- **Exam duration**: 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct**.
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

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1. (45 total points) Answer the following unrelated miscellaneous questions.

(a) (10 points) Consider the following nonlinear system:
\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)x_2(t) - 2x_1(t) \\
\dot{x}_2(t) &= x_1(t) - x_2(t) - 1.
\end{align*}
\]

Find two equilibrium points of the nonlinear system. By two equilibrium points I mean:
\[
x^{(1)}_e = \begin{bmatrix} x^{(1)}_e \\ x^{(1)}_e \end{bmatrix}, \quad \text{and} \quad x^{(2)}_e = \begin{bmatrix} x^{(2)}_e \\ x^{(2)}_e \end{bmatrix}.
\]

The equilibrium points for this system are:
- \( x^{(1)}_e = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \),
- \( x^{(2)}_e = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \).

(b) (10 points) You are given a matrix \( A \) with the characteristic polynomial
\[
\pi_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3)^4 = 0.
\]

In other words, \( A \) has three distinct eigenvalues \( \lambda_{1,2,3} \) of different algebraic multiplicity. Given that
\[
\dim N(A - \lambda_2 I) = 2, \quad \dim N(A - \lambda_3 I) = 2,
\]

obtain all possible Jordan canonical forms for \( A \). You have to be clear and precise. Explain your answer.

The dimension of the nullspace for each eigenvector determines the number of Jordan blocks for eigenvalues \( \lambda_2 \) and \( \lambda_3 \):
- For eigenvalue \( \lambda_1 \), the only possible Jordan block is
  \[
  J_{\lambda_1} = [\lambda_1].
  \]
- For eigenvalue \( \lambda_2 \), the only possible Jordan block is
  \[
  J_{\lambda_2} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix}
  \]
  since the geometric multiplicity is equal to the algebraic one, then there will be two Jordan blocks for \( \lambda_2 \). Since the total size of these two Jordan blocks is equal to 2, then the only possible Jordan block form for \( \lambda_2 \) is
  \[
  J_{\lambda_2} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix}.
  \]
- For eigenvalue \( \lambda_3 \), the geometric multiplicity is equal to 2, hence there are two Jordan blocks with a total size of 4. The possible combinations are hence
  \[
  J_{\lambda_3}^{(1)} = \begin{bmatrix} \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}.
  \]
or

\[ J^{(2)}_{\lambda_2} = \begin{bmatrix} \lambda_3 & 1 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}. \]

Therefore, and given the problem description, there can only be two possible combinations of the Jordan form of \( A \), given as follows:

\[ J^{(1)} = \text{blkdiag}(J_{\lambda_1}, J_{\lambda_2}, J_{\lambda_3}^{(1)}) \]

or

\[ J^{(2)} = \text{blkdiag}(J_{\lambda_1}, J_{\lambda_2}, J_{\lambda_3}^{(2)}). \]

(c) (10 points) Consider that

\[ A = uv^\top = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}. \]

Note that \( A \) is a rank one matrix. Derive \( e^{At} \) for any \( u \) and \( v \) and then compute \( e^{At} \) for the matrix given above and for \( t = \frac{1}{v^\top u} = \frac{1}{32} \).

If \( A \) is a rank-1 matrix, we can write

\[ e^{At} = I + \frac{A}{v^\top u} \left[ e^{(v^\top u)t} - 1 \right]. \]

Notice that

\[ v^\top u = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32, \]

hence

\[ e^{At} = I_3 + \frac{A}{32} \left[ e^{(v^\top u)t} - 1 \right] = I + \frac{A}{32} \left[ e^1 - 1 \right] \approx I + 0.05A. \]

(d) (10 points) Is the following system defined by

\[ y(t) = (u(t))^{1.1} + u(t + 1) \]

causal or non-causal? Linear or nonlinear? Time-invariant or time-varying? You have to prove your answers. A one-word answer is not enough.

The system is nonlinear due to the \((u(t))^{1.1}\) (which is a nonlinear function in terms of the input), causal because the output depends on future inputs, and time-invariant. You have to prove these results, though. :)
(e) (5 points) The transfer function matrix of the state space system

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

can be written as

\[ H(s) = C(sI - A)^{-1}B + D \]

for any MIMO or SISO system. Find the transfer function \( H(s) \) when

\[
A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C = [1 \ 0], \quad D = [0 \ 0].
\]

Your \( H(s) \) should be \( \in \mathbb{R}^{1 \times 2} \)

\[
H(s) = C(sI - A)^{-1}B + D = [1 \ 0] \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \frac{1}{s - 2}. \]
2. (35 total points) The state-space representation of a dynamical system is given as follows:
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

with
\[
A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \end{bmatrix}, x_0 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, D = 0.
\]

(a) (5 points) By finding the eigenvalues, eigenvectors of the \( A \) matrix, compute \( e^{At} \) via the diagonal transformation. You have to clearly show your work.

\[
\begin{align*}
A &= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^{-1} \\
\Rightarrow e^{At} &= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0.5 - 0.5e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}.
\end{align*}
\]

(b) (5 points) Assume that the control input is \( u(t) = 0 \), compute \( x(t) \) and \( y(t) \).

\[
\begin{align*}
x(t) &= e^{At}x_0 = \begin{bmatrix} 1 & 0.5 - 0.5e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix}.
\end{align*}
\]

\[
y(t) = Cx(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} = -1.
\]

(c) (20 points) Assume that the input is \( u(t) = 1 + 2e^{-2t} \), compute \( x(t), y(t) \).

\[
\begin{align*}
x(t) &= e^{A(t-t_0)}x_{10} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) d\tau = \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) d\tau.
\end{align*}
\]

\[
\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) d\tau = \int_{t_0}^{t} \begin{bmatrix} 1 & 0.5 - 0.5e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1 + 2e^{-2\tau}) d\tau = \begin{bmatrix} 0.75 + 0.5t - 0.75e^{-2t} + te^{-2t} \\ -0.5 + 0.5e^{-2t} - 2te^{-2t} \end{bmatrix}.
\]

Hence,
\[
x(t) = \begin{bmatrix} -1.5e^{-2t} - 0.5 \\ 3e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.75 + 0.5t - 0.75e^{-2t} + te^{-2t} \\ -0.5 + 0.5e^{-2t} - 2te^{-2t} \end{bmatrix} = \begin{bmatrix} 0.25 + 0.5t - 2.25e^{-2t} + te^{-2t} \\ -0.5 + 3.5e^{-2t} - 2te^{-2t} \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.
\]
and

\[ y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} x(t) = t - e^{-2t}. \]

(d) (5 points) Given your answers to the previous question, compute \( x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \) as \( t \to \infty \). Which state blows up? Also, find \( y(\infty) \).

\[
\begin{align*}
x(\infty) &= \begin{bmatrix} \infty \\ -0.5 \end{bmatrix} = \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix}, & y(\infty) = \infty.
\end{align*}
\]

The first state blows up (this state corresponds to the unstable mode with eigenvalue \( \lambda_1 = 0 \)) and the second state converges to -0.5 (this state corresponds to the stable mode with eigenvalue \( \lambda_2 = -2 \)).
3. (20 total points) In this problem, we will study the equilibrium of Susceptible-Infectious-Susceptible (SIS) in epidemics—similar to what we discussed in class. The dynamics of a simplified SIS model can be written as

\[
\begin{align*}
\frac{dS}{dt} &= -\beta \frac{SI}{N(t)} + \gamma I \\
\frac{dI}{dt} &= \beta \frac{SI}{N(t)} - \gamma I
\end{align*}
\]

where \( S(t) \) is the number of people that are susceptible at time \( t \) and \( I(t) \) is the number of infected people at time \( t \), where \( N(t) \) is the total number of people which is a time-varying quantity.

Assume that the number of people is fixed, that is \( S(t) + I(t) = N(t) \) where \( N(t) \) is the time-varying population of the SIS dynamics.

(a) (10 points) Given the above assumption, reduce the above dynamical system from 2 states \((S(t), I(t))\) to a dynamic system with only one state \( I(t) \). You should obtain something like

\[
\dot{I}(t) = f(I(t), \beta, N(t), \gamma)
\]

where \( f(\cdot) \) is the function that you should determine.

First, we can substitute \( S(t) = N(t) - I(t) \) into the second differential equation, we obtain

\[
\frac{dI}{dt} = \beta \frac{(N(t) - I(t))I(t)}{N(t)} - \gamma I(t) = -\beta \frac{I^2(t)}{N(t)} + (\beta - \gamma)I(t) = f(I(t), \beta, N(t), \gamma)
\]

(b) (5 points) What is the non-trivial (different than zero) time-varying equilibrium of the system? In other words, what is \( I_{eq}(t) \)?

Setting

\[
f(I_{eq}(t), \beta, N(t), \gamma) = -\beta \frac{I^2_{eq}(t)}{N(t)} + (\beta - \gamma)I_{eq}(t) = 0
\]

we obtain

\[
I_{eq}(t) = \frac{\beta - \gamma}{\beta} N(t)
\]

as the non-trivial time-varying equilibrium.

(c) (5 points) The linearized dynamics of \( I(t) \) can be written as:

\[
\dot{I}_{lin}(t) = \frac{\partial f(t)}{\partial I(t)} \bigg|_{I(t) = I_{eq}(t)} \cdot I_{lin}(t).
\]

where \( \bigg|_{I(t) = I_{eq}(t)} \) means “evaluated at \( I(t) = I_{eq}(t) \)”. In other words, the linearized dynamic system can be written as

\[
\dot{x}(t) = \alpha(t) \cdot x(t)
\]
where \( x(t) \) is the linearized state \( I_{lin}(t) \), and \( \alpha(t) = \frac{\partial f(t)}{\partial I(t)} \bigg|_{I(t)=I_{eq}(t)} \). Analyze the stability of this equilibrium point and explain what happens as \( t \to \) as any of these parameters \( \beta, N(t), \gamma \) change.

Applying the linearization, we get
\[
\frac{\partial f(t)}{\partial I(t)} \bigg|_{I(t)=I_{eq}(t)} = -2 \frac{\beta}{N(t)} I_{eq}(t) + (\beta - \gamma) = -2 \frac{\beta}{\beta} \cdot \frac{\beta - \gamma}{\beta} N(t) + (\beta - \gamma) = \gamma - \beta.
\]

Hence, we can write
\[
\dot{I}_{lin}(t) = (\gamma - \beta) I_{lin}(t).
\]
If \( \gamma - \beta < 0 \), then the time-varying equilibrium point is a stable operating point. Otherwise if \( \gamma - \beta > 0 \), the equilibrium point \( I_{eq}(t) \) is an unstable operating point. Finally, if \( \gamma = \beta \), the operating point yields a marginally stable system.

Does the stability of the linearized system depend on \( N(t) \)?

Interestingly, the equilibrium point \( I_{eq}(t) \) does not depend on the time-varying quantity \( N(t) \) which is the total time-varying population of susceptible and infected people.