Your Name: ____________________

Your Signature: ____________________

- **Exam duration:** 1 hour and 20 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- No bathroom break allowed.
- **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 13 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

Question Number	Maximum Points	Your Score
1	20	
2	25	
3	15	
4	20	
5	20	
Total	100	

1. (20 total points) Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} = TJT^{-1}, \; B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) (15 points) Find $x(n)$ for any n if $u(k) = 2$ and $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

You might find this summation rule useful:

$$\sum_{j=0}^{k-1} j\alpha^{j-1} = \frac{d}{d\alpha}\sum_{j=0}^{k-1} \alpha^j = \frac{d}{d\alpha}\left[\frac{1-\alpha^k}{1-\alpha}\right] = \frac{1 - k\alpha^{k-1} + (k-1)\alpha^k}{(1-\alpha)^2}$$

(b) (5 points) What happens to $x(n)$ as $n \to \infty$?

2. (25 total points) You are given the following CT LTI system

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Assume that the control input is constant between two sampling instances, i.e.,

$$u(t) = u(kT) =: u(k), \textbf{ for } kT \leq t \leq (k+1)T, k = 0, 1, \ldots, k_f,$$

where T is the sampling time.

(a) (20 points) We wish to discretize the above continuous time system, and obtain:

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k).$$

Find the discretized state space matrices via the **two discretization method** we discussed in class. **You should derive these methods**. Recall that the second discretization method provides more accurate approximations.

(b) (5 points) Obtain $\tilde{A}, \tilde{B}$ given that

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -\pi \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T = \texttt{samplingTime} = 1$$

using either of the discretization methods.

3. (15 total points) Determine the stability of these systems (marginal, asymptotic, unstable). You have to clearly justify your answer.

   (a) (5 points)

$$x(k+1) = \begin{bmatrix} 0.4 & 1 \\ 2 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 10000000 \\ 0 \end{bmatrix} u(k)$$

(b) (5 points)

$$\dot{x}(t) = T \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} T^{-1} x(t) + \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} u(t)$$

(c) (5 points)

$$\dot{x}(t) = \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t)$$

4. (20 total points) You are given the following nonlinear dynamical system:

$$\dot{x}_1(t) = x_1(t)\sin(x_2^2(t)) + x_1^2 x_2(t)u(t) \quad (1)$$
$$\dot{x}_2(t) = x_1(t)e^{-x_2(t)} + \sin(u^2(t)) \quad (2)$$
$$y(t) = 2x_1(t)x_2(t) + x_2^2(t) + u(t). \quad (3)$$

   (a) (15 points) Obtain the linearized state space representation of the following nonlinear system around $x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix}$ and $u_e = u^*$. These equilibrium quantities are assumed to be given. You should obtain A, B, C, D for

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \quad \tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t).$$

   where $\tilde{x}(t) = x(t) - x_e$ and $\tilde{u}(t) = u(t) - u_e$. Note that A, B, C, D will be a function of the x_e and u_e.

(b) (5 points) Given A, B, C, D, determine the stability of the system around this equilibrium point:

$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \, u_e = 0.$$

5. (20 total points) Consider an LTI CT system

$$\dot{x}(t) = \begin{bmatrix} -2+2t & 4 \\ -1 & 2+2t \end{bmatrix} x(t).$$

   (a) (15 points) Obtain the state transition matrix $\phi(t, t_0)$ for the above system. To receive full credit, you have to clearly show your steps.

(b) (5 points) Is this system asymptotically stable?