left(\frac{t}{2 T_{0}}\right)=u\left(t+T_{0}\right)-u\left(t-T_{0}\right) \rightarrow 2 T_{0} \operatorname{sinc}\left(\omega T_{0}\right) .
$$

In the particular $T_{0}=1 / 2$ case, we obtain $\operatorname{rect}(t) \xrightarrow{\mathcal{F}} \operatorname{sinc}(\omega / 2)$. Note that $\operatorname{sinc}(\omega)$ is the function that is not limited neither absolute integrable.
D. We consider a continuous-in-time signal that itself represents a step-function with values of the sampled sequence

$$
\begin{equation*}
\widetilde{x}(t)=x(n T), \quad t \in A_{n}=(n T-T / 2, n T+T / 2], \quad n \in Z . \tag{82}
\end{equation*}
$$

The function $\widetilde{x}(t)$ can be represented as the infinite sum of the pulse signals

$$
\widetilde{x}(t)=\sum_{n \in Z} x(n T) r e c t\left(\frac{t}{T}-n\right) .
$$

Each function $r(t)=\operatorname{rect}\left(\frac{t}{T}-n\right)$ represents the performance of the following time-transformations $t: \rightarrow t / T$ and $t \rightarrow t-n T$ of the $\operatorname{rect}(t)$ function. Therefore, the Fourier transform is defined by these transformation as it shown in the diagram:

$$
\begin{array}{llllll}
\operatorname{rect}(t) & \rightarrow z(t) & \operatorname{rect}\left(\frac{t}{T}\right) & \rightarrow z(t-n T) & = & \operatorname{rect}\left(\frac{t}{T}-n\right) \\
\downarrow \mathcal{F} & \downarrow \mathcal{F} & \downarrow \mathcal{F} & \downarrow \mathcal{F} & =\downarrow \mathcal{F} \\
\operatorname{sinc}\left(\frac{\omega}{2}\right) \rightarrow Z(\omega) & =T \operatorname{sinc}\left(\frac{\omega T}{2}\right) \rightarrow & Z(\omega) e^{-j \omega n T} & = & T \operatorname{sinc}\left(\frac{\omega T}{2}\right) e^{-j \omega n T}
\end{array}
$$

Therefore, the Fourier transform of the function $\widetilde{x}(t)$ is calculated as

$$
\begin{aligned}
\tilde{X}(\omega) & =\sum_{n \in Z} x(n T) T \operatorname{sinc}\left(\frac{\omega T}{2}\right) e^{-j \omega n T} \\
& =\sum_{n \in Z} x(n T) e^{-j \omega n T} \cdot\left[T \operatorname{sinc}\left(\frac{\omega T}{2}\right)\right]=T X\left(e^{j \omega T}\right) \operatorname{sinc}\left(\frac{\omega T}{2}\right) .
\end{aligned}
$$

On the other hand, if $x(n T)$ is the sampled signal $x(t)$ with the sampling period $T$, then according to (74)

$$
X\left(e^{j \omega T}\right)=\frac{1}{T} X(\omega)+\frac{1}{T} \sum_{m= \pm 1, \pm 2, \ldots} X\left(\omega+m \frac{2 \pi}{T}\right)
$$

Therefore

$$
\begin{aligned}
\widetilde{X}(\omega) & =T X\left(e^{j \omega T}\right) \operatorname{sinc}\left(\frac{\omega T}{2}\right) \\
& =X(\omega) \operatorname{sinc}\left(\frac{\omega T}{2}\right)+\left[\sum_{m \neq 0} X\left(\omega+m \frac{2 \pi}{T}\right)\right] \operatorname{sinc}\left(\frac{\omega T}{2}\right) .
\end{aligned}
$$

and we can see that in general
Even in the case when $x(t)$ has a spectrum bounded by $\Omega<\pi / T$, the following holds for $\omega \in(-\Omega, \Omega)$

$$
\tilde{X}(\omega)=X(\omega) \operatorname{sinc}\left(\frac{\omega T}{2}\right) \neq X(\omega) \quad(\text { if } \omega \neq 0)
$$

which shows the error of the digital-to-analog convertor.
F. (Periodic sampling step function)

We consider the mathematical representation of the sampling

$$
x(t) \rightarrow x_{s}(t)=x(t) \cdot s(t)= \begin{cases}x(n T), & \text { if } t=n T \\ 0, & \text { otherwise }\end{cases}
$$

when the continuous-time signal is transformed into a continuous-time signal with values of the original signal at points $n T$. The modulated signal $s(t)$ is periodic impulse function with period $T$

$$
\begin{aligned}
s(t) & =\{\ldots, \delta(t+2 T), \delta(t+T), \delta(t), \delta(t-T), \delta(t-2 T), \ldots\} \\
& =\sum_{n \in Z} \delta(t-n T)=\frac{1}{T} \sum_{n \in Z} e^{-j n \omega_{s} t}
\end{aligned}
$$

where $\omega_{s}=2 \pi / T$ is the sampling frequency.
In the Fourier domain, we following diagram holds for the function $s(t)$

$$
\begin{array}{rcc}
s(t) & = & \frac{1}{T} \sum_{n \in Z} e^{-j n \omega_{s} t} \\
\downarrow \mathcal{F} & \downarrow \mathcal{F} \\
S(\omega) & = & \frac{1}{T} \sum_{n \in Z} 2 \pi \delta\left(\omega-n \omega_{s}\right)
\end{array}
$$

Next, because of the property of the convolution in the frequency domain, we obtain the following

$$
\begin{array}{ccc}
x_{s}(t) & = & x(t) \cdot s(t) \\
\downarrow \mathcal{F} & = & \downarrow \mathcal{F} \\
& & \frac{1}{2 \pi} \int_{-\infty}^{\infty} X(\omega-\tau) S(\tau) d \tau
\end{array}
$$

and

$$
\begin{aligned}
X_{s}(\omega) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} X(\omega-\tau)\left[\frac{1}{T} \sum_{n \in Z} 2 \pi \delta\left(\tau-n \omega_{s}\right)\right] d \tau=\frac{1}{T} \sum_{n \in Z} \int_{-\infty}^{\infty} X(\omega-\tau) \delta\left(\tau-n \omega_{s}\right) d \tau \\
& =\frac{1}{T} \sum_{n \in Z} X\left(\omega-n \omega_{s}\right)=\frac{1}{T} X(\omega)+\frac{1}{T} \sum_{n \in Z \backslash\{0\}} X\left(\omega-n \omega_{s}\right)
\end{aligned}
$$

We can see again that the Fourier transform of the function $x_{s}(t)$ consists of periodic copies (withe period $\omega_{s}$ ) of the scaled Fourier transform of the original signal $x(t)$. If the Fourier transform of $x(t)$ is bound limited, $X(\omega)=0$ for $\omega \notin(-\Omega, \Omega)$, then these copies do not overlap if $\omega_{s}>2 \Omega$. In this case, we have

$$
X(\omega)=T X_{s}(\omega), \quad \forall \omega \in(-\Omega, \Omega)
$$

In other words, the "lowpass" filter

$$
Y_{l p}(\omega)=\left\{\begin{array}{ll}
T, & \text { if }|\omega|<\omega_{c u t} \\
0, & \text { otherwise }
\end{array} \quad \xrightarrow{\mathcal{F}^{-1}} \quad y_{l p}(t)=\frac{T \omega_{c u t}}{\pi} \operatorname{sinc}\left(\omega_{c u t} t\right)\right.
$$

with a cutoff frequency $\omega_{\text {cut }}=\omega_{\text {cutoff }}$ from the interval $\left(\Omega, \omega_{s} / 2\right)$ can be used to recover the Fourier transform

$$
X(\omega)=Y_{l p}(\omega) X_{s}(\omega)
$$



Fig. 14. Fourier transform calculation by the lowpass filter $Y(\omega)$.
In the case when $\omega_{s}<2 \Omega$ (or $T>\pi / \Omega$ ), the Fourier transform of the signal $x(t)$ cannot be recovered from the Fourier transform of the signal $x_{s}(t)$, because of overlapping of copies $X\left(\omega-n \omega_{s}\right)$. Thus, $X(\omega) \neq T X_{s}(\omega)$, or $X(\omega) \neq Y_{l p}(\omega) X_{s}(\omega)$ for any lowpass filter.

Consider now the pulse train sampling of $x(t)$

$$
x_{p}(t)=\sum_{n \in Z} x(n T) \delta(t-n T)
$$

that is continuous-time function with non zero values at points $t=n T$. In many cases of $x(t)$, the bandwidth of the function exceeds $2 \pi / T$ and it still may possible to recover the continuous signal from the samples $x(n T)$. For instance, if $x(t)$ is a piece -wise linear function, then by the impulse response

$$
h_{\Delta}(t)= \begin{cases}1-|t| / T, & \text { if }|t| \leq T \\ 0, & \text { otherwise }\end{cases}
$$

the linear interpolation can be performed to reconstruct $x(t)$. In this case, in the reconstructed continuous-time signal $x_{p}(t) * h_{\Delta}(t)$ the samples $x(n T)$ are connected with straight lines.

## VI. Fourier transform of periodic signals

We recall here the relation between the Fourier transform and Fourier series of a periodic signal $x(t)$ with period $T$

$$
x(t)=\sum_{n=-\infty}^{\infty} c_{n} e^{j n \omega_{0} t} \quad\left(\omega_{0}=2 \pi / T\right)
$$

where the coefficients of the series are defined as

$$
c_{n}=\frac{1}{T} \int_{0}^{T} x(t) e^{-j n \omega_{0} t} d t, \quad n=0, \pm 1, \pm 2, \ldots
$$

According to properties of the Fourier transform, we obtain

$$
\begin{array}{|ccc|}
\hline x(t) & = & \sum_{n=-\infty}^{\infty} c_{n} e^{j n \omega_{0} t}  \tag{83}\\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} \\
X(\omega) & = & \sum_{n=-\infty}^{\infty} c_{n} 2 \pi \delta\left(\omega-n \omega_{0}\right) \\
\hline
\end{array}
$$

Thus the Fourier transform of the periodic signal is the set of uniformly spaced delta-impulses (the space interval equals to the fundamental frequency $\omega_{0}$ ).

Example 1: Consider $x(t)=2 \cos \left(\omega_{0} t\right)$. This periodic signal has only one frequency and that is the fundamental frequency $\omega_{0}$. The coefficients of the Fourier series

$$
c_{n}=\left\{\begin{array}{lr}
1, & n=1 \\
1, & n=-1 \\
0, & \text { otherwise }
\end{array}\right.
$$

Therefore

$$
\begin{aligned}
x(t) & \rightarrow X(\omega)=c_{1} \cdot 2 \pi \delta\left(\omega-1 \omega_{0}\right)+c_{-1} \cdot 2 \pi \delta\left(\omega+1 \omega_{0}\right) \\
X(\omega) & =2 \pi\left[\delta\left(\omega-\omega_{0}\right)+\delta\left(\omega+\omega_{0}\right)\right]
\end{aligned}
$$

as shown in Fig. 15. The signal $x(t)$ is bounded in spectrum by frequency $\omega_{b}=\omega_{0}$.


Fig. 15. Fourier transform of $x(t)$.
Consider now the signal composed by sinusoidal signals with two frequencies

$$
x(t)=2 \cos \left(\omega_{0} t\right)-\frac{1}{2} \cos \left(2 \omega_{0} t\right)
$$

The fundamental frequency of the signal equals $\omega_{0}$ and coefficients of the Fourier series are defined as

$$
c_{n}=\left\{\begin{array}{rr}
1, & n= \pm 1 \\
-1 / 4, & n= \pm 2 \\
0, & \text { otherwise } .
\end{array}\right.
$$

Then

$$
\begin{gathered}
X(\omega)=c_{1} \cdot 2 \pi \delta\left(\omega-\omega_{0}\right)+c_{-1} \cdot 2 \pi \delta\left(\omega+\omega_{0}\right)+c_{2} \cdot 2 \pi \delta\left(\omega-2 \omega_{0}\right)+c_{-2} \cdot 2 \pi \delta\left(\omega+2 \omega_{0}\right) \\
X(\omega)=2 \pi\left[\delta\left(\omega-\omega_{0}\right)+\delta\left(\omega+\omega_{0}\right)\right]-\frac{\pi}{2}\left[\delta\left(\omega-2 \omega_{0}\right)+\delta\left(\omega+2 \omega_{0}\right)\right] .
\end{gathered}
$$

The spectrum of the signal $x(t)$ is bounded by frequency $\omega_{b}=2 \omega_{0}$. Figure 16 shows the Fourier transform of this signal.


Fig. 16. Fourier transform of $x(t)$.
Consider another signals composed by two sinusoidal signals with frequencies $1 / 2$ and $1 / 3$ respectively

$$
x(t)=2 \cos \left(\frac{1}{3} t\right)-\frac{1}{2} \cos \left(\frac{1}{2} t\right) .
$$

The fundamental frequency of the signal equals $\omega_{0}=1 / 6$,since

$$
x(t)=2 \cos \left(2 \cdot \frac{1}{6} t\right)-\frac{1}{2} \cos \left(3 \cdot \frac{1}{6} t\right)
$$

The coefficients of the Fourier series are defined by

$$
c_{n}=\left\{\begin{array}{rr}
1, & n= \pm 2 \\
-1 / 4, & n= \pm 3 \\
0, & \text { otherwise. }
\end{array}\right.
$$

Then

$$
\begin{gathered}
X(\omega)=c_{2} \cdot 2 \pi \delta\left(\omega-2 \cdot \frac{1}{6}\right)+c_{-2} \cdot 2 \pi \delta\left(\omega+2 \cdot \frac{1}{6}\right)+c_{3} \cdot 2 \pi \delta\left(\omega-3 \cdot \frac{1}{6}\right)+c_{-3} \cdot 2 \pi \delta\left(\omega+3 \cdot \frac{1}{6}\right) \\
=c_{2} \cdot 2 \pi \delta\left(\omega-\frac{1}{3}\right)+c_{-2} \cdot 2 \pi \delta\left(\omega+\frac{1}{3}\right)+c_{3} \cdot 2 \pi \delta\left(\omega-\frac{1}{2}\right)+c_{-3} \cdot 2 \pi \delta\left(\omega+\frac{1}{2}\right) \\
X(\omega)=2 \pi\left[\delta\left(\omega-\frac{1}{3}\right)+\delta\left(\omega+\frac{1}{3}\right)\right]-\frac{\pi}{4}\left[\delta\left(\omega-\frac{1}{2}\right)+\delta\left(\omega+\frac{1}{2}\right)\right] .
\end{gathered}
$$

The signal has a bounded spectrum by frequency $\omega_{b}=1 / 2$.

## A. Discrete to continuous-time signal transformation

The first time when we face with relation between the discrete and continuous Fourier transforms is the case when a discrete-time signal $x(n)$ is represented as a continuous signal

$$
x_{c}(t)=\sum_{n=-\infty}^{\infty} x(n) \delta(t-n T)
$$

and $T$ is a given number. The following diagram holds in this case

$$
\begin{array}{|ccccll|}
\hline x(n) & \Rightarrow & x_{c}(t) & = & \sum_{n=-\infty}^{\infty} x(n) \delta(t-n T) & \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} & \downarrow \mathcal{F} \\
X\left(e^{j \omega}\right) & \Rightarrow & X_{c}(\omega) & = & \sum_{n=-\infty}^{\infty} x(n) e^{-j n \omega T} & \leftarrow\left(e^{-j \omega T}\right)^{n} \\
\hline
\end{array}
$$

Thus

$$
x(n) \rightarrow X\left(e^{j \omega}\right), \quad x_{c}(t) \rightarrow X_{c}(\omega)=X\left(e^{j \omega T}\right)
$$

which means that the Fourier transform of the continuous-time version of the signal is the discrete Fourier transform after the frequency transformation $\omega \rightarrow \omega T$.
$X\left(e^{j \omega}\right)$ is a periodic function with period $P_{d}=2 \pi$, and $X_{c}(\omega)$ is a periodic function with period $P_{c}=2 \pi / T$

$$
X_{c}(\omega)=X_{c}\left(\omega+k \frac{2 \pi}{T}\right), \quad-\frac{\pi}{T}<\omega<\frac{\pi}{T}, \quad k=0, \pm 1, \ldots .
$$

Note that $X\left(e^{j \omega T}\right)$ is the Fourier transform of the continuous signal, not "sampled signal"

$$
X\left(e^{j \omega T}\right) \rightarrow x(n T)
$$

and we are not considering the sampled signal in this example.

## B. Sampling Theorem

Consider a proof of the sampling theorem, which is similar to the considered above case when the discrete-time signal is represented by the continuous-time signal. For the sampling process with sampling period $T$

$$
x(t) \rightarrow x(n T), \quad n=0, \pm 1, \ldots
$$

we consider the continuous-time representation

$$
x_{c}(t)=\sum_{n} x(n T) \delta(t-n T)=x(t) \sum_{n} \delta(t-n T)=x(t) \cdot p(t)
$$

where the impulse train

$$
p(t)=\sum_{n} \delta(t-n T)
$$

is a periodic function (period is $T$ ).
According to the property of convolution (and duality), the following is valid

$$
x_{c}(t)=x(t) \cdot p(t) \rightarrow \frac{1}{2 \pi} X(\omega) * P(\omega)=X_{c}(\omega)
$$

Thus

$$
X_{c}(\omega)=\frac{1}{2 \pi} X(\omega) * \frac{2 \pi}{T} \sum_{n} \delta\left(\omega-n \omega_{0}\right)=\frac{1}{T} \sum_{n} X\left(\omega-n \omega_{0}\right)
$$

where $\omega_{0}$ is sampling frequency (rate), $\omega_{0}=2 \pi / T$.

## C. Examples of sampling

1. Consider the sinusoidal signal $x(t)=\cos (\pi / 3 t)$, which Fourier transform equals

$$
X(\omega)=2 \pi \frac{1}{2}\left[\delta\left(\omega-\frac{\pi}{3}\right)+\delta\left(\omega+\frac{\pi}{3}\right)\right] .
$$

The spectrum of this signal is bounded by frequency $\omega_{b}=\pi / 3$, i.e.

$$
X(\omega)=0, \quad \text { if }|\omega|>\omega_{b}
$$

Let $T>0$ be a sampling period. Then according to relation of the Fourier transforms of the sampled and continuous signals

$$
\begin{aligned}
X\left(e^{j \omega T}\right) & =X_{c}(\omega)=\frac{1}{T} \sum_{n} X\left(\omega-n \frac{2 \pi}{T}\right) \\
& =\frac{1}{T} \sum_{n} \pi\left[\delta\left(\left[\omega-\frac{\pi}{3}\right]-n \frac{2 \pi}{T}\right)+\delta\left(\left[\omega+\frac{\pi}{3}\right]-n \frac{2 \pi}{T}\right)\right]
\end{aligned}
$$

or

$$
\begin{equation*}
X\left(e^{j \omega T}\right)=X_{c}(\omega)=\frac{\omega_{s}}{2} \sum_{n}\left[\delta\left(\omega-\frac{\pi}{3}-n \omega_{s}\right)+\delta\left(\omega+\frac{\pi}{3}-n \omega_{s}\right)\right] \tag{84}
\end{equation*}
$$

where the sampling frequency (rate) $\omega_{s}=2 \pi / T$.
We now analyze which value of $T=1 / 2 s, T=2 s, T=3 s$, and $T=6 s$ can be used for proper sampling procedure. For that we will chose such values of $T$ for which the condition of not overlapping $\pi / T>\omega_{b}$ holds. We have the following four inequalities

$$
\frac{\pi}{T}=2 \pi>\omega_{b}, \quad \frac{\pi}{T}=\frac{\pi}{2}>\omega_{b}, \quad \frac{\pi}{T}=\frac{\pi}{3}=\omega_{b}, \quad \frac{\pi}{T}=\frac{\pi}{6}<\omega_{b} .
$$

Therefore, values of time sampling equal $0.5 s, 2 s$, and $3 s$ can be used for sampling the signal $x(t)$.
2. Consider the input signal to be $x(t)=5 \operatorname{sinc}(5 \pi t)$, for which the Fourier transform is bounded by $\omega_{b}=5 \pi$. Indeed, the following table of calculations holds


Thus $X(\omega)=\operatorname{rect}\left(\frac{\omega}{10 \pi}\right), X(\omega)=0$ if $|\omega|>5 \pi$, and $\omega_{b}=5 \pi$.
Consider the following candidates for time sampling $T=1 s, T=0.25 s$, and $T=0.1 \mathrm{~s}$. We again chose such values of $T$ for which the condition of not overlapping $\pi / T>5 \pi$ holds. We have the following

$$
\frac{\pi}{T}=\pi<5 \pi, \quad \frac{\pi}{T}=4 \pi<5 \pi, \quad \frac{\pi}{T}=10 \pi>5 \pi
$$

Thus, $T=0.1 s$ can be considered for good time sampling of the signal $5 \operatorname{sinc}(5 \pi t)$.

## D. DTFT of Periodic Sequences ${ }^{1}$

Let $N$ be a period of a sequence $x(n)$, and let $x_{0}(n)$ be the following part of $x(n)$ of one period

$$
x_{0}(n)= \begin{cases}x(n), & n=0:(N-1)  \tag{86}\\ 0, & \text { otherwise }\end{cases}
$$

The sequence $x(n)$ can be written as $\left(u_{0}(n)=\delta[n]\right)$

$$
\begin{equation*}
x(n)=\sum_{k \in Z} x_{0}(n-k N)=\sum_{k \in Z} u_{0}(n-k N) * x_{0}(n)=x_{0}(n) * \sum_{k \in Z} u_{0}(n-k N) . \tag{87}
\end{equation*}
$$

As we know the train of discrete impulses corresponds in the frequency domain to the train of delta functions

$$
p(n)=\sum_{k} u_{0}(n-k N) \rightarrow \omega_{0} \sum_{k \in Z} \delta\left(\omega-k \omega_{0}\right)
$$

where $\omega_{0}=2 \pi / N$. Therefore, the DTFT of the sequence $x(n)$ can be expressed as follows

$$
\begin{aligned}
x(n) & =x_{0}(n) * \sum_{k \in Z} u_{0}(n-k N) \rightarrow X_{0}\left(e^{j \omega}\right) \cdot\left[\omega_{0} \sum_{k \in Z} \delta\left(\omega-k \omega_{0}\right)\right] \\
X\left(e^{j \omega}\right) & =\omega_{0} \sum_{k \in Z} X_{0}\left(e^{j \omega}\right) \delta\left(\omega-k \omega_{0}\right)=\omega_{0} \sum_{k \in Z} X_{0}\left(e^{j k \omega_{0}}\right) \delta\left(\omega-k \omega_{0}\right) .
\end{aligned}
$$

Thus

$$
\begin{equation*}
X\left(e^{j \omega}\right)=\frac{2 \pi}{N} \sum_{k \in Z} X_{0}\left(e^{j k \frac{2 \pi}{N}}\right) \delta\left(\omega-k \frac{2 \pi}{N}\right) \tag{88}
\end{equation*}
$$

The DTFT is a periodic function with period $2 \pi$, and the inverse DTFT can be calculated as follows

$$
\begin{align*}
x(n) & =\frac{1}{2 \pi} \int_{0}^{2 \pi} X\left(e^{j \omega}\right) e^{j n \omega} d \omega \\
& =\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{2 \pi}{N} \sum_{k \in Z} X_{0}\left(e^{j k \frac{2 \pi}{N}}\right) \delta\left(\omega-k \frac{2 \pi}{N}\right) e^{j n \omega} d \omega \\
& =\frac{1}{N} \sum_{k \in Z} \int_{0}^{2 \pi} \underbrace{X_{0}\left(e^{j k \frac{2 \pi}{N}}\right) e^{j n \omega}} \delta\left(\omega-k \frac{2 \pi}{N}\right) d \omega  \tag{89}\\
& =\left.\frac{1}{N} \sum_{k=0}^{N-1} \underbrace{X_{0}\left(e^{j k \frac{2 \pi}{N}}\right) e^{j n \omega}}\right|_{\omega=k \frac{2 \pi}{N}}=\frac{1}{N} \sum_{k=0}^{N-1} X_{0}\left(e^{j k \frac{2 \pi}{N}}\right) e^{j n k \frac{2 \pi}{N}} .
\end{align*}
$$

Thus

$$
\begin{equation*}
x(n)=\frac{1}{N} \sum_{k=0}^{N-1} X_{0}\left(e^{j k \frac{2 \pi}{N}}\right) e^{j n k \frac{2 \pi}{N}}, \quad n=0, \pm 1, \pm 2, \ldots \tag{90}
\end{equation*}
$$

The limits of summation in (90) equal to 0 and $N-1$, since frequency-points $k 2 \pi / N$ lie outside the interval of integration $[0,2 \pi)$ in (89), when $k<0$ and $k \geq N$.
${ }^{1}$ Addition to $\S 12.3$ of the text book.
VII. Discrete linear convolutions .............................................................EE-3523

## A. Array method

Consider the case when two discrete-time signals to be convolved are of finite length, for instance

$$
\begin{gathered}
\{x(n)\}=\{\ldots, 0, x(-3), x(-2), x(-1), x(0), x(1), x(2), \ldots, x(100), 0, \ldots\} \\
\{h(n)\}=\{\ldots, 0, h(-1), h(0), h(1), h(2), h(3), h(4), \ldots, h(9), 0, \ldots\} .
\end{gathered}
$$

Then to calculate the convolution we compose the following table

| $x(-3)$ | $x(-2)$ | $x(-1)$ | $x(0)$ | $x(1)$ | $x(2)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $h(-1)$ | $h(0)$ | $h(1)$ | $h(2)$ | $h(3)$ | $h(4)$ |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
| $\frac{x(-3)}{\downarrow} h(-1)$ | $x(-2) h(-1)$ | $x(-1) h(-1)$ | $x(0) h(-1)$ | $x(1) h(-1)$ | $x(2) h(-1)$ |
|  | $\frac{x(-3)}{+} h(0)$ | $x(-2) h(0)$ | $x(-1) h(0)$ | $x(0) h(0)$ | $x(1) h(0)$ |
|  | $\downarrow$ | $\frac{x(-3) h(1)}{+}$ | $x(-2) h(1)$ | $x(-1) h(1)$ | $x(0) h(1)$ |
|  |  | $\downarrow$ | $\frac{x(-3) h(2)}{+}$ | $x(-2) h(2)$ | $x(-1) h(2)$ |
| $(n=-3-1)$ |  |  | $\downarrow$ | $\frac{x(-3) h(3)}{+}$ | $x(-2) h(3)$ |
|  |  |  |  | $\downarrow$ | $\frac{x(-3) h(4)}{+}$ |
|  |  |  |  |  |  |
| $y(n=-4)$ | $y(-3)$ | $y(-2)$ | $y(-1)$ | $y(0)$ | $y(1)$ |

Consider the following example given below in the table.

| $x(n)$ | 1 | $\underline{3}$ | 2 | 4 | 7 | 5 | 3 | 4 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $h(n)$ | -2 | 1 | $\underline{2}$ | 1 | -1 | -2 | 1 | 3 |
|  |  |  |  |  |  |  |  |  |
|  | -2 | -6 | -4 | -8 | -14 | -10 |  |  |
|  | $\downarrow$ | 1 | 3 | 2 | 4 | 7 |  |  |
|  |  | + | 2 | 6 | 4 | 8 |  |  |
|  |  | $\downarrow$ | + | 1 | 3 | 2 |  |  |
|  |  |  | $\downarrow$ | + | -1 | -3 |  |  |
|  |  |  |  | $\downarrow$ | + | -2 |  |  |
|  |  |  |  |  | $\downarrow$ | + |  |  |
|  |  |  |  |  |  | $\downarrow$ | + |  |
|  |  |  |  |  |  |  | $\downarrow$ | + |
|  |  |  |  |  |  |  |  |  |
| $y(n)$ | -2 | -5 | 1 | $\underline{1}$ | -4 | 2 |  |  |

Thus

$$
y=\{-2,-5,1, \underline{1},-4,2, \quad, \quad, \ldots\}, \quad(y(-3)=-2) .
$$



Fig. 16. Interval of definition of the 1-D linear convolution.

## B. Circular convolution

For two given discrete-time signals $f(n)$ and $h(n)$, the linear convolution is defined as

$$
\begin{equation*}
g(n)=(f * h)(n)=\sum_{m} h(n-m) f(m) \tag{90}
\end{equation*}
$$

In the case, when $f(n)$ is defined in the finite interval $[a, b]$ and the signal $h(n)$ is defined in the interval $[c, d]$, where $a, b, c, d \in Z$, the linear convolution $g(n)$ is defined in the interval $[a+c, b+d]$. Thus, the length of the linear convolution is calculated as

$$
\begin{aligned}
\operatorname{length}(f * h) & =(b+d)-(a+c)+1 \\
& =(b-a+1)+(d-c+1)-1 \\
& =\operatorname{length}(f)+\operatorname{length}(h)-1 .
\end{aligned}
$$

Figure 16 illustrate this property. In particular case, when the length of the signals $f(n)$ and $h(n)$ are equal, the convolution yields the signal of length 2 length $(f)-1$.

We now consider another concept of the linear operation, that operates over signals of equal length $N$ and yields the signal of the same length. The periodic convolution of two discrete-time signals $f(n)$ and $h(n)$ is defined as

$$
\begin{equation*}
y(n)=(f \otimes h)(n)=\sum_{m=0}^{N-1} h(n-m) f(m)=\sum_{m=0}^{N-1} h([n-m] \bmod N) f(m) \tag{91}
\end{equation*}
$$

where argument $n-m$ is considered modulo $N$.
This linear operation is also called the circular convolution, because of the simple method of calculations by using two concentric circles. For that, $N$ values of the signal $f$ are spaced equally around the outer circle in counter clock-wise direction. Then, $N$ values of the signal $h$ are spaced equally around the inner circle in clock-wise direction. On each stage of the linear convolution calculation, the values on the inner circle are rotated counter clock-wise through the angle $2 \pi / N$ and the sum of products of the corresponding values of (inside) two circles is calculated. As an example, Figure 17(a) shows the data of two sequences of length six placed inside the sectors of two circles. The calculation on this figure corresponds to the calculation $y(0)$. Figure 17(b) shows the calculation of the linear circular convolution $y(1)$, for what the data of the first circle have been rotated clockwise by $60^{\circ}$.

(a) 1-D DCC (step 1)

(b) 1-D DCC (step 2)

Fig. 17. Method of calculation of the 1-D discrete circular convolution of length 6.

## C. DCC and DLC

There is the simple relation between the circular convolution and linear convolution, that yields a signal of length $M=N_{1}+N_{2}-1$, where $N_{1}=$ length $(f)$ and $N_{2}=$ length $(h)$. One can calculate $M$ values of the linear convolution $g(n)$ by means of the circular convolution. For that the signals $f(n)$ and $h(n)$ are extended to the same length, by zero-padding

$$
\begin{aligned}
& \{f(n)\} \rightarrow\{\bar{f}(n)\}=\{f(0), f(1), f(2), \ldots, f\left(N_{1}-1\right), \underbrace{0,0,0, \ldots, 0}_{N_{2}-1 \text { times }}\} \\
& \{h(n)\} \rightarrow\{\bar{h}(n)\}=\{h(0), h(1), h(2), \ldots, h\left(N_{2}-1\right), \underbrace{0,0,0, \ldots, 0}_{N_{1}-1 \text { times }}\} .
\end{aligned}
$$

The circular convolution of the two zero-padding sequences results in the $M$ values of the linear convolution. That is

$$
\begin{equation*}
g(n)=(f * h)(n)=(\bar{f} \otimes \bar{h})(n), \quad n=0:(M-1) \tag{92}
\end{equation*}
$$



Fig. 18. Method of calculation of the 1-D discrete linear convolution by the circular convolution.
VIII. Discrete-time Fourier transform of periodic signals EE-3523

Consider a periodic sequence (discrete-time signal) $x(n)=x(n+N)$ with fundamental period $N>0$. Let $\widehat{x}(n)$ be one period of the signal

$$
\widehat{x}=\{0, \ldots, 0, x(0), x(1), \ldots, x(N-1), 0, \ldots\}
$$

Using the discrete unit impulse $u_{0}(n),{ }^{1}$ we can write

$$
\begin{aligned}
\widehat{x}(n) & =x(0) u_{0}(n)+x(1) u_{0}(n-1)+x(2) u_{0}(n-2)+\ldots+x(N-1) u_{0}(n-N+1) \\
& =\sum_{k=0}^{N-1} x(k) u_{0}(n-k) .
\end{aligned}
$$

The periodic signal

$$
x=\{\ldots,\{\widehat{x}(n) ; n=0:(N-1)\},\{\widehat{x}(n) ; n=0:(N-1)\},\{\widehat{x}(n) ; n=0:(N-1)\}, \ldots\}
$$

can be written as

$$
x(n)=\sum_{k=-\infty}^{\infty} \widehat{x}(n-k N)=\widehat{x}(n) * \sum_{k=-\infty}^{\infty} u_{0}(n-k N) .
$$

The Fourier transform of the periodic signal $x(n)$

$$
\begin{array}{|ccccc|lc|}
\hline x(n) & = & \widehat{x}(n) & * & \sum_{k=-\infty}^{\infty} u_{0}(n-k N) & n & N  \tag{94}\\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} & \downarrow & & & \\
X\left(e^{j \omega}\right) & = & \widehat{X}\left(e^{j \omega}\right) & \cdot & \frac{2 \pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega-k \frac{2 \pi}{N}\right) & \downarrow & \downarrow \\
\hline & \omega_{0}=\frac{2 \pi}{N} \\
\hline
\end{array}
$$

We use here the fact that the discrete-time train signal $t(n)=\sum_{k} u_{0}(n-k N)$ can be represented as the continuous-time train signal $t(t)=\sum_{k} \delta(t-k N)$. And as we know from Section VI(A), the Fourier transform of the train signal coincides with the discrete Fourier transform of the discretetime train signal, i.e. $T(\omega)=T\left(e^{j \omega}\right)$

$$
\begin{aligned}
& t(n)=\sum_{k=-\infty}^{\infty} u_{0}(n-k N) \rightarrow^{\mathcal{D F}} \rightarrow T\left(e^{j \omega}\right)=T(\omega) \\
& t(t)=\sum_{k=-\infty}^{\infty} \delta(t-k N) \rightarrow^{\mathcal{C F}} \rightarrow T(\omega)=\frac{2 \pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega-k \frac{2 \pi}{N}\right) .
\end{aligned}
$$

Thus, we obtain from (94) that

$$
X\left(e^{j \omega}\right)=\frac{2 \pi}{N} \sum_{k=-\infty}^{\infty} \hat{X}\left(e^{j \omega}\right) \delta\left(\omega-k \frac{2 \pi}{N}\right)=\frac{2 \pi}{N} \sum_{k=-\infty}^{\infty} \hat{X}\left(e^{j \frac{2 \pi}{N} k}\right) \delta\left(\omega-k \frac{2 \pi}{N}\right)
$$

$$
{ }^{1} u_{0}(n)=1 \text { if } n=0, \text { and } u_{0}(n)=0 \text { if } n \neq 0 .
$$

The discrete Fourier transform $\widehat{X}\left(e^{j \frac{2 \pi}{N} k}\right)$ is the periodic sequence relative $k$ and the period is $N$

$$
\widehat{X}\left(e^{j \frac{2 \pi}{N} k}\right)=\sum_{k=0}^{N-1} \widehat{x}(n) e^{-j \frac{2 \pi}{N} k n}
$$

This discrete transform is called in general the $N$-point discrete Fourier transform (DFT)

$$
\begin{equation*}
X_{k}=\sum_{k=0}^{N-1} x_{n} e^{-j \frac{2 \pi}{N} k n}=\sum_{k=0}^{N-1} x_{n} W^{n k}, \quad k=0:(N-1) \tag{95}
\end{equation*}
$$

where $W=e^{-j \frac{2 \pi}{N}}$. The inverse $N$-point discrete Fourier transform is defined as

$$
x_{n}=\frac{1}{N} \sum_{k=0}^{N-1} X_{k} e^{j \frac{2 \pi}{N} k n}=\frac{1}{N} \sum_{k=0}^{N-1} X_{k} W^{-k n}, \quad n=0:(N-1) .
$$

The pair of $N$-point discrete Fourier transforms are implemented in MATLAB by fast algorithms as functions $f f t$ and $i f f t$. The $N$-point discrete Fourier transform is defined by cosine and sine waveforms of $N$ frequencies $\omega=2 \pi n / N$ uniformly spaced on the unit circle.

Example 1: Consider the following periodic sequence

$$
x=\{\ldots, 1,2,3,4,1,2,3,4,1,2,3,4, \ldots\}
$$

for which $N=4$ and $x(0)=1$. Then one period of the sequence is $\widehat{x}=\{\underline{1}, 2,3,4\}$ and its 4 -point DFT is calculated as follows

$$
\begin{aligned}
X_{k} & =\sum_{k=0}^{3} x_{n} W^{n k}=x_{0} W^{0 k}+x_{1} W^{1 k}+x_{2} W^{2 k}+x_{3} W^{3 k}, \quad k=0: 3 \\
& =1+2 W^{1 k}+3 W^{2 k}+4 W^{3 k} \\
& =1+2 e^{-j \frac{2 \pi}{4} k}+3 e^{-j \frac{2 \pi}{4} 2 k}+4 e^{-j \frac{2 \pi}{4} 3 k} \\
& =1+2(-j)^{k}+3(-1)^{k}+4(j)^{k} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& X_{0}=1+2+3+4=10 \quad \text { (power of signal) } \\
& X_{1}=1-2 j-3+4 j=-2+2 j \\
& X_{2}=1-2+3-4=-2 \\
& X_{3}=1+2 j-3-4 j=-2-2 j=\bar{X}_{1} .
\end{aligned}
$$

The main application of the $N$-point discrete Fourier transform in linear filtering, namely when calculating the circular convolution of two discrete-time signals of finite length $N$,

$$
\begin{equation*}
y_{n}=\sum_{k=0}^{N-1} x_{k} h_{n-k \bmod N} \rightarrow Y_{k}=X_{k} H_{k}, \quad k=0:(N-1) . \tag{96}
\end{equation*}
$$

And because the linear convolution can be reduced to the circular convolution, the method of the $N$-point DFT can be used for fast calculating the linear convolution.


Fig. 20. (a) Signal $x(t)$. (b) Correlogram of signals $x(t)$ and $y_{1}(t)$. (c) Signals $y_{1}(t)$ and $y_{2}(t)$. (d) Correlogram of signals $x(t)$ and $y_{2}(t)$.

## A. Correlation operations

There are tools to measure if two signals are "similar," dependent, or/and correlated with each other.

## A. 1 Correlogram

One can use the correlogram which is the plot of one signal versus another. In the ideal case when the signals are the same, the correlogram is the straight line. In general, if correlogram tends to a straight line then two signals are said to be highly correlated. The closer the correlogram to a straight line, the more correlated the signals are (but not always).

Example 2: Consider the signal

$$
x(t)=2 \cos \left(\omega_{0} t\right)+0.1 \sin \left(\omega_{1} t-0.25\right)=2 y_{1}(t)+0.1 y_{2}(t)
$$

where $\omega_{0}=2$ and $\omega_{1}=100 \omega_{0}$. Figure 20 shows the signal $x(t)$ in part (a) along with signals $y_{1}(t)$ and $y_{2}(t)$ in part (c), and the correlograms of signal $x(t)$ with these functions in parts b and d . The correlogram in part b shows that the signals $x(t)$ and $y_{1}(t)$ are highly correlated.

Example 3: Consider the following cosinusoidal signals

$$
x(t)=\cos \left(\omega_{0} t\right), \quad y(t)=\sin \left(\omega_{1} t-\vartheta\right)
$$

where the phase of the second signal $\vartheta=\pi / 4$ and $\vartheta=\pi / 2$. Figure 21 shows the signal $x(t)$ in part (a) along with signal $y(t)$ with two phases in part (c), and the correlograms of signal $x(t)$ with this function in part b. One can see form the correlogram that the principle axis of the ellipse is directed in plane by the angle equal to the phase of the signal $y(T)$.

## A. 2 Correlation function

The correlogram is a good tool for visualization, but not for precise mathematical way of expressing the relation between two signals. Such mathematical tool exists and well-known as the correlation function.


Fig. 21. (a) Signal $x(t)$.(b) Signals $y(t)$ with phase $\pi / 4$ and $\pi / 2$.(c) Correlograms of signals $x(t)$ with $y(t)$.

The correlation function shows if the signals have similarity or they are correlated, and shows how the signals are correlated in a given position, and how they correlated when one signal is shifted. We consider the correlation function as a main measure for similarity of signals.

There are two operations (or definitions) of correlation that are defined on the two "different" signals or the same signal

$$
\begin{equation*}
R_{x y}(n)=r_{x y}(n)=\sum_{k=0}^{N-1} x_{k} y_{n+k}, \quad R_{x}(n)=r_{x}(n)=\sum_{k=0}^{N-1} x_{k} x_{n+k} \tag{97}
\end{equation*}
$$

which in the case of continuous-time signals have respectively the forms

$$
\begin{equation*}
R_{x y}(t)=r_{x y}(t)=\int_{-\infty}^{\infty} x(\tau) y(t+\tau) d \tau, \quad R_{x}(t)=r_{x}(t)=\int_{-\infty}^{\infty} x(\tau) x(t+\tau) d \tau \tag{98}
\end{equation*}
$$

(We here consider only the real signals or functions).
The operation of correlation of signals can be also effectively calculated by the $N$-point DFT. Indeed, we recall that the operation of the linear convolution

$$
(x * y)(t)=\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d \tau, \quad t \in(-\infty, \infty)
$$

For the linear convolution, one of the signals is flipped in time and then a shift of the signal is performed. There is no need in time-inverting for the correlation. Therefore, we can write the correlation by using the convolution operation as

$$
R_{x y}(t)=x(-t) * y(t), \quad R_{x y}(n)=x(-n) * y(n)
$$

We also recall that for the frequency domain the following holds $X(\omega) \leftarrow x(t) \rightarrow x(-t) \rightarrow \bar{X}(\omega)$. Therefore $R_{x y}(t) \rightarrow R_{x y}(\omega)=\bar{X}(\omega) Y(\omega)$.
XI. The z-transform

The $z$-transform can be considered as a generation of the discrete Fourier transform (DFT). The $z$-transform is to the DFT what the Laplace transform is to the Fourier transform. The $z$-transform is used for analysis of discrete-time signals and systems. For many discrete signals $x(n)$, the DFT cannot be defined, but the $z$-transform with many similar properties takes place.

Let $x(n)$ be a sequence defined (may be) for all integers $n=0, \pm 1, \pm 2, \ldots$. The $z$-transform is defined by

$$
\begin{equation*}
x(n) \xrightarrow{z} X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad z \in C^{2}, \tag{109}
\end{equation*}
$$

where $z$ is a complex number for which $X(z)$ exists. In other words, the concept of the $z$-transform refers to as the formula (109) with region of convergence (ROC) of $X(z)$. We recall that the formula for DFT is defined by

$$
X\left(e^{j \omega}\right)=\sum_{n=-\infty}^{\infty} x(n) e^{-j \omega n}=\sum_{n=-\infty}^{\infty} x(n)\left(e^{j \omega}\right)^{-n}, \quad \omega \in(-\pi, \pi),
$$

and one can see a similarity when substituting $e^{j \omega}$ by $z$. The DFT is defined on points $z=e^{j \omega}$ lying on the unit circle. $X\left(e^{j \omega}\right)=X(z)_{\mid z=e^{j \omega}}$, i.e. the DFT is the $z$-transform considered on the unit circle $O_{1}$. In general, the region of convergence of the $z$-transform may not contain this circle.

The following properties can be derived directly from the definition of the transform.

1. If $z_{0} \in R O C(X)$, then $z \in R O C(X)$ for any point such that $|z|=\left|z_{0}\right|$. Indeed

$$
|X(z)| \leq \sum_{n=-\infty}^{\infty}\left|x(n) z^{-n}\right|=\sum_{n=-\infty}^{\infty}\left|x(n) z_{0}^{-n}\right|<\infty .
$$

2. If $z_{1} \neq z_{2} \in R O C(X)$, then $z \in R O C(X)$ for any point such that $\left|z_{1}\right| \leq|z| \leq\left|z_{2}\right|$ (we assume that $\left|z_{1}\right| \leq\left|z_{2}\right|$. In other words, ROC of the $z$-transform consists of circles of different radii.

The $z$-transform can be derived as the response $y(n)$ of the LTI system with $h(n)$ impulse response to a complex exponential discrete-time signal $x(n)=A z^{n}$ :

$$
\begin{aligned}
x(n) & =A z^{n} \rightarrow y(n)=\sum_{k=-\infty}^{\infty} h(k) x(n-k)=\sum_{k=-\infty}^{\infty} A z^{n-k} h(k) \\
& =A z^{n} \sum_{k=-\infty}^{\infty} h(k) z^{-k}=x(n) \underline{H(z)} .
\end{aligned}
$$

Thus, the transfer function of the LTI system is the $z$-transform of the impulse response. This relation yields also the property of convolution

$$
\begin{equation*}
x(n) * h(n) \rightarrow X(z) H(z), \quad z \in R O C(X) \cap R O C(H) . \tag{110}
\end{equation*}
$$

Example 1: Let sequence

$$
x(n) \neq 0, \quad n \in\left[N_{1}, N_{2}\right],
$$

where $N_{1}<N_{2}$ integer numbers. Then,

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=\sum_{n=N_{1}}^{N_{2}} x(n) z^{-n}=z^{-N_{1}} \sum_{n=0}^{N_{2}-N_{1}} x(n) z^{-n}
$$

which is defined for all $z$ except may be $z=0$.
Example 2: Let sequence $x(n)=u_{0}(n)$. Then, the $z$-transform

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=x(0) z^{0}=1, \quad \text { for all } z
$$

If sequence is defined as $x(n)=u_{0}\left(n-n_{0}\right)$, where $n_{0}$ is an integer, then the $z$-transform

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=x\left(n_{0}\right) z^{-n_{0}}=z^{-n_{0}}, \quad \text { for all } z \neq 0
$$

Example 3: Let sequence be

$$
x(n)=u(n)= \begin{cases}1, & n \geq 0 \\ 0, & n<0\end{cases}
$$

Then, the $z$-transform

$$
\begin{aligned}
X(z) & =\sum_{n-0}^{\infty} x(n) z^{-n} \\
& =1+z^{-1}+z^{-2}+\ldots+z^{-n}+\ldots=\frac{1}{1-z^{-1}}
\end{aligned}
$$

which converges at all points $z$ such that $|z|>1$. The point $z=1$ is a special point of $X(z)$.
Example 4: Given a real $\omega$, consider the complex exponential sequence

$$
x(n)=e^{j \omega n} u(n)=\left\{\begin{array}{cc}
e^{j \omega n}, & n=0,1,2, \ldots \\
0, & n<0
\end{array}\right.
$$

The $z$-transform of this sequence is

$$
X(z)=\sum_{n=0}^{\infty} e^{j \omega n} z^{-n}=\sum_{n=0}^{\infty}\left(e^{j \omega} z^{-1}\right)^{n}=\frac{1}{1-e^{j \omega} z^{-1}}
$$

and converges at points $z$ such that

$$
\left|e^{j \omega} z^{-1}\right|=\left|z^{-1}\right|<1
$$

Example 5: Consider the sequence

$$
x(n)=a^{n} u(n)=\left\{\begin{array}{cc}
a^{n}, \quad n=0,1,2, \ldots \\
0, & n=-1,-2, \ldots
\end{array}\right.
$$

for a given positive number $a$.
The $z$-transform of $x(n)$ is calculated as

$$
X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}=\sum_{n=0}^{\infty} a^{n} z^{-n}=\sum_{n=0}^{\infty}\left(a z^{-1}\right)^{n}=\frac{1}{1-a z^{-1}}
$$

and the series converges at points $z$ such that

$$
\left|a z^{-1}\right|<1 \quad \Rightarrow \quad|z|>a .
$$

Example 6: Consider the discrete-time signal

$$
x(n)=\left[2\left(\frac{2}{3}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right] u(n) .
$$

The $z$-transform is calculated as follows

$$
\begin{aligned}
& X(z)=2 \sum_{n=0}^{\infty}\left(\frac{2}{3} z^{-1}\right)^{n}-\sum_{n=0}^{\infty}\left(\frac{1}{4} z^{-1}\right)^{n} \\
& X(z)=\left.2 \frac{1}{1-\frac{2}{3} z^{-1}}\right|_{|z|>\frac{2}{3}}-\left.\frac{1}{1-\frac{1}{4} z^{-1}}\right|_{|z|>\frac{1}{4}}
\end{aligned}
$$

for all $z$ such that $|z|>2 / 3$.

## A. Inverse formula

In the case of the discrete Fourier transform (when $z=e^{j \omega}$ ), the inverse formula

$$
\begin{aligned}
x(n) & =\frac{1}{2 \pi} \int_{-\pi}^{\pi} X\left(e^{j \omega}\right) e^{j \omega n} d \omega=\frac{1}{2 \pi} \int_{-\pi}^{\pi} X\left(e^{j \omega}\right) e^{j \omega n} e^{-j \omega} \frac{1}{j} d e^{j \omega} \\
& =\frac{1}{2 \pi j} \int_{-\pi}^{\pi} X\left(e^{j \omega}\right) e^{j \omega(n-1)} d e^{j \omega}=\frac{1}{2 \pi j} \int_{e^{-j \pi}}^{e^{j \pi}} X(z) z^{(n-1)} d z=\frac{1}{2 \pi j} \oint_{O_{1}} X(z) z^{n-1} d z
\end{aligned}
$$

where the last integral is taken over the unit circle $O_{1}$.
For $z$-transform, the inverse formula is similar,

$$
x(n)=\frac{1}{2 \pi j} \oint_{C} X(z) z^{n-1} d z
$$

where $C$ is any contour that embraces the original point $(0,0)$, for example the circle $O_{r}$ of radius $r>0$.

## B. Properties of $z$-transform

1. (Linearity)

If

$$
x_{1}(n) \rightarrow X_{1}(z), \quad x_{2}(n) \rightarrow X_{2}(z)
$$

then

$$
x_{1}(n)+k x_{2}(n) \rightarrow X_{1}(z)+k X_{2}(z)
$$

for any constant $k$.
2. (Delay)

$$
\begin{array}{ccc}
x(n) & \rightarrow & x\left(n-n_{0}\right) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X(z) z^{-n_{0}}
\end{array}
$$

for any integer $n_{0}$.
3. (Time-reversal)

$$
\begin{array}{rlc}
x(n) & \rightarrow & x(-n) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X\left(z^{-1}\right),
\end{array} \quad z^{-1} \in R O C(X)
$$

for instance if $\operatorname{ROC}(X(z))=\{z ;|z|>2\}$ then $R O C\left(X\left(z^{-1}\right)\right)=\{z ;|z|<1 / 2\}$.
Example 7: The following diagram holds for a given real number $a>0$ :

$$
\begin{array}{cccc}
\frac{1}{1-\underbrace{a z^{-1}}_{q}} & = & \frac{z}{\frac{z-a}{z-a}} & = \\
\downarrow z^{-1} & & -\frac{a^{-1} z}{1-\underbrace{a^{-1} z}_{q}} \\
a^{n} u(n) & \leftarrow & x(n) & \rightarrow \\
\downarrow & -a^{n} u(-n-1) \\
\Downarrow & & & \Downarrow \\
\{z ;|z|>a\} & \leftarrow & R O C & \rightarrow
\end{array}\{z ;|z|<a\}
$$

Example 8: Consider the signal

$$
x(n)=2^{n} u(n)+5^{n} u(-n)
$$

The $z$-transform for this signal is defined as follows

$$
X(z)=\left.\frac{1}{1-2 z^{-1}}\right|_{|z|>2}-\left.\frac{1}{1-\frac{1}{5} z}\right|_{|z|<5}
$$

for all $z$ such that $2<|z|<5$.
Application (Difference equation)

$$
\begin{array}{ccccccc}
y(n) & +a_{1} y(n-1) & +a_{2} y(n-2) & = & x(n) & + & b_{1} x(n-1) \\
\downarrow z & \downarrow z & & \downarrow z & & \downarrow & \\
Y(z) & +a_{1} Y(z) z^{-1} & +a_{2} Y(z) z^{-2} & = & \downarrow(z) & +b_{1} X(z) z^{-1}
\end{array}
$$

Therefore

$$
Y(z)\left[1+a_{1} z^{-1}+a_{2} z^{-2}\right]=X(z)\left[1+b_{1} z^{-1}\right]
$$

and we obtain

$$
Y(z)=\frac{1+b_{1} z^{-1}}{1+a_{1} z^{-1}+a_{2} z^{-2}} X(z) . \quad(Y(z)=H(z) X(z) ?)
$$

The equation holds only for $z$ lying in the intersection of ROCs of two $z$-transforms.
4. (Convolution)

$$
\begin{array}{cccc}
y(n) & = & x(n) & * \\
\downarrow z & \downarrow(n) \\
\downarrow & \downarrow & \downarrow z \\
Y(z) & =X(z) & \cdot & H(z)
\end{array} \quad \text { if } \quad R O C_{X} \cap R O C_{H} \neq \emptyset .
$$

Example 9: For given two numbers $a \neq b$, we consider two sequences

$$
\begin{aligned}
& x(n)=a^{n} u(n)=\left\{\begin{array}{cc}
a^{n}, & n=0,1,2, \ldots \\
0, & n=-1,-2, \ldots
\end{array}\right. \\
& h(n)=b^{n} u(n)=\left\{\begin{array}{cc}
b^{n}, & n=0,1,2, \ldots \\
0, & n=-1,-2, \ldots
\end{array}\right.
\end{aligned}
$$

and the linear convolution

$$
y(n)=\sum_{m=-\infty}^{\infty} h(m) x(n-m)=\sum_{m=0}^{n} b^{m} a^{n-m}
$$

Using the results of Example 6, we obtain the following diagram for $z$-transform:

$$
\begin{array}{ccccc}
y(n) & = & x(n) & * & h(n) \\
\downarrow z & & \downarrow z & \downarrow & \downarrow z \\
Y(z) & = & \frac{1}{1-a z^{-1}} & & \\
1-b z^{-1}
\end{array}
$$

Therefore

$$
Y(z)=A \frac{1}{1-a z^{-1}}+B \frac{1}{1-b z^{-1}}, \quad|z|>\max (|a|,|b|)
$$

where $A=-a /(b-a)$ and $B=b /(b-a)$.
Example 10: Using results of Example 5, we obtain

$$
\begin{array}{cccc}
Y(z) & =A \frac{1}{1-a z^{-1}} & +B \frac{1}{1-b z^{-1}} \\
\uparrow z & \uparrow z & \uparrow & \uparrow z \\
y(n) & = & A x(n) & +
\end{array} \quad B h(n)
$$

and the direct formula for the linear convolution can be derived as following

$$
\begin{aligned}
y(n) & =x(n) * h(n)=A x(n)+B h(n) \\
& =A a^{n} u(n)+B b^{n} u(n)=\left[A a^{n}+B b^{n}\right] u(n)=\frac{b^{n+1}-a^{n+1}}{b-a} u(n)
\end{aligned}
$$

5. (Multiplication)

$$
\begin{array}{ccc}
x_{1}(n) & x_{2}(n) & y(n)=x_{1}(n) \cdot x_{2}(n) \\
\downarrow z & \downarrow z & \downarrow z \\
X_{1}(z) & X_{2}(z) & Y(z)=\frac{1}{2 \pi j} \oint_{C} X_{1}(v) X_{2}\left(\frac{z}{v}\right) v^{-1} d v
\end{array}
$$

which, in particular case for the discrete Fourier transform takes the form

$$
Y\left(e^{j \omega}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} X_{1}\left(e^{j \theta}\right) X_{2}\left(e^{j(\omega-\theta)}\right) d \theta
$$

## C. Single-sided Z-transform

Let $x(n)$ be a sequence defined for all integers $n=0, \pm 1, \pm 2, \ldots$. The transform

$$
x(n) \xrightarrow{z} X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in R O C(X) \subseteq C^{2},
$$

is called a single-sided $z$-transform of $x(n)$.
This is the linear transform with the following property.
6. (Delay)

$$
\begin{array}{ccc}
x(n) & \rightarrow & x(n-1) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X(z) z^{-1}+x(-1)
\end{array}
$$

Indeed

$$
\begin{aligned}
x(n-1) & \rightarrow \sum_{n=0}^{\infty} x(n-1) z^{-n}=\sum_{n=-1}^{\infty} x(n) z^{-(n+1)} \\
& =x(-1)+\sum_{n=0}^{\infty} x(n) z^{-n-1}=x(-1)+z^{-1} \sum_{n=0}^{\infty} x(n) z^{-n}
\end{aligned}
$$

Similarly, in the case for the second delay, we obtain

$$
\begin{array}{ccc}
x(n) & \rightarrow & x(n-2) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X(z) z^{-2}+x(-2)+x(-1) z^{-1}
\end{array}
$$

Application (Difference equation with initial condition)
Given a coefficient $a$, we consider the difference equation with condition $y(-1)=A$ :

$$
\begin{array}{cccc}
y(n) & + & a y(n-1) & = \\
\downarrow z & & \downarrow(n) \\
Y(z) & +a\left[Y(z) z^{-1}+y(-1)\right] & = & \downarrow z \\
& X(z)
\end{array}
$$

Therefore,

$$
\begin{equation*}
Y(z)\left[1+a z^{-1}\right]+a A=X(z) \quad \Rightarrow \quad Y(z)=\frac{X(z)-a A}{1+a z^{-1}} . \tag{111}
\end{equation*}
$$

For example, if

$$
x(n)=e^{j \omega n} u_{1}(n),
$$

then we can substitute in (111)

$$
X(z)=\frac{1}{1-e^{j \omega} z^{-1}}, \quad|z|>1 .
$$

Taking, the inverse $z$-transform of $Y(z)$, we can determine the response $y(n)$.

## D. Digital filters

The general expression for transfer function of the digital filter is

$$
H(z)=\frac{Y(z)}{X(z)}
$$

where $X(z)$ and $Y(z)$ are $z$-transforms of the input $x(n)$ and output $y(n)$ sequences.
The relation between input and output is described by the difference equation of order $N$,

$$
y(n)+a_{1} y(n-1)+a_{2} y(n-2)+\ldots+a_{N} y(n-N)=x(n)+b_{1} x(n-1)+\ldots+b_{N} x(n-N)
$$

which has the following form in terms of $z$-transform,

$$
Y(z)\left[1+a_{1} z^{-1}+a_{2} z^{-2}+\ldots+a_{N} z^{-N}\right]=X(z)\left[1+b_{1} z^{-1}+\ldots+b_{N} z^{-N}\right]
$$

Therefore, transfer function of the digital filter is

$$
H(z)=\frac{Y(z)}{X(z)}=\frac{1+b_{1} z^{-1}+b_{2} z^{-2}+\ldots+b_{N} z^{-N}}{1+a_{1} z^{-1}+a_{2} z^{-2}+\ldots+a_{N} z^{-N}}=\frac{\sum_{k=0}^{N} b_{k} z^{-k}}{1+\sum_{m=1}^{N} a_{m} z^{-m}} \quad\left(a_{0}=1\right)
$$

This formula can be written as

$$
\begin{gathered}
H(z)=\frac{Y(z)}{X(z)}=H_{1}(z) H_{2}(z)=\frac{W(z)}{X(z)} \frac{Y(z)}{W(z)} \\
H_{1}(z)=\frac{1}{1+\sum_{m=1}^{N} a_{m} z^{-m}}, \quad H_{2}(z)=\sum_{k=0}^{N} b_{k} z^{-k} \quad\left(b_{0}=1\right) \\
H_{1}(z)=\frac{W(z)}{X(z)} \Rightarrow w(n)+a_{1} w(n-1)+a_{2} w(n-2)+\ldots+a_{N} w(n-N)=x(n) \\
H_{2}(z)=\frac{Y(z)}{W(z)} \Rightarrow y(n)=w(n)+b_{1} w(n-1)+b_{2} w(n-2)+\ldots+b_{N} w(n-N)
\end{gathered}
$$

Example 11: We now derive the difference equation of a LTI system that has the following transfer function

$$
H(z)=\frac{Y(z)}{X(z)}=\frac{z-\frac{1}{2}}{z^{2}-z+\frac{2}{9}}=z^{-1} \frac{1-\frac{1}{2} z^{-1}}{1-z^{-1}+\frac{2}{9} z^{-2}} .
$$

For the causal system, the transfer function can be written as

$$
H(z)=z^{-1}\left[\frac{A}{1-\frac{1}{3} z^{-1}}+\frac{B}{1-\frac{2}{3} z^{-1}}\right]
$$

where $A=1 / 2$ and $B=1 / 2$. The impulse response for the sum in the square brackets is defined as

$$
\left[A\left(\frac{1}{3}\right)^{n}+B\left(\frac{2}{3}\right)^{n}\right] u(n) \rightarrow h(n)=\left[A\left(\frac{1}{3}\right)^{n-1}+B\left(\frac{2}{3}\right)^{n-1}\right] u(n-1) .
$$

We also have

$$
Y(z)\left[1-z^{-1}+\frac{2}{9} z^{-2}\right]=X(z)\left[z^{-1}-\frac{1}{2} z^{-2}\right]
$$

and in the time domain

$$
\begin{equation*}
y(n)-y(n-1)+\frac{2}{9} y(n-2)=x(n-1)-\frac{1}{2} x(n-2) . \tag{112}
\end{equation*}
$$

The block diagram of realization for this system is given in Fig. 23.
State variable model The LTI discrete system can be described by the following model with two state-variables:

$$
\begin{aligned}
x_{1}(n+1) & =x_{2}(n) \\
x_{2}(n+1) & =-\frac{2}{9} x_{1}(n)+x_{2}(n)+x(n)
\end{aligned}
$$

or in the matrix form as

$$
\left[\begin{array}{l}
x_{1}(n+1) \\
x_{2}(n+1)
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-\frac{2}{9} & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(n) \\
x_{2}(n)
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] x(n) .
$$

The relation of the state-variables with the output is described by

$$
y(n)=-\frac{1}{2} x_{1}(n)+x_{2}(n)=\left[\begin{array}{ll}
-\frac{1}{2} & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(n) \\
x_{2}(n)
\end{array}\right]+\left[\begin{array}{l}
0 \\
0
\end{array}\right] x(n) .
$$



Fig. 23. Diagram of realization of the system (form II).

## E. The Laplace, Fourier and z-transforms

The $z$-transform is the discrete-time counter-part of the Laplace transform and a generalization of the Fourier transform of a sampled signal. Like Laplace transform, the $z$-transform allows for representing a linear and time invariant system in terms of the locations of the poles and zeros of the system transfer function in the complex $z$-plane. The roots of the transfer function, i.e. poles and zeros, describe the behavior of the system.

The common for the Laplace, Fourier, and $z$-transform is the use of the complex exponential functions as the basis functions of these transforms

$$
\begin{equation*}
z=e^{p}=e^{\sigma+j \omega}=\underline{e}^{\sigma} e^{j \omega}=\underline{r} e^{j \omega} \Rightarrow z^{n}=\underline{r}^{n}\left(e^{j \omega}\right)^{n} . \tag{113}
\end{equation*}
$$

For continuous-time right-sided signal $x(t)$, the Laplace transform is defined as

$$
\begin{equation*}
X(p)=\int_{0}^{\infty} x(t) e^{-p t} d t, \quad p=\sigma+j \omega . \tag{114}
\end{equation*}
$$

By sampling the continuous-time signal $x(t) \rightarrow x(n T)$ with sampling period assumed to be $T=1 s$, the Laplace transform becomes

$$
X(p)=\int_{0}^{\infty} x(t) e^{-p t} d t \rightarrow \sum_{n=0}^{\infty} x(n T) e^{-p n T} \Delta T=\sum_{n=0}^{\infty} x(n)\left(e^{p}\right)^{-n}=X\left(e^{p}\right)
$$

Substituting the variable $e^{p}$ with variable $z$, we obtain the single-sided $z$-transform

$$
X\left(e^{p}\right) \rightarrow X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}
$$

The $z$-transform of the sampled (discrete-time) signal can be written as (see (113))

$$
X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}=\sum_{n=0}^{\infty} x(n) \underline{r}^{-n}\left(e^{j \omega}\right)^{-n}=\sum_{n=0}^{\infty} x(n)\left(e^{j \omega}\right)^{-n}=X\left(e^{j \omega}\right)
$$

when $r=1$. Thus, the $z$-transform becomes the discrete Fourier transform when $z$ is considered to be on the unit circle, $|z|=1$.

The Fourier transform of a time-continuous right-sided signal $x(t)$ is a linear combination of complex exponentials $e^{j \omega t}$, where $\omega$ is a frequency (real variable)

$$
X(\omega)=\int_{0}^{\infty} x(t) e^{-j \omega t} d t
$$

and the Laplace transform (114) can be considered as a generalization of the Fourier transform when $p=j \omega$ and the vertical $j \omega$-axis of imaginary numbers belongs to the region of convergence of the Laplace transform.

We consider a few more examples for the bilateral $z$-transform

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad z \in R O C(X)
$$

1. 

$$
x(n)=u_{0}(n+7)+u_{0}(n-7) .
$$

Then

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=z^{7}+z^{-7} \rightarrow X\left(e^{j \omega}\right)=e^{j 7 \omega}+e^{-j 7 \omega}=2 \cos (7 \omega) .
$$

2. 

$$
x(n)=\left\{\begin{array}{ll}
\alpha^{n}, & n \geq 0 \\
\beta^{n}, & n<0
\end{array}=\alpha^{n} u(n)+\beta^{n} u(-n-1), \quad(0<\alpha<\beta) .\right.
$$

Then

$$
\begin{aligned}
X(z) & =\sum_{n=-\infty}^{\infty} x(n) z^{-n}=Z\left[\alpha^{n} u(n)\right]+Z\left[\beta^{n} u(-n-1)\right] \\
& =\left[\frac{1}{1-\alpha z^{-1}}\right]-\left[\frac{\beta^{-1} z}{1-\beta^{-1} z}\right]
\end{aligned}
$$

for $z$ such that $|z|>\alpha$ and $|z|<\beta$. Thus $\operatorname{ROC}(X)=\{z ; \alpha<|z|<\beta\}$.
3.

We now consider the $z$-transform of the discrete-time signal $x(n)$ multiplied by $n$, i.e. $y(n)=$ $n x(n)$.

$$
\begin{aligned}
Y(z) & =\sum_{n=-\infty}^{\infty} y(n) z^{-n}=\sum_{n=-\infty}^{\infty} n x(n) z^{-n}=\sum_{n=-\infty}^{\infty} x(n)\left(z^{-n}\right)^{\prime}(-z) \\
& =-z\left(\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right)^{\prime}=-z X^{\prime}(z) .
\end{aligned}
$$

For instance, if $x(n)=3^{n} u(n)$, then

$$
y(n)=n x(n)=n 3^{n} u(n) \xrightarrow{z}-\left.z\left(\frac{1}{1-3 z^{-1}}\right)^{\prime}\right|_{|z|>3}=\left.\frac{3 z^{-1}}{\left(1-3 z^{-1}\right)^{2}}\right|_{|z|>3} .
$$

E. 1 Initial and final value theorems

Consider the right-sided discrete-time signal $x(n)$ and its single-sided $z$-transform.

1. When $z$ approaches to infinity, we obtain

$$
\lim _{z \rightarrow \infty} X(z)=\lim _{z \rightarrow \infty}\left[\sum_{n=0}^{\infty} x(n) z^{-n}=x(0)+z^{-1} \cdot \sum_{n=1}^{\infty} x(n) z^{-n}\right]=x(0) .
$$

2. When $z$ approaches to 1 , we can do the following calculations

$$
\begin{aligned}
Z[x(n+1)]-Z[x(n)] & =\lim _{N \rightarrow \infty} \sum_{n=0}^{N}[x(n+1)-x(n)] z^{-n} \\
z[X(z)-x(0)]-X(z) & =\lim _{N \rightarrow \infty} \sum_{n=0}^{N}[x(n+1)-x(n)] z^{-n}
\end{aligned}
$$

and when $z \rightarrow 1$, we obtain

$$
\lim _{z \rightarrow 1}(z-1) X(z)-x(0)=\lim _{N \rightarrow \infty} \sum_{n=0}^{N-1}[x(n+1)-x(n)]=\lim _{N \rightarrow \infty}[x(N)-x(0)]=\lim _{N \rightarrow \infty} x(N)-x(0)
$$

Thus

$$
\lim _{z \rightarrow 1}(z-1) X(z)=\lim _{n \rightarrow \infty} x(n), \quad\left(\text { if } \lim _{n \rightarrow \infty} x(n) \exists\right)
$$

Example 12: As we know $\lim _{n \rightarrow \infty} \sin \left(n \omega_{0}\right)$ does not exist, but for the $z$-transform of this signal

$$
x(n)=\sin \left(n \omega_{0}\right) \xrightarrow{z} X(z)=\frac{z^{-1} \sin \left(\omega_{0}\right)}{1-2 z^{-1} \cos \left(\omega_{0}\right)+z^{-2}}
$$

we obtain

$$
\lim _{z \rightarrow 1}(z-1) X(z)=\lim _{z \rightarrow 1}(z-1) \frac{z \sin \left(\omega_{0}\right)}{1-2 z \cos \left(\omega_{0}\right)+z^{2}}=0
$$

## E. 2 Inverse systems

For a linear time invariant system with impulse response $h(n)$, the inverse system is defined by the impulse response $h_{1}(n)$ such that

$$
[x(n) * h(n)] * h_{1}(n)=x(n) \quad \Rightarrow \quad h(n) * h_{1}(n)=u_{0}(n)
$$

This relation is expressed in terms of the $z$-transforms as

$$
H(z) H_{1}(z)=1, \quad z \in R O C(H) \cap R O C\left(H_{1}\right)
$$

Thus the inverse system is defined by the transfer function

$$
\begin{equation*}
H_{1}(z)=\frac{1}{H(z)}, \quad z \in R O C(H) \cap\{z ; H(z) \neq 0\} \tag{115}
\end{equation*}
$$

If the system is causal, the impulse response is right-sided and the transfer function $H(z)$ is defined by the single-sided $z$-transform. If the poles of the system are inside the unit circle, then the system is stable. Consider as an example the transfer function

$$
H(z)=\frac{\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}
$$

According to (115), the poles $1 / 4$ and $1 / 3$ of $H(z)$ becomes zeros of $H_{1}(z)$, and zero $1 / 2$ of $H(z)$ becomes pole of $H_{1}(z)$. Indeed, we have

$$
\begin{aligned}
H_{1}(z) & =\frac{1}{H(z)}=\frac{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{2}\right)}=\left(z-\frac{1}{3}\right)+\frac{1}{4} \frac{\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{2}\right)} \\
& =\left(z-\frac{1}{3}\right)+\frac{1}{4}+\frac{1}{24} \frac{1}{z-\frac{1}{2}}=z-\frac{1}{12}+\frac{1}{12} \frac{\frac{1}{2} z^{-1}}{1-\frac{1}{2} z^{-1}}
\end{aligned}
$$

If the system is stable, is the inverse system stable, too? If yes, poles of $H_{1}(z)$ being zeros of $H(z)$ should be inside the unit circle. Thus, the poles and zeros of a stable and causal system are inside the unit circle.

Example 13: Consider a causal system described by the difference equation

$$
y(n)-\frac{1}{2} y(n-1)=x(n)
$$

Then

$$
H(z)=\left.\frac{1}{1-\frac{1}{2} z^{-1}}\right|_{|z|>1 / 2} \quad \rightarrow \quad h(n)=\left(\frac{1}{2}\right)^{n} u(n)
$$

For the inverse system, we have the following

$$
H_{1}(z)=\frac{1}{H(z)}=1-\frac{1}{2} z^{-1} \quad \xrightarrow{z^{-1}} \quad h(n)=u_{0}(n)-\frac{1}{2} u_{0}(n-1)
$$

Example 14: Consider the following difference equation described a LTI causal system

$$
y(n)+\frac{1}{4} y(n-1)-\frac{1}{8} y(n-2)=-2 x(n)+\frac{5}{4} x(n-1)
$$

In terms of $z$-transform, this equation has the form

$$
Y(z)\left[1+\frac{1}{4} z^{-1}-\frac{1}{8} z^{-2}\right]=X(z)\left[-2+\frac{5}{4} z^{-1}\right]
$$

The transfer function (or filter) is

$$
\begin{aligned}
H(z) & =\frac{Y(z)}{X(z)}=-2 \frac{1-\frac{5}{8} z^{-1}}{\left(1+\frac{1}{2} z^{-1}\right)\left(1-\frac{1}{4} z^{-1}\right)} \\
& =\left.A \frac{1}{1+\frac{1}{2} z^{-1}}\right|_{|z|>1 / 2}+\left.B \frac{1}{1-\frac{1}{4} z^{-1}}\right|_{|z|>1 / 4}
\end{aligned}
$$

where $A$ and $B$ are constants. Therefore, the system impulse response equals

$$
h(n)=A\left(-\frac{1}{2}\right)^{n} u(n)+B\left(\frac{1}{4}\right)^{n} u(n)
$$

(the system is stable and causal).
We now consider the inverse system which has the following transfer function

$$
H_{1}(z)=\frac{1}{H(z)}=-\frac{1}{2} \frac{\left(1+\frac{1}{2} z^{-1}\right)\left(1-\frac{1}{4} z^{-1}\right)}{1-\frac{5}{8} z^{-1}}
$$

which can be written as

$$
H_{1}(z)=-\frac{1}{2}\left[-\frac{2}{5}+\frac{1}{5} z^{-1}+\frac{\frac{7}{5}}{1-\frac{5}{8} z^{-1}}\right]
$$

Therefore, the impulse response for the inverse system is

$$
h_{1}(n)=\frac{1}{5}\left[u_{0}(n)-\frac{1}{10} u_{0}(n-1)-\frac{7}{10}\left(\frac{5}{8}\right)^{n} u(n)\right]
$$

(the inverse system is stable and causal).

## $F$. The $z$-transform in examples

Example 1:

$$
\begin{gathered}
x_{n}=x(n)=\left\{\begin{array}{rc}
1 & n=-1 \\
2 & n=0 \\
-1 & n=1 \\
1 & n=2 \\
0 & \text { otherwise }
\end{array}\right. \\
X(z)=z+2-z^{-1}+z^{-2} \\
X\left(e^{j \omega}\right)=e^{j \omega}+2-e^{-j \omega}+e^{-2 j \omega}
\end{gathered}
$$

$z$-transform exists for all $z \neq 0$.

Example 2: For a real number $a>0$, the following diagram holds

$$
\begin{array}{ccc}
x(n)=a^{n} u(n) & \rightarrow & y(n)=-a^{n} u(-n-1) \\
\downarrow z & \downarrow z \\
X(z)=\frac{1}{1-a z^{-1}} & \rightarrow & Y(z)=\frac{1}{1-a z^{-1}} \\
R O C=\{z ;|z|>a\} & & R O C=\{z ;|z|<a\}
\end{array}
$$

The sequences have the formula for $z$-transforms but different regions of convergence.

Example 3:

$$
\begin{array}{ccccc}
y(n) & = & -u(-n-1) & + & \left(\frac{1}{2}\right)^{n} u(n) \\
\downarrow z & \downarrow z & \downarrow z & a^{n} x(n) \quad(a \neq 0) \\
Y(z) & \frac{1}{1-z^{-1}} & & \downarrow & \\
R O C=\left\{z ; \frac{1}{2}<|z|<1\right\} & & R O C=\{z ;|z|<1\} & \cap & R O C=\left\{z ;|z|>\frac{1}{2}\right\}
\end{array}
$$

Example 4: Find the ROC associated with the $z$-transform of each of the following signal:

$$
\begin{aligned}
x(n) & =\left(\frac{1}{2}\right)^{n} u(n)+\left(\frac{1}{4}\right)^{n} u(n) \\
R O C & =\left\{z ;|z|>\frac{1}{2}\right\} \cap\left\{z ;|z|>\frac{1}{4}\right\}=\left\{z ;|z|>\frac{1}{2}\right\} \\
x(n) & =\left(\frac{1}{2}\right)^{n} u(n)+\left(\frac{1}{4}\right)^{n} u(-n) \\
R O C & =\left\{z ;|z|>\frac{1}{2}\right\} \cap\left\{z ;|z|<\frac{1}{4}\right\}=\emptyset
\end{aligned}
$$

$$
\begin{aligned}
x(n) & =\left(\frac{1}{4}\right)^{n} u(n)+\left(\frac{1}{2}\right)^{n} u(-n) \\
R O C & =\left\{z ;|z|>\frac{1}{4}\right\} \cap\left\{z ;|z|<\frac{1}{2}\right\}=\left\{z ; \frac{1}{4}<|z|<\frac{1}{2}\right\} \\
x(n) & =\left(\frac{1}{2}\right)^{n} u(-n)+\left(\frac{1}{4}\right)^{n} u(-n) \\
R O C & =\left\{z ;|z|<\frac{1}{2}\right\} \cap\left\{z ;|z|<\frac{1}{4}\right\}=\left\{z ;|z|<\frac{1}{4}\right\}
\end{aligned}
$$

Example 5: In the ring $1<|z|<2$, consider the function

$$
X(z)=\frac{1-z^{-1}+z^{-2}}{\left(1-\frac{1}{2} z^{-1}\right)\left(1-2 z^{-1}\right)\left(1-z^{-1}\right)} .
$$

The following representation holds

$$
X(z)=\frac{A}{1-\frac{1}{2} z^{-1}}+\frac{B}{1-2 z^{-1}}+\frac{C}{1-z^{-1}}
$$

where

$$
\begin{aligned}
A & =X(z)\left(1-\frac{1}{2} z^{-1}\right)_{\mid z=1 / 2} \\
B & =X(z)\left(1-2 z^{-1}\right)_{\mid z=2} \\
C & =X(z)\left(1-z^{-1}\right)_{\mid z=1}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& X(z) \quad=\frac{A}{1-\frac{1}{2} z^{-1}}+\frac{B}{1-2 z^{-1}}+\frac{C}{1-z^{-1}} \\
& \uparrow \begin{array}{cc}
\uparrow & \uparrow z \quad \uparrow z \quad \uparrow z
\end{array} \\
& x(n) \quad=A\left(\frac{1}{2}\right)^{n} u(n)+-B 2^{n} u(-n-1)+-C u(n) \\
& \{z ; 1<|z|<2\}=\{z ;|z|>1 / 2\} \cap \quad\{z ;|z|<2\} \quad \cap \quad\{z ;|z|>1\}
\end{aligned}
$$

Example 6:

$$
X(z)=\frac{z^{3}-9 z^{2}-6 z+4}{2 z^{2}-2 z-4}, \quad|z|<2 .
$$

The following representation holds for this function

$$
\begin{aligned}
X(z) & =\frac{1}{2} \frac{\left[z^{3}-z^{2}-2 z\right]-8 z^{2}-2 z+4}{z^{2}-z-2} \\
& =\frac{1}{2}\left[z-4 \frac{2 z^{2}+z-1}{z^{2}-z-2}\right]=\frac{1}{2}\left[z-4 \frac{\left[2 z^{2}-2 z-4\right]+3 z+3}{z^{2}-z-2}\right] \\
& =\frac{1}{2}\left[z-8-12 \frac{z+1}{z^{2}-z-2}\right]=\frac{1}{2}\left[z-8-12 \frac{z+1}{(z+1)(z-2)}\right] \\
& =\frac{1}{2}\left[z-8-\frac{12}{z-2}\right]=\frac{1}{2}\left[z-8+\frac{6}{1-2^{-1} z}\right]
\end{aligned}
$$

Therefore

$$
\begin{array}{ccccccc}
X(z) & = & \frac{1}{2} z & - & 4 & + & \frac{3}{1-2^{-1} z} \\
\uparrow z & & \uparrow z & & \uparrow z & & \uparrow z
\end{array}
$$

Example 7:

$$
X(z)=\frac{2+z^{-1}}{1-\frac{1}{2} z^{-1}}, \quad|z|>\frac{1}{2} .
$$

We have the following

$$
\begin{aligned}
X(z) & =2 \frac{1+\frac{1}{2} z^{-1}}{1-\frac{1}{2} z^{-1}}=2 \frac{-1+\frac{1}{2} z^{-1}+2}{1-\frac{1}{2} z^{-1}} \\
& =2\left[-1+\frac{2}{1-\frac{1}{2} z^{-1}}\right]=-2+\frac{4}{1-\frac{1}{2} z^{-1}}
\end{aligned}
$$

and

Example 8:

$$
\begin{array}{ccccc}
X(z) & = & -2 & - & \frac{4}{1-\frac{1}{2} z^{-1}} \\
\uparrow z & & \uparrow z & & \uparrow z \\
x(n) & = & -2 u_{0}(n) & +4 \cdot\left(\frac{1}{2}\right)^{n} u(n) \\
\left\{z ;|z|>\frac{1}{2}\right\} & = & \{\forall z\} & \cap\left\{z ;|z|>\frac{1}{2}\right\}
\end{array}
$$

$$
\begin{array}{ccc}
y(n)=\cos \left(\omega_{0} n\right) x(n), \quad a>0 . \\
2 y(n) & = & {\left[e^{j \omega_{0} n}=a^{n}\right] x(n)} \\
\downarrow z & & +a^{n} e^{-j \omega_{0} n} u(n) \\
\downarrow Y(z) & \downarrow & X\left(\frac{z}{e^{j \omega_{0}}}\right) \\
2 Y & + & X\left(\frac{z}{e^{-j \omega_{0}}}\right)
\end{array}
$$

Therefore

$$
Y(z)=\frac{1}{2}\left[X\left(e^{-j \omega_{0}} z\right)+X\left(e^{j \omega_{0}} z\right)\right] .
$$

Consider for example $a>0$ and the sequence

$$
\begin{array}{ccc}
x(n)= & a^{n} \cos \left(\omega_{0} n\right) u(n), \quad a>0 . \\
y(n) & = & \cos \left(\omega_{0} n\right)\left[a^{n} u(n)\right] \\
\downarrow z & \downarrow z \\
Y(z) & = & \frac{1}{2}\left[\frac{1}{\left.1-a e^{-j \omega_{0} z^{-1}}+\frac{1}{1-a e^{j \omega_{0} z^{-1}}}\right]}\right. \\
\{z ;|z|>a\} & \{z ;|z|>a\} \\
Y(z)= & \frac{1-z^{-1} a \cos \left(\omega_{0}\right)}{1-2 a \cos \left(\omega_{0}\right) z^{-1}+a^{2} z^{-2}}
\end{array}
$$

XI. The z-transform

The $z$-transform can be considered as a generation of the discrete Fourier transform (DFT). The $z$-transform is to the DFT what the Laplace transform is to the Fourier transform. The $z$-transform is used for analysis of discrete-time signals and systems. For many discrete signals $x(n)$ the DFT cannot be defined, but the $z$-transform with many similar properties takes place.

Let $x(n)$ be a sequence defined (may be) for all integers $n=0, \pm 1, \pm 2, \ldots$. The $z$-transform is defined by

$$
\begin{equation*}
x(n) \xrightarrow{z} X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad z \in C^{2}, \tag{109}
\end{equation*}
$$

where $z$ is a complex number for which $X(z)$ exists. In other words, the concept of the $z$-transform refers to as the formula (109) with region of convergence (ROC) of $X(z)$. We recall that the formula for DFT is defined by

$$
X\left(e^{j \omega}\right)=\sum_{n=-\infty}^{\infty} x(n) e^{-j \omega n}=\sum_{n=-\infty}^{\infty} x(n)\left(e^{j \omega}\right)^{-n}, \quad \omega \in(-\pi, \pi),
$$

and one can see a similarity when substituting $e^{j \omega}$ by $z$. The DFT is defined on points $z=e^{j \omega}$ lying on the unit circle. $X\left(e^{j \omega}\right)=X(z)_{\mid z=e^{j \omega}}$, i.e. the DFT is the $z$-transform considered on the unit circle $O_{1}$. In general, the region of convergence of the $z$-transform may not contain this circle.

The following properties can be derived directly from the definition of the transform.

1. If $z_{0} \in R O C(X)$, then $z \in R O C(X)$ for any point such that $|z|=\left|z_{0}\right|$. Indeed

$$
|X(z)| \leq \sum_{n=-\infty}^{\infty}\left|x(n) z^{-n}\right|=\sum_{n=-\infty}^{\infty}\left|x(n) z_{0}^{-n}\right|<\infty .
$$

2. If $z_{1} \neq z_{2} \in R O C(X)$, then $z \in R O C(X)$ for any point such that $\left|z_{1}\right| \leq|z| \leq\left|z_{2}\right|$ (we assume that $\left|z_{1}\right| \leq\left|z_{2}\right|$. In other words, ROC of the $z$-transform consists of circles of different radii.

The $z$-transform can be derived as the response $y(n)$ of the LTI system with $h(n)$ impulse response to a complex exponential discrete-time signal $x(n)=A z^{n}$ :

$$
\begin{aligned}
x(n) & =A z^{n} \rightarrow y(n)=\sum_{k=-\infty}^{\infty} h(k) x(n-k)=\sum_{k=-\infty}^{\infty} A z^{n-k} h(k) \\
& =A z^{n} \sum_{k=-\infty}^{\infty} h(k) z^{-k}=x(n) \underline{H(z)} .
\end{aligned}
$$

Thus, the transfer function of the LTI system is the $z$-transform of the impulse response. This relation yields also the property of convolution

$$
\begin{equation*}
x(n) * h(n) \rightarrow X(z) H(z), \quad z \in R O C(X) \cap R O C(H) . \tag{110}
\end{equation*}
$$

Example 1: Let sequence

$$
x(n) \neq 0, \quad n \in\left[N_{1}, N_{2}\right],
$$

where $N_{1}<N_{2}$ integer numbers. Then,

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=\sum_{n=N_{1}}^{N_{2}} x(n) z^{-n}=z^{-N_{1}} \sum_{n=0}^{N_{2}-N_{1}} x(n) z^{-n}
$$

which is defined for all $z$ except may be $z=0$.
Example 2: Let sequence $x(n)=u_{0}(n)$. Then, the $z$-transform

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=x(0) z^{0}=1, \quad \text { for all } z
$$

If sequence is defined as $x(n)=u_{0}\left(n-n_{0}\right)$, where $n_{0}$ is an integer, then the $z$-transform

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=x\left(n_{0}\right) z^{-n_{0}}=z^{-n_{0}}, \quad \text { for all } z \neq 0
$$

Example 3: Let sequence be

$$
x(n)=u(n)= \begin{cases}1, & n \geq 0 \\ 0, & n<0\end{cases}
$$

Then, the $z$-transform

$$
\begin{aligned}
X(z) & =\sum_{n-0}^{\infty} x(n) z^{-n} \\
& =1+z^{-1}+z^{-2}+\ldots+z^{-n}+\ldots=\frac{1}{1-z^{-1}}
\end{aligned}
$$

which converges at all points $z$ such that $|z|>1$. The point $z=1$ is a special point of $X(z)$.
Example 4: Given a real $\omega$, consider the complex exponential sequence

$$
x(n)=e^{j \omega n} u(n)=\left\{\begin{array}{cc}
e^{j \omega n}, & n=0,1,2, \ldots \\
0, & n<0
\end{array}\right.
$$

The $z$-transform of this sequence is

$$
X(z)=\sum_{n=0}^{\infty} e^{j \omega n} z^{-n}=\sum_{n=0}^{\infty}\left(e^{j \omega} z^{-1}\right)^{n}=\frac{1}{1-e^{j \omega} z^{-1}}
$$

and converges at points $z$ such that

$$
\left|e^{j \omega} z^{-1}\right|=\left|z^{-1}\right|<1
$$

Example 5: Consider the sequence

$$
x(n)=a^{n} u(n)=\left\{\begin{array}{cc}
a^{n}, & n=0,1,2, \ldots \\
0, & n=-1,-2, \ldots
\end{array}\right.
$$

for a given positive number $a$.
The $z$-transform of $x(n)$ is calculated as

$$
X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}=\sum_{n=0}^{\infty} a^{n} z^{-n}=\sum_{n=0}^{\infty}\left(a z^{-1}\right)^{n}=\frac{1}{1-a z^{-1}}
$$

and the series converges at points $z$ such that

$$
\left|a z^{-1}\right|<1 \quad \Rightarrow \quad|z|>a
$$

Example 6: Consider the discrete-time signal

$$
x(n)=\left[2\left(\frac{2}{3}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right] u(n) .
$$

The $z$-transform is calculated as follows

$$
\begin{aligned}
& X(z)=2 \sum_{n=0}^{\infty}\left(\frac{2}{3} z^{-1}\right)^{n}-\sum_{n=0}^{\infty}\left(\frac{1}{4} z^{-1}\right)^{n} \\
& X(z)=\left.2 \frac{1}{1-\frac{2}{3} z^{-1}}\right|_{|z|>\frac{2}{3}}-\left.\frac{1}{1-\frac{1}{4} z^{-1}}\right|_{|z|>\frac{1}{4}}
\end{aligned}
$$

for all $z$ such that $|z|>2 / 3$.

## A. Inverse formula

In the case of the discrete Fourier transform (when $z=e^{j \omega}$ ), the inverse formula

$$
\begin{aligned}
x(n) & =\frac{1}{2 \pi} \int_{-\pi}^{\pi} X\left(e^{j \omega}\right) e^{j \omega n} d \omega=\frac{1}{2 \pi} \int_{-\pi}^{\pi} X\left(e^{j \omega}\right) e^{j \omega n} e^{-j \omega} \frac{1}{j} d e^{j \omega} \\
& =\frac{1}{2 \pi j} \int_{-\pi}^{\pi} X\left(e^{j \omega}\right) e^{j \omega(n-1)} d e^{j \omega}=\frac{1}{2 \pi j} \int_{e^{-j \pi}}^{e^{j \pi}} X(z) z^{(n-1)} d z=\frac{1}{2 \pi j} \oint_{O_{1}} X(z) z^{n-1} d z
\end{aligned}
$$

where the last integral is taken over the unit circle $O_{1}$.
For $z$-transform, the inverse formula is similar,

$$
x(n)=\frac{1}{2 \pi j} \oint_{C} X(z) z^{n-1} d z
$$

where $C$ is any contour that embraces the original point $(0,0)$, for example the circle $O_{r}$ of radius $r>0$.

## B. Properties of $z$-transform

1. (Linearity)

If

$$
x_{1}(n) \rightarrow X_{1}(z), \quad x_{2}(n) \rightarrow X_{2}(z)
$$

then

$$
x_{1}(n)+k x_{2}(n) \rightarrow X_{1}(z)+k X_{2}(z)
$$

for any constant $k$.
2. (Delay)

$$
\begin{array}{ccc}
x(n) & \rightarrow & x\left(n-n_{0}\right) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X(z) z^{-n_{0}}
\end{array}
$$

for any integer $n_{0}$.
3. (Time-reversal)

$$
\begin{array}{rlc}
x(n) & \rightarrow & x(-n) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X\left(z^{-1}\right), \quad z^{-1} \in R O C(X)
\end{array}
$$

for instance if $\operatorname{ROC}(X(z))=\{z ;|z|>2\}$ then $R O C\left(X\left(z^{-1}\right)\right)=\{z ;|z|<1 / 2\}$.
Example 7: The following diagram holds for a given real number $a>0$ :

$$
\begin{array}{cccc}
\frac{1}{1-\underbrace{a z^{-1}}_{q}} & = & \frac{z}{\frac{z-a}{z-a}} & = \\
\downarrow & -\frac{a^{-1} z}{1-\underbrace{a^{-1} z}_{q}} \\
\downarrow z^{-1} & & \downarrow & \downarrow z^{-1} \\
a^{n} u(n) & \leftarrow & x(n) & \rightarrow \\
\Downarrow & -a^{n} u(-n-1) \\
\Downarrow & & & \Downarrow \\
\{z ;|z|>a\} & \leftarrow & R O C & \rightarrow
\end{array}\{z ;|z|<a\}
$$

Example 8: Consider the signal

$$
x(n)=2^{n} u(n)+5^{n} u(-n)
$$

The $z$-transform for this signal is defined as follows

$$
X(z)=\left.\frac{1}{1-2 z^{-1}}\right|_{|z|>2}-\left.\frac{1}{1-\frac{1}{5} z}\right|_{|z|<5}
$$

for all $z$ such that $2<|z|<5$.
Application (Difference equation)

$$
\begin{array}{cccccc}
y(n)+a_{1} y(n-1) & +a_{2} y(n-2) & =x(n) & + & b_{1} x(n-1) \\
\downarrow z & \downarrow z & & \downarrow & \downarrow & \\
Y(z)+a_{1} Y(z) z^{-1} & +a_{2} Y(z) z^{-2} & =X(z) & +b_{1} X(z) z^{-1}
\end{array}
$$

Therefore

$$
Y(z)\left[1+a_{1} z^{-1}+a_{2} z^{-2}\right]=X(z)\left[1+b_{1} z^{-1}\right]
$$

and we obtain

$$
Y(z)=\frac{1+b_{1} z^{-1}}{1+a_{1} z^{-1}+a_{2} z^{-2}} X(z) . \quad(Y(z)=H(z) X(z) ?)
$$

The equation holds only for $z$ lying in the intersection of ROCs of two $z$-transforms.
4. (Convolution)

$$
\begin{array}{cccc}
y(n) & =x(n) & * & h(n) \\
\downarrow z & \downarrow z & \downarrow & \downarrow z \\
Y(z) & =X(z) & \cdot & H(z)
\end{array}
$$

Example 9: For given two numbers $a \neq b$, we consider two sequences

$$
\begin{aligned}
& x(n)=a^{n} u(n)=\left\{\begin{array}{rc}
a^{n}, & n=0,1,2, \ldots \\
0, & n=-1,-2, \ldots
\end{array}\right. \\
& h(n)=b^{n} u(n)=\left\{\begin{array}{cc}
b^{n}, & n=0,1,2, \ldots \\
0, & n=-1,-2, \ldots
\end{array}\right.
\end{aligned}
$$

and the linear convolution

$$
y(n)=\sum_{m=-\infty}^{\infty} h(m) x(n-m)=\sum_{m=0}^{n} b^{m} a^{n-m}
$$

Using the results of Example 6, we obtain the following diagram for $z$-transform:

$$
\begin{array}{ccccc}
y(n) & = & x(n) & * & h(n) \\
\downarrow z & \downarrow z & \downarrow & \downarrow z \\
Y(z) & = & \frac{1}{1-a z^{-1}} & \cdot & \frac{1}{1-b z^{-1}}
\end{array}
$$

Therefore

$$
Y(z)=A \frac{1}{1-a z^{-1}}+B \frac{1}{1-b z^{-1}}, \quad|z|>\max (|a|,|b|)
$$

where $A=-a /(b-a)$ and $B=b /(b-a)$.
Example 10: Using results of Example 5, we obtain

$$
\begin{array}{cccc}
Y(z) & =A \frac{1}{1-a z^{-1}} & + & B \frac{1}{1-b z^{-1}} \\
\uparrow z & \uparrow z & \uparrow & \uparrow z \\
y(n) & = & A x(n) & +
\end{array}
$$

and the direct formula for the linear convolution can be derived as following

$$
\begin{aligned}
y(n) & =x(n) * h(n)=A x(n)+B h(n) \\
& =A a^{n} u_{1}(n)+B b^{n} u_{1}(n)=\left[A a^{n}+B b^{n}\right] u_{1}(n)=\frac{b^{n+1}-a^{n+1}}{b-a} u_{1}(n)
\end{aligned}
$$

5. (Multiplication)

$$
\begin{array}{ccc}
x_{1}(n) & x_{2}(n) & y(n)= \\
\downarrow z & \downarrow z & x_{1}(n) \cdot x_{2}(n) \\
X_{1}(z) & X_{2}(z) & Y(z)=\frac{1}{2 \pi j} \oint_{C} X_{1}(v) X_{2}\left(\frac{z}{v}\right) v^{-1} d v
\end{array}
$$

which, in particular case for the discrete Fourier transform takes the form

$$
Y\left(e^{j \omega}\right)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} X_{1}\left(e^{j \theta}\right) X_{2}\left(e^{j(\omega-\theta)}\right) d \theta
$$

## I. Single-sided Z-transform

Let $x(n)$ be a sequence defined for all integers $n=0, \pm 1, \pm 2, \ldots$. The transform

$$
x(n) \xrightarrow{z} X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in R O C(X) \subseteq C^{2},
$$

is called a single-sided $z$-transform of $x(n)$.
This is the linear transform with the following property.
6. (Delay)

$$
\begin{array}{ccc}
x(n) & \rightarrow & x(n-1) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X(z) z^{-1}+x(-1)
\end{array}
$$

Indeed

$$
\begin{aligned}
x(n-1) & \rightarrow \sum_{n=0}^{\infty} x(n-1) z^{-n}=\sum_{n=-1}^{\infty} x(n) z^{-(n+1)} \\
& =x(-1)+\sum_{n=0}^{\infty} x(n) z^{-n-1}=x(-1)+z^{-1} \sum_{n=0}^{\infty} x(n) z^{-n}
\end{aligned}
$$

Similarly, in the case for the second delay, we obtain

$$
\begin{array}{ccc}
x(n) & \rightarrow & x(n-2) \\
\downarrow z & & \downarrow z \\
X(z) & \rightarrow & X(z) z^{-2}+x(-2)+x(-1) z^{-1}
\end{array}
$$

Application (Difference equation with initial condition)
Given a coefficient $a$, we consider the difference equation with condition $y(-1)=A$ :

$$
\begin{array}{ccccc}
y(n) & + & a y(n-1) & = & x(n) \\
\downarrow z & & \downarrow z & \downarrow z \\
Y(z) & +a\left[Y(z) z^{-1}+y(-1)\right] & = & X(z)
\end{array}
$$

Therefore,

$$
\begin{equation*}
Y(z)\left[1+a z^{-1}\right]+a A=X(z) \quad \Rightarrow \quad Y(z)=\frac{X(z)-a A}{1+a z^{-1}} . \tag{111}
\end{equation*}
$$

For example, if

$$
x(n)=e^{j \omega n} u_{1}(n),
$$

then we can substitute in (111)

$$
X(z)=\frac{1}{1-e^{j \omega} z^{-1}}, \quad|z|>1 .
$$

Taking, the inverse $z$-transform of $Y(z)$, we can determine the response $y(n)$.

## A. Digital filters

The general expression for transfer function of the digital filter is

$$
H(z)=\frac{Y(z)}{X(z)}
$$

where $X(z)$ and $Y(z)$ are $z$-transforms of the input $x(n)$ and output $y(n)$ sequences.
The relation between input and output is described by the difference equation of order $N$,

$$
y(n)+a_{1} y(n-1)+a_{2} y(n-2)+\ldots+a_{N} y(n-N)=x(n)+b_{1} x(n-1)+\ldots+b_{N} x(n-N)
$$

which has the following form in terms of $z$-transform,

$$
Y(z)\left[1+a_{1} z^{-1}+a_{2} z^{-2}+\ldots+a_{N} z^{-N}\right]=X(z)\left[1+b_{1} z^{-1}+\ldots+b_{N} z^{-N}\right]
$$

Therefore, transfer function of the digital filter is

$$
H(z)=\frac{Y(z)}{X(z)}=\frac{1+b_{1} z^{-1}+b_{2} z^{-2}+\ldots+b_{N} z^{-N}}{1+a_{1} z^{-1}+a_{2} z^{-2}+\ldots+a_{N} z^{-N}}=\frac{\sum_{k=0}^{N} b_{k} z^{-k}}{1+\sum_{m=1}^{N} a_{m} z^{-m}} \quad\left(a_{0}=1\right)
$$

This formula can be written as

$$
\begin{gathered}
H(z)=\frac{Y(z)}{X(z)}=H_{1}(z) H_{2}(z)=\frac{W(z)}{X(z)} \frac{Y(z)}{W(z)} \\
H_{1}(z)=\frac{1}{1+\sum_{m=1}^{N} a_{m} z^{-m}}, \quad H_{2}(z)=\sum_{k=0}^{N} b_{k} z^{-k} \quad\left(b_{0}=1\right) \\
H_{1}(z)=\frac{W(z)}{X(z)} \Rightarrow w(n)+a_{1} w(n-1)+a_{2} w(n-2)+\ldots+a_{N} w(n-N)=x(n) \\
H_{2}(z)=\frac{Y(z)}{W(z)} \Rightarrow y(n)=w(n)+b_{1} w(n-1)+b_{2} w(n-2)+\ldots+b_{N} w(n-N)
\end{gathered}
$$

Example 11: We now derive the difference equation of a LTI system that has the following transfer function

$$
H(z)=\frac{Y(z)}{X(z)}=\frac{z-\frac{1}{2}}{z^{2}-z+\frac{2}{9}}=z^{-1} \frac{1-\frac{1}{2} z^{-1}}{1-z^{-1}+\frac{2}{9} z^{-2}} .
$$

For the causal system, the transfer function can be written as

$$
H(z)=z^{-1}\left[\frac{A}{1-\frac{1}{3} z^{-1}}+\frac{B}{1-\frac{2}{3} z^{-1}}\right]
$$

where $A=1 / 2$ and $B=1 / 2$. The impulse response for the sum in the square brackets is defined as

$$
\left[A\left(\frac{1}{3}\right)^{n}+B\left(\frac{2}{3}\right)^{n}\right] u(n) \rightarrow h(n)=\left[A\left(\frac{1}{3}\right)^{n-1}+B\left(\frac{2}{3}\right)^{n-1}\right] u(n-1) .
$$

We also have

$$
Y(z)\left[1-z^{-1}+\frac{2}{9} z^{-2}\right]=X(z)\left[z^{-1}-\frac{1}{2} z^{-2}\right]
$$

and in the time domain

$$
\begin{equation*}
y(n)-y(n-1)+\frac{2}{9} y(n-2)=x(n-1)-\frac{1}{2} x(n-2) . \tag{112}
\end{equation*}
$$

The block diagram of realization for this system is given in Fig. 23.
State variable model The LTI discrete system can be described by the following model with two state-variables:

$$
\begin{aligned}
x_{1}(n+1) & =x_{2}(n) \\
x_{2}(n+1) & =-\frac{2}{9} x_{1}(n)+x_{2}(n)+x(n)
\end{aligned}
$$

or in the matrix form as

$$
\left[\begin{array}{l}
x_{1}(n+1) \\
x_{2}(n+1)
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-\frac{2}{9} & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(n) \\
x_{2}(n)
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right] x(n) .
$$

The relation of the state-variables with the output is described by

$$
y(n)=-\frac{1}{2} x_{1}(n)+x_{2}(n)=\left[\begin{array}{ll}
-\frac{1}{2} & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(n) \\
x_{2}(n)
\end{array}\right]+\left[\begin{array}{l}
0 \\
0
\end{array}\right] x(n) .
$$



Fig. 23. Diagram of realization of the system (form II).

## B. The Laplace, Fourier and $z$-transforms

The $z$-transform is the discrete-time counter-part of the Laplace transform and a generalization of the Fourier transform of a sampled signal. Like Laplace transform the $z$-transform allows to represent a system in terms of the locations of the poles and zeros of the system transfer function in the complex $z$-plane. The roots of the transfer function, i.e. poles and zeros, describe the behavior of the system.

The common for the Laplace, Fourier, and $z$-transform is the use of the complex exponential functions as the basis functions of these transforms

$$
\begin{equation*}
z=e^{p}=e^{\sigma+j \omega}=\underline{e}^{\sigma} e^{j \omega}=\underline{r} e^{j \omega} \Rightarrow z^{n}=\underline{r}^{n}\left(e^{j \omega}\right)^{n} . \tag{113}
\end{equation*}
$$

For continuous-time right-sided signal $x(t)$, the Laplace transform is defined as

$$
\begin{equation*}
X(p)=\int_{0}^{\infty} x(t) e^{-p t} d t, \quad p=\sigma+j \omega . \tag{114}
\end{equation*}
$$

By sampling the continuous-time signal $x(t) \rightarrow x(n T)$ with sampling period assumed to be $T=1 s$, the Laplace transform becomes

$$
X(p)=\int_{0}^{\infty} x(t) e^{-p t} d t \rightarrow \sum_{n=0}^{\infty} x(n T) e^{-p n T} \Delta T=\sum_{n=0}^{\infty} x(n)\left(e^{p}\right)^{-n}=X\left(e^{p}\right)
$$

Substituting the variable $e^{p}$ with variable $z$, we obtain the single-sided $z$-transform

$$
X\left(e^{p}\right) \rightarrow X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}
$$

The $z$-transform of the sampled (discrete-time) signal can be written as (see (113))

$$
X(z)=\sum_{n=0}^{\infty} x(n) z^{-n}=\sum_{n=0}^{\infty} x(n) \underline{r}^{-n}\left(e^{j \omega}\right)^{-n}=\sum_{n=0}^{\infty} x(n)\left(e^{j \omega}\right)^{-n}=X\left(e^{j \omega}\right)
$$

when $r=1$. Thus, the $z$-transform becomes the discrete Fourier transform when $z$ is considered to be on the unit circle, $|z|=1$.

The Fourier transform of a time-continuous right-sided signal $x(t)$ is a linear combination of complex exponentials $e^{j \omega t}$, where $\omega$ is a frequency (real variable)

$$
X(\omega)=\int_{0}^{\infty} x(t) e^{-j \omega t} d t
$$

and the Laplace transform (114) can be considered as a generalization of the Fourier transform when $p=j \omega$ and the vertical $j \omega$-axis of imaginary numbers belongs to the region of convergence of the Laplace transform.

We consider a few more examples for the bilateral $z$-transform

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}, \quad z \in R O C(X)
$$

1. 

$$
x(n)=u_{0}(n+7)+u_{0}(n-7) .
$$

Then

$$
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}=z^{7}+z^{-7} \rightarrow X\left(e^{j \omega}\right)=e^{j 7 \omega}+e^{-j 7 \omega}=2 \cos (7 \omega) .
$$

2. 

$$
x(n)=\left\{\begin{array}{ll}
\alpha^{n}, & n \geq 0 \\
\beta^{n}, & n<0
\end{array}=\alpha^{n} u(n)+\beta^{n} u(-n-1), \quad(0<\alpha<\beta) .\right.
$$

Then

$$
\begin{aligned}
X(z) & =\sum_{n=-\infty}^{\infty} x(n) z^{-n}=Z\left[\alpha^{n} u(n)\right]+Z\left[\beta^{n} u(-n-1)\right] \\
& =\left[\frac{1}{1-\alpha z^{-1}}\right]-\left[\frac{\beta^{-1} z}{1-\beta^{-1} z}\right]
\end{aligned}
$$

for $z$ such that $|z|>\alpha$ and $|z|<\beta$. Thus $\operatorname{ROC}(X)=\{z ; \alpha<|z|<\beta\}$.
3.

We now consider the $z$-transform of the discrete-time signal $x(n)$ multiplied by $n$, i.e. $y(n)=$ $n x(n)$.

$$
\begin{aligned}
Y(z) & =\sum_{n=-\infty}^{\infty} y(n) z^{-n}=\sum_{n=-\infty}^{\infty} n x(n) z^{-n}=\sum_{n=-\infty}^{\infty} x(n)\left(z^{-n}\right)^{\prime}(-z) \\
& =-z\left(\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right)^{\prime}=-z X^{\prime}(z) .
\end{aligned}
$$

For instance, if $x(n)=3^{n} u(n)$, then

$$
y(n)=n x(n)=n 3^{n} u(n) \xrightarrow{z}-\left.z\left(\frac{1}{1-3 z^{-1}}\right)^{\prime}\right|_{|z|>3}=\left.\frac{3 z^{-1}}{\left(1-3 z^{-1}\right)^{2}}\right|_{|z|>3} .
$$

B. 1 Initial and final value theorems

Consider the right-sided discrete-time signal $x(n)$ and its $z$-transform.

1. When $z$ approaches to infinity, we obtain

$$
\lim _{z \rightarrow \infty} X(z)=\lim _{z \rightarrow \infty}\left[\sum_{n=0}^{\infty} x(n) z^{-n}=x(0)+z^{-1} \cdot \sum_{n=1}^{\infty} x(n) z^{-n}\right]=x(0) .
$$

2. When $z$ approaches to 1 , we can do the following calculations

$$
\begin{aligned}
Z[x(n+1)]-Z[x(n)] & =\lim _{N \rightarrow \infty} \sum_{n=0}^{N}[x(n+1)-x(n)] z^{-n} \\
z[X(z)-x(0)]-X(z) & =\lim _{N \rightarrow \infty} \sum_{n=0}^{N}[x(n+1)-x(n)] z^{-n}
\end{aligned}
$$

and when $z \rightarrow 1$, we obtain

$$
\lim _{z \rightarrow 1}(z-1) X(z)-x(0)=\lim _{N \rightarrow \infty} \sum_{n=0}^{N-1}[x(n+1)-x(n)]=\lim _{N \rightarrow \infty}[x(N)-x(0)]=\lim _{N \rightarrow \infty} x(N)-x(0)
$$

Thus

$$
\lim _{z \rightarrow 1}(z-1) X(z)=\lim _{n \rightarrow \infty} x(n), \quad\left(\text { if } \lim _{n \rightarrow \infty} x(n) \exists\right)
$$

Example 12: As we know $\lim _{n \rightarrow \infty} \sin \left(n \omega_{0}\right)$ does not exist, but for the $z$-transform of this signal

$$
x(n)=\sin \left(n \omega_{0}\right) \xrightarrow{z} X(z)=\frac{z^{-1} \sin \left(\omega_{0}\right)}{1-2 z^{-1} \cos \left(\omega_{0}\right)+z^{-2}}
$$

we obtain

$$
\lim _{z \rightarrow 1}(z-1) X(z)=\lim _{z \rightarrow 1}(z-1) \frac{z \sin \left(\omega_{0}\right)}{1-2 z \cos \left(\omega_{0}\right)+z^{2}}=0
$$

## B. 2 Inverse systems

For a linear time invariant system with impulse response $h(n)$, the inverse system is defined by the impulse response $h_{1}(n)$ such that

$$
[x(n) * h(n)] * h_{1}(n)=x(n) \quad \Rightarrow \quad h(n) * h_{1}(n)=u_{0}(n)
$$

This relation is expressed in terms of the $z$-transforms as

$$
H(z) H_{1}(z)=1, \quad z \in R O C(H) \cap R O C\left(H_{1}\right)
$$

Thus the inverse system is defined by the transfer function

$$
\begin{equation*}
H_{1}(z)=\frac{1}{H(z)}, \quad z \in R O C(H) \cap\{z ; H(z) \neq 0\} \tag{115}
\end{equation*}
$$

If the system is causal, the impulse response is right-sided and the transfer function $H(z)$ is defined by the single-sided $z$-transform. If the poles of the system are inside the unit circle, then the system is stable. Consider as an example the transfer function

$$
H(z)=\frac{\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}
$$

According to (115), the poles $1 / 4$ and $1 / 3$ of $H(z)$ becomes zeros of $H_{1}(z)$, and zero $1 / 2$ of $H(z)$ becomes pole of $H_{1}(z)$. Indeed, we have

$$
\begin{aligned}
H_{1}(z) & =\frac{1}{H(z)}=\frac{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{2}\right)}=\left(z-\frac{1}{3}\right)+\frac{1}{4} \frac{\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{2}\right)} \\
& =\left(z-\frac{1}{3}\right)+\frac{1}{4}+\frac{1}{24} \frac{1}{z-\frac{1}{2}}=z-\frac{1}{12}+\frac{1}{12} \frac{\frac{1}{2} z^{-1}}{1-\frac{1}{2} z^{-1}}
\end{aligned}
$$

If the system is stable, is the inverse system stable, too? If yes, poles of $H_{1}(z)$ being zeros of $H(z)$ should be inside the unit circle. Thus, the poles and zeros of a stable and causal system are inside the unit circle.

Example 13: Consider a causal system described by the difference equation

$$
y(n)-\frac{1}{2} y(n-1)=x(n)
$$

Then

$$
H(z)=\left.\frac{1}{1-\frac{1}{2} z^{-1}}\right|_{|z|>1 / 2} \quad \rightarrow \quad h(n)=\left(\frac{1}{2}\right)^{n} u(n)
$$

For the inverse system, we have the following

$$
H_{1}(z)=\frac{1}{H(z)}=1-\frac{1}{2} z^{-1} \quad \xrightarrow{z^{-1}} \quad h(n)=u_{0}(n)-\frac{1}{2} u_{0}(n-1)
$$

Example 14: Consider the following difference equation described a LTI causal system

$$
y(n)+\frac{1}{4} y(n-1)-\frac{1}{8} y(n-2)=-2 x(n)+\frac{5}{4} x(n-1)
$$

In terms of $z$-transform, this equation has the form

$$
Y(z)\left[1+\frac{1}{4} z^{-1}-\frac{1}{8} z^{-2}\right]=X(z)\left[-2+\frac{5}{4} z^{-1}\right]
$$

The transfer function (or filter) is

$$
\begin{aligned}
H(z) & =\frac{Y(z)}{X(z)}=-2 \frac{1-\frac{5}{8} z^{-1}}{\left(1+\frac{1}{2} z^{-1}\right)\left(1-\frac{1}{4} z^{-1}\right)} \\
& =\left.A \frac{1}{1+\frac{1}{2} z^{-1}}\right|_{|z|>1 / 2}+\left.B \frac{1}{1-\frac{1}{4} z^{-1}}\right|_{|z|>1 / 4}
\end{aligned}
$$

where $A$ and $B$ are constants. Therefore, the system impulse response equals

$$
h(n)=A\left(-\frac{1}{2}\right)^{n} u(n)+B\left(\frac{1}{4}\right)^{n} u(n)
$$

(the system is stable and causal).
We now consider the inverse system which has the following transfer function

$$
H_{1}(z)=\frac{1}{H(z)}=-\frac{1}{2} \frac{\left(1+\frac{1}{2} z^{-1}\right)\left(1-\frac{1}{4} z^{-1}\right)}{1-\frac{5}{8} z^{-1}}
$$

which can be written as

$$
H_{1}(z)=-\frac{1}{2}\left[-\frac{2}{5}+\frac{1}{5} z^{-1}+\frac{\frac{7}{5}}{1-\frac{5}{8} z^{-1}}\right]
$$

Therefore, the impulse response for the inverse system is

$$
h_{1}(n)=\frac{1}{5}\left[u_{0}(n)-\frac{1}{10} u_{0}(n-1)-\frac{7}{10}\left(\frac{5}{8}\right)^{n} u(n)\right]
$$

(the inverse system is stable and causal).

## C. The $z$-transform in examples

Example 1:

$$
\begin{gathered}
x_{n}=x(n)=\left\{\begin{array}{rc}
1 & n=-1 \\
2 & n=0 \\
-1 & n=1 \\
1 & n=2 \\
0 & \text { otherwise }
\end{array}\right. \\
X(z)=z+2-z^{-1}+z^{-2} \\
X\left(e^{j \omega}\right)=e^{j \omega}+2-e^{-j \omega}+e^{-2 j \omega}
\end{gathered}
$$

$z$-transform exists for all $z \neq 0$.

Example 2: For a real number $a>0$, the following diagram holds

$$
\begin{array}{ccc}
x(n)=a^{n} u(n) & \rightarrow & y(n)=-a^{n} u(-n-1) \\
\downarrow z & \downarrow z \\
X(z)=\frac{1}{1-a z^{-1}} & \rightarrow & Y(z)=\frac{1}{1-a z^{-1}} \\
R O C=\{z ;|z|>a\} & & R O C=\{z ;|z|<a\}
\end{array}
$$

The sequences have the formula for $z$-transforms but different regions of convergence.

Example 3:

$$
\begin{array}{ccccc}
y(n) & = & -u(-n-1) & + & \left(\frac{1}{2}\right)^{n} u(n) \\
\downarrow z & \downarrow z & \downarrow z & a^{n} x(n) \quad(a \neq 0) \\
Y(z) & \frac{1}{1-z^{-1}} & & \downarrow & \\
R O C=\left\{z ; \frac{1}{2}<|z|<1\right\} & & R O C=\{z ;|z|<1\} & \cap & R O C=\left\{z ;|z|>\frac{1}{2}\right\}
\end{array}
$$

Example 4: Find the ROC associated with the $z$-transform of each of the following signal:

$$
\begin{aligned}
x(n) & =\left(\frac{1}{2}\right)^{n} u(n)+\left(\frac{1}{4}\right)^{n} u(n) \\
R O C & =\left\{z ;|z|>\frac{1}{2}\right\} \cap\left\{z ;|z|>\frac{1}{4}\right\}=\left\{z ;|z|>\frac{1}{2}\right\} \\
x(n) & =\left(\frac{1}{2}\right)^{n} u(n)+\left(\frac{1}{4}\right)^{n} u(-n) \\
R O C & =\left\{z ;|z|>\frac{1}{2}\right\} \cap\left\{z ;|z|<\frac{1}{4}\right\}=\emptyset
\end{aligned}
$$

$$
\begin{aligned}
x(n) & =\left(\frac{1}{4}\right)^{n} u(n)+\left(\frac{1}{2}\right)^{n} u(-n) \\
R O C & =\left\{z ;|z|>\frac{1}{4}\right\} \cap\left\{z ;|z|<\frac{1}{2}\right\}=\left\{z ; \frac{1}{4}<|z|<\frac{1}{2}\right\} \\
x(n) & =\left(\frac{1}{2}\right)^{n} u(-n)+\left(\frac{1}{4}\right)^{n} u(-n) \\
R O C & =\left\{z ;|z|<\frac{1}{2}\right\} \cap\left\{z ;|z|<\frac{1}{4}\right\}=\left\{z ;|z|<\frac{1}{4}\right\}
\end{aligned}
$$

Example 5: In the ring $1<|z|<2$, consider the function

$$
X(z)=\frac{1-z^{-1}+z^{-2}}{\left(1-\frac{1}{2} z^{-1}\right)\left(1-2 z^{-1}\right)\left(1-z^{-1}\right)}
$$

The following representation holds

$$
X(z)=\frac{A}{1-\frac{1}{2} z^{-1}}+\frac{B}{1-2 z^{-1}}+\frac{C}{1-z^{-1}}
$$

where

$$
\begin{aligned}
A & =X(z)\left(1-\frac{1}{2} z^{-1}\right)_{\mid z=1 / 2} \\
B & =X(z)\left(1-2 z^{-1}\right)_{\mid z=2} \\
C & =X(z)\left(1-z^{-1}\right)_{\mid z=1}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& X(z) \quad=\frac{A}{1-\frac{1}{2} z^{-1}}+\frac{B}{1-2 z^{-1}}+\frac{C}{1-z^{-1}} \\
& \uparrow z \uparrow z \quad \uparrow z \quad \uparrow z \\
& x(n)=A\left(\frac{1}{2}\right)^{n} u(n)+-B 2^{n} u(-n-1)+-C u(n) \\
& \{z ; 1<|z|<2\}=\{z ;|z|>1 / 2\} \cap \quad\{z ;|z|<2\} \cap\{z ;|z|>1\}
\end{aligned}
$$

Example 6:

$$
X(z)=\frac{z^{3}-9 z^{2}-6 z+4}{2 z^{2}-2 z-4}, \quad|z|<2 .
$$

The following representation holds for this function

$$
\begin{aligned}
X(z) & =\frac{1}{2} \frac{\left[z^{3}-z^{2}-2 z\right]-8 z^{2}-2 z+4}{z^{2}-z-2} \\
& =\frac{1}{2}\left[z-4 \frac{2 z^{2}+z-1}{z^{2}-z-2}\right]=\frac{1}{2}\left[z-4 \frac{\left[2 z^{2}-2 z-4\right]+3 z+3}{z^{2}-z-2}\right] \\
& =\frac{1}{2}\left[z-8-12 \frac{z+1}{z^{2}-z-2}\right]=\frac{1}{2}\left[z-8-12 \frac{z+1}{(z+1)(z-2)}\right] \\
& =\frac{1}{2}\left[z-8-\frac{12}{z-2}\right]=\frac{1}{2}\left[z-8+\frac{6}{1-2^{-1} z}\right]
\end{aligned}
$$

Therefore

$$
\begin{array}{ccccccc}
X(z) & = & \frac{1}{2} z & - & 4 & + & \frac{3}{1-2^{-1} z} \\
\uparrow z & & \uparrow z & & \uparrow z & & \uparrow z
\end{array}
$$

Example 7:

$$
X(z)=\frac{2+z^{-1}}{1-\frac{1}{2} z^{-1}}, \quad|z|>\frac{1}{2}
$$

We have the following

$$
\begin{aligned}
X(z) & =2 \frac{1+\frac{1}{2} z^{-1}}{1-\frac{1}{2} z^{-1}}=2 \frac{-1+\frac{1}{2} z^{-1}+2}{1-\frac{1}{2} z^{-1}} \\
& =2\left[-1+\frac{2}{1-\frac{1}{2} z^{-1}}\right]=-2+\frac{4}{1-\frac{1}{2} z^{-1}}
\end{aligned}
$$

and

Example 8:

$$
\begin{array}{ccccc}
X(z) & = & -2 & - & \frac{4}{1-\frac{1}{2} z^{-1}} \\
\uparrow z & & \uparrow z & & \uparrow z \\
x(n) & = & -2 u_{0}(n) & +4 \cdot\left(\frac{1}{2}\right)^{n} u(n) \\
\left\{z ;|z|>\frac{1}{2}\right\} & = & \{\forall z\} & \cap\left\{z ;|z|>\frac{1}{2}\right\}
\end{array}
$$

$$
\begin{array}{ccc}
y(n)=\cos \left(\omega_{0} n\right) x(n), & a>0 . \\
2 y(n) & = & {\left[e^{j \omega_{0} n}=a^{n}\right] x(n)} \\
\downarrow z & & a^{n} e^{-j \omega_{0} n} u(n) \\
\downarrow z & \downarrow z \\
2 Y(z) & = & X\left(\frac{z}{e^{j \omega_{0}}}\right)
\end{array}+\quad X\left(\frac{z}{e^{-j \omega_{0}}}\right)
$$

Therefore

$$
Y(z)=\frac{1}{2}\left[X\left(e^{-j \omega_{0}} z\right)+X\left(e^{j \omega_{0}} z\right)\right] .
$$

Consider for example $a>0$ and the sequence

$$
\begin{array}{ccc}
x(n)= & a^{n} \cos \left(\omega_{0} n\right) u(n), \quad a>0 . \\
y(n) & = & \cos \left(\omega_{0} n\right)\left[a^{n} u(n)\right] \\
\downarrow z & \downarrow z \\
Y(z) & = & \frac{1}{2}\left[\frac{1}{\left.1-a e^{-j \omega_{0} z^{-1}}+\frac{1}{1-a e^{j \omega_{0}} z^{-1}}\right]}\right. \\
\{z ;|z|>a\} & \{z ;|z|>a\} \\
Y(z)= & \frac{1-z^{-1} a \cos \left(\omega_{0}\right)}{1-2 a \cos \left(\omega_{0}\right) z^{-1}+a^{2} z^{-2}}
\end{array}
$$

