• **Exam duration**: 1 hour and 20 minutes.
• This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
• No bathroom break allowed.
• **If we find that a laptop, phone, tablet or any electronic device near or on a person and even if the electronics device is switched off, it will lead to a straight zero in the finals.**
• **No calculators** of any kind are allowed.
• In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct**.
• Place a box around your final answer to each question.
• If you need more room, use the backs of the pages and indicate that you have done so.
• This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

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1. (20 total points) Consider the discrete-time LTI dynamical system model

\[ x(k+1) = Ax(k) + Bu(k), \]

where

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} = TJT^{-1}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

(a) (15 points) Find \( x(n) \) for any \( n \) if \( u(k) = 2 \) and \( x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

You might find this summation rule useful:

\[
\sum_{j=0}^{k-1} j^\alpha = \frac{d}{d\alpha} \sum_{j=0}^{k-1} \alpha^j = \frac{1 - k\alpha^{k-1} + (k-1)\alpha^k}{(1-\alpha)^2}
\]

\[
x(k) = A^kx(0) + \sum_{j=0}^{k-1} A^{k-1-j}Bu(j) = A^kx(0) + \sum_{j=0}^{k-1} A^jBu(k-1-j)
\]

- First, we can write \( A \) as:

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} = TJT^{-1}
\]

- Find \( A^k = TJ^kT^{-1} \), with \( J^k = \begin{bmatrix} 0.5^k & k0.5^{k-1} \\ 0 & 0.5^k \end{bmatrix} \), then:

\[
x(k) = TJ^kT^{-1}x(0) + T \sum_{j=0}^{k-1} J^jBu(k-1-j)
\]

- \( T^{-1}Bu(k-1-j) = 2T^{-1}B = 2v = 2 \begin{bmatrix} 0 & 1 \end{bmatrix}^\top \) constant, hence:

\[
x(k) = TJ^kT^{-1}x(0) + T \left( \sum_{j=0}^{k-1} J^j \right) (2v) = T \left( \sum_{j=0}^{k-1} J^j \right) (2v)
\]

- Recall that

\[
J^k = \begin{bmatrix} 0.5^k & k0.5^{k-1} \\ 0 & 0.5^k \end{bmatrix} \Rightarrow \sum_{j=0}^{k-1} J^j = \sum_{j=0}^{k-1} \begin{bmatrix} 0.5^j & j0.5^{j-1} \\ 0 & 0.5^j \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{k-1} 0.5^j & \sum_{j=0}^{k-1} j0.5^{j-1} \\ \sum_{j=0}^{k-1} 0.5^j & \sum_{j=0}^{k-1} 0.5^j \end{bmatrix}
\]

- Hence,

\[
x(k) = T \left( \sum_{j=0}^{k-1} J^j \right) (2v) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sum_{j=0}^{k-1} 0.5^j & \sum_{j=0}^{k-1} j0.5^{j-1} \\ \sum_{j=0}^{k-1} 0.5^j & \sum_{j=0}^{k-1} 0.5^j \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]
(b) (5 points) What happens to \( x(n) \) as \( n \to \infty \)?

Finding the limit of

\[
\lim_{k \to \infty} x(k) = \lim_{k \to \infty} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 - 2k0.5^{k-1} + 2(k - 1)0.5^k \\ \frac{(1 - 0.5)^2}{4 - 4 \cdot 0.5^k} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix}
\]
2. (25 total points) You are given the following CT LTI system

\[ \dot{x}(t) = Ax(t) + Bu(t). \]

Assume that the control input is constant between two sampling instances, i.e.,

\[ u(t) = u(kT) =: u(k), \text{ for } kT \leq t \leq (k+1)T, k = 0,1,\ldots,k_f, \]

where \( T \) is the sampling time.

(a) (20 points) We wish to discretize the above continuous time system, and obtain:

\[ x(k + 1) = \tilde{A}x(k) + \tilde{B}u(k). \]

Find the discretized state space matrices via the two discretization method we discussed in class. You should derive these methods. Recall that the second discretization method provides more accurate approximations.

**First method:**
- Use the derivative rule:

\[ \dot{x}(t) = \lim_{T \to 0} \frac{x(t + T) - x(t)}{T} \]

- You can use this approximation:

\[ \frac{x(t + T) - x(t)}{T} = Ax(t) + Bu(t) \Rightarrow x(t + T) = x(t) + ATx(t) + BTu(t) \]

- Hence,

\[ x(t + T) = (I + AT)x(t) + BTu(t) \]

- Now, if we compute \( x(t) \) and \( y(t) \) only at \( t = kT \) for \( k = 0,1,\ldots \), then the dynamical system equation for the discretized, approximate system is:

\[
  x((k+1)T) = (I + AT)x(kT) + BTu(kT)
\]

\[
  y(kT) = \tilde{C}x(kT) + \tilde{D}u(kT)
\]

**Second method:**
- Recall the solution to the state-equation:

\[ x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \]

- Setting \( t = kT \) in the previous equation, then we can write:

\[
  x(k) := x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau
\]

\[
  x(k + 1) := x((k + 1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau
\]
• Note that the above equation can be written as:

\[
x(k+1) = e^{AT} \left( e^{AkT}x(0) + \int_{0}^{kT} e^{A(kT-\tau)} Bu(\tau) d\tau \right) + \int_{kT}^{(k+1)T} e^{A(kT+\tau-\tau)} Bu(\tau) d\tau
\]

• Recall that we’re assuming that:

\[u(t) = u(kT) =: u(k) \quad \text{for} \quad kT \leq t \leq (k+1)T, \quad k = 0, 1, \ldots, k_f\]

i.e., the input is constant between two sampling instances.

• Look at \(x(k)\) and let \(\alpha = kT + T - \tau\), then:

\[
x(k+1) = e^{AT} x(k) + \left( \int_{0}^{T} e^{A\alpha} d\alpha \right) Bu(k).
\]

Hence,

\[
\tilde{A} = e^{AT}, \tilde{B} = \left( \int_{0}^{T} e^{A\alpha} d\alpha \right) B.
\]

(b) (5 points) Obtain \(\tilde{A}, \tilde{B}\) given that

\[
A = \begin{bmatrix} 2 & 0 \\ 0 & -\pi \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T = \text{samplingTime} = 1
\]

using either of the discretization methods.

• First method:

\[
\tilde{A} = I + TA = \begin{bmatrix} 3 & 0 \\ 0 & 1 - \pi \end{bmatrix}, \tilde{B} = TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

• Second method:

\[
\tilde{A} = \exp(AT) = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-\pi} \end{bmatrix}, \tilde{B} = \left( \int_{0}^{T} e^{A\alpha} d\alpha \right) B = \begin{bmatrix} 3.2 \\ 0 \end{bmatrix}
\]
3. (15 total points) Determine the stability of these systems (marginal, asymptotic, unstable). You have to clearly justify your answer.

(a) (5 points)

\[ x(k + 1) = \begin{bmatrix} 0.4 & 1 \\ 2 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 10000000 \\ 0 \end{bmatrix} u(k) \]

You have to find the eigenvalues of \( A \). The eigenvalues of \( A \) are: \( \{-0.42, 2.82\} \), hence \( A \) is unstable since one eigenvalue is outside the unit disk. Therefore, this system is unstable.

(b) (5 points)

\[ \dot{x}(t) = T \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} T^{-1} x(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t) \]

Unstable since 0.1 is an unstable eigenvalue in the Jordan block.

(c) (5 points)

\[ \dot{x}(t) = \begin{bmatrix} -0.4 & 1 & 1 & 0 \\ 0 & -0.4 & 1 & 0 \\ 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) \]

Marginally stable, since \( \text{eig}(A) = \{-0.4, -0.4, -0.4, 0\} \). Hence, all eigenvalues of \( A \) are in the closed LHP, and the eigenvalue on the \( j\omega \) axis has Jordan block of size 1. Hence, the system is marginally stable.
4. (20 total points) You are given the following nonlinear dynamical system:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) \sin(x_2^2(t)) + x_1^2 x_2(t) u(t) \\
\dot{x}_2(t) &= x_1(t) e^{-x_2(t)} + \sin(u^2(t)) \\
y(t) &= 2x_1(t) x_2(t) + x_2^2(t) + u(t).
\end{align*}
\]  

(a) (15 points) Obtain the linearized state space representation of the following nonlinear system around \(x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix}\) and \(u_e = u^*\). These equilibrium quantities are assumed to be given. You should obtain \(A, B, C, D\) for

\[
\begin{align*}
\dot{\Delta x}(t) &= A \Delta x(t) + B \Delta u(t) \\
y(t) &= C \Delta x(t) + D \Delta u(t)
\end{align*}
\]

where \(\Delta x(t) = x(t) - x_e\) and \(\Delta u(t) = u(t) - u_e\). Note that \(A, B, C, D\) will be a function of the \(x_e\) and \(u_e\).

\[
\begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \sin(x_{2e}^2) + 2x_{1e}x_{2e}u_e & x_{1e}^2u_e + 2x_{1e}x_{2e}\cos(x_{2e}^2) \\ e^{-x_{2e}} & -x_{1e}e^{-x_{2e}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} \Delta x_{1e}x_{2e} \\ 2u_e\cos(u_e^2) \end{bmatrix} \Delta u(t)
\]  

(b) (5 points) Given \(A, B, C, D\), determine the stability of the system around this equilibrium point:

\[
x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_e = 0.
\]

For the given linearization point, we obtain

\[
A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D = 1.
\]

The \(A\)-matrix is unstable because it’s Jordan form is of size 2, which means \(e^{At}\) would go to infinity as \(t\) goes to infinity. Hence, the above operating point is an unstable operating point.
5. (20 total points) Consider an LTI CT system
\[ \dot{x}(t) = \begin{bmatrix} -2 + 2t & 4 \\ -1 & 2 + 2t \end{bmatrix} x(t). \]

(a) (15 points) Obtain the state transition matrix \( \phi(t, t_0) \) for the above system. To receive full credit, you have to clearly show your steps.

We can write
\[
\begin{bmatrix} -2 + 2t & 4 \\ -1 & 2 + 2t \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} + 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_1 + \beta(t)A_2.
\]

Note that \( A_1 A_2 = A_2 A_1 \), and
\[ A_1^2 = 0. \]

Hence, \( A_1 \) is nilpotent of order 2. Hence,
\[
e^{A_1(t-t_0)} = I + (t-t_0)A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (t-t_0) \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 - 2(t-t_0) & 4(t-t_0) \\ -(t-t_0) & 1 + 2(t-t_0) \end{bmatrix}.
\]

In addition, we can write
\[
e^{A_2(t-t_0)} = e^{t_0^2-t_0^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Therefore,
\[
\phi(t, t_0) = e^{t_0^2-t_0^2} \begin{bmatrix} 1 - 2(t-t_0) & 4(t-t_0) \\ -(t-t_0) & 1 + 2(t-t_0) \end{bmatrix}.
\]

(b) (5 points) Is this system asymptotically stable?

No; \( \lim_{t \to \infty} \phi(t, t_0) = \infty. \)