Module 03
Linear Algebra Review & Solutions to State Space

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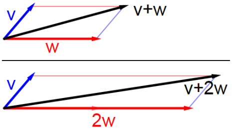
Vector Space (aka Linear Space)

A (real) vector space V is a set with two operations:

- Vector sum $+ : V + V \to V$
- Scalar multiplication $\cdot : \mathbb{R} \times V \to V$

that has the following properties

1. Commutative: $x + y = y + x, \forall x, y \in V$
2. Associative: $(x + y) + z = x + (y + z), \forall x, y, z \in V$
3. Zero element: $\exists! 0 \in V$ such that $0 + x = x, \forall x \in V$
4. Inverse: $\forall x \in V, \exists (-x) \in V$ such that $x + (-x) = 0$
5. $(\alpha\beta)x = \alpha(\beta x), \forall \alpha, \beta \in \mathbb{R}, x \in V$
6. $\alpha(x + y) = \alpha x + \alpha y, \forall a \in \mathbb{R}, x, y \in V$
7. $(\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in \mathbb{R}, x \in V$



Examples of Vector Space

1. $\mathbb{R}^n$ with vector sum and scalar multiplication
2. $\mathbb{R}^{m\times n}$: the set of all m-by-n matrices
3. P_n: the set of all real polynomials in s with degree up to n:

$$\mathcal{P}_n := \{a_n s^n + \cdots + a_1 s + a_0 \,|\, a_0, \ldots, a_n \in \mathbb{R}\}$$

4. Give an index set $\mathcal{I}$, the set of all mappings from $\mathcal{I}$ to $\mathbb{R}^n$:

$$\mathcal{F}(\mathcal{I}; \mathbb{R}^n) := \{f : \mathcal{I} \to \mathbb{R}^n\}$$

5. $\{f : \mathbb{R}_+ \to \mathbb{R}^n \,|\, f \text{ is differentiable}\}$
6. The set of all functions $f(t)$, $t \geq 0$, with a Laplace transform
7. The set of all square integrable functions $f : \mathbb{R}_+ \to \mathbb{R}$
8. The set of all solutions $x(t) \in \mathbb{R}^n$, $t \geq 0$, to autonomous LTI system

$$\dot{x} = Ax, \quad x(0) = x_0$$

Supspaces and Product Spaces

Definition (Subspace)

W is a subspace of vector space V if $W \subset V$ and W itself is a vector space under the same vector sum and scalar multiplication operations

Example:

- $\text{span}\,\{v_1, v_2, \ldots, v_k\} := \{\alpha_1 v_1 + \cdots + \alpha_k v_k : \alpha_i \in \mathbb{R}\} \subset V$
- Diagonal and symmetric matrices
- $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_\infty$

Definition (Product space)

Given two vector spaces V_1 and V_2, their direct product is the vector space $V_1 \times V_2 := \{(v_1, v_2) \,|\, v_1 \in V_1, v_2 \in V_2\}$

Example:

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
- $\mathcal{F}(\mathcal{I}; \mathbb{R}^2) = \mathcal{F}(\mathcal{I}; \mathbb{R}) \times \mathcal{F}(\mathcal{I}; \mathbb{R})$

Bases and Dimension of Vector Spaces

$v_1, \ldots, v_k$ in vector space V are linearly independent if for $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$,

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \quad \Rightarrow \quad \alpha_1 = \cdots = \alpha_k = 0$$

A set of vectors $\{v_1, \ldots, v_k\}$ is a basis of the vector space V if

- $v_1, \ldots, v_k$ are linear independent in V
- $V = \text{span}\,\{v_1, \ldots, v_k\}$

Or equivalently,

- each $v \in V$ has a unique expression $v = \alpha_1 v_1 + \cdots + \alpha_k v_k$
- $(\alpha_1, \ldots, \alpha_k)$ is the coordinate of v in this basis

Definition (Dimension)

The dimension of a vector space V is the number of vectors in any of its basis, and is denoted $\dim V$.

Examples of finite and infinite dimensional vector spaces:

Linear Maps

A map $f : V \to W$ between two vector spaces V and W is linear if

$$f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$$

Example:

- $x \in \mathbb{R}^n \mapsto Ax \in \mathbb{R}^m$ for some matrix $A \in \mathbb{R}^{m\times n}$
- Projection $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto x_i \in \mathbb{R}$
- $X \in \mathbb{R}^{m\times n} \mapsto X^T \in \mathbb{R}^{n\times m}$
- $X \in \mathbb{R}^{n\times n} \mapsto A_1X + XA_2 \in \mathbb{R}^{n\times n}$ for constant $A_1, A_2 \in \mathbb{R}^{n\times n}$
- A continuous function on $[0, 1] \mapsto \int_0^t f(x)\, dx \in \mathbb{R}$
- Polynomial $p(s) \in \mathcal{P}_n \mapsto p'(s) \in \mathcal{P}_{n-1}$
- Solutions (zero-state, zero-input responses) of an LTI system

A linear map $f : V \to W$ must map $0 \in V$ to $0 \in W$

The composition of two linear maps $f : V \to W$ and $g : W \to U$ is also linear: $g \circ f : v \in V \mapsto g(f(v)) \in U$

One-To-One Mapping

Matrix $A \in \mathbb{R}^{m \times n}$ considered as a linear map $\mathbb{R}^n$ to $\mathbb{R}^m$ has null space $\mathcal{N}(A) = \{x \in \mathbb{R}^n \,|\, Ax = 0\}$

- Set of all vectors orthogonal to all rows of A
- Characterize ambiguity in solving equation $Ax = y$

$A \in \mathbb{R}^{m \times n}$ is one-to-one if and only if

- Columns of A are linearly independent
- Rows of A span $\mathbb{R}^n$
- A has rank n (full column rank)
- A has a left inverse: $\exists B \in \mathbb{R}^{n \times m}$ such that $BA = I_n$

Matrix Rank

The rank of a matrix $A \in \mathbb{R}^{m \times n}$ is its maximum number of linearly independent columns (or rows), or equivalently, $\dim \mathcal{R}(A)$

- $\text{Rank}\,(A) \leq \min(m, n)$
- $\text{Rank}\,(A) = \text{Rank}\,(A^T)$
- $\text{Rank}\,(A) + \dim \mathcal{N}(A) = n$ (conservation of dimension)

Matrix $A \in \mathbb{R}^{m \times n}$ is full rank if $\text{Rank}\,(A) = \min(m, n)$, which means

- (for skinny matrices) independent column or injective maps
- (for fat matrices) independent rows or surjective maps
- (for square matrices) nonsingular or bijective maps

Matrix Transpose

When $A \in \mathbb{R}^{m\times n}$ is considered as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, its transpose $A^T \in \mathbb{R}^{n\times m}$ is a linear map from $\mathbb{R}^m$ back to $\mathbb{R}^n$

The following are equivalent

1. A is one-to-one
2. A^T is onto
3. $\det A^T A \neq 0$
4. $A^T A \in \mathbb{R}^{n\times n}$ is bijective

The following are equivalent

1. A is onto
2. A^T is one-to-one
3. $\det AA^T \neq 0$
4. $AA^T \in \mathbb{R}^{m\times m}$ is bijective

More generally, for any $A \in \mathbb{R}^{m\times n}$

- $\mathcal{R}(A^T) = \mathcal{N}(A)^\perp$
- $\mathcal{N}(A^T) = \mathcal{R}(A)^\perp$

Inner Products

For $x, y \in \mathbb{R}^n$, their inner product is

$$\langle x, y \rangle := x^T y = x_1 y_1 + \cdots + x_n y_n$$

For $x, y, z \in \mathbb{R}^n$

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle x, y \rangle$
- $\langle x, x \rangle = \|x\|^2 \geq 0$, where $\|x\|$ is the Euclidean norm of x:

$$\|x\| := \sqrt{x^T x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Theorem (Cauchy-Schwartz Inequality)

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \quad \forall x, y \in \mathbb{R}^n$$

Eigenvalues and Eigenvectors

Eigenvalues/Eigenvectors of a matrix

- Evalues/vectors are **only defined for square[1] matrices**
- For a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, we always have n evalues/evectors
  - Some of these evalues might be distinct, real, repeated, imaginary
  - To find evalues($\boldsymbol{A}$), solve this equation ($\boldsymbol{I}_n$: identity matrix of size n)

$$\det(\lambda \boldsymbol{I}_n - A) = 0 \text{ or } \det(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = 0 \Rightarrow a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

- **Example**: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$
- **Eigenvectors**: A number λ and a non-zero vector $\boldsymbol{v}$ satisfying

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \Rightarrow (\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{v} = 0$$

  are called an eigenvalue and an eigenvector of $\boldsymbol{A}$
  - λ is an eigenvalue of an $n \times n$-matrix $\boldsymbol{A}$ if and only if $\lambda \boldsymbol{I}_n - \boldsymbol{A}$ is not invertible, which is equivalent to

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}_{\mathrm{n}}) = 0.$$

[1] A square matrix has equal number of rows and columns.

Matrix Inverse

- Inverse of a generic 2by2 matrix:

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  – Notice that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = I_2$

- Inverse of a generic 3by3 matrix:

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^{T} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} A & D & G \\ B & E & H \\ C & F & I \end{bmatrix}$$

$$\begin{array}{lll} A = (ei - fh) & D = -(bi - ch) & G = (bf - ce) \\ B = -(di - fg) & E = (ai - cg) & H = -(af - cd) \\ C = (dh - eg) & F = -(ah - bg) & I = (ae - bd) \end{array}$$

$$\det(\mathbf{A}) = aA + bB + cC.$$

  – Notice that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = I_3$

Linear Algebra — Example 1

- Find the eigenvalues, eigenvectors, and inverse of matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

  - Eigenvalues: $\lambda_{1,2} = 5, -2$
  - Eigenvectors: $\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, \boldsymbol{v}_2 = \begin{bmatrix} -\frac{4}{3} & 1 \end{bmatrix}^\top$
  - Inverse: $\boldsymbol{A}^{-1} = -\frac{1}{10}\begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}$

- Write $\boldsymbol{A}$ in the matrix **diagonal transformation,** i.e., $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$ where $\boldsymbol{D}$ is the diagonal matrix containing the eigenvalues of $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}^{-1}$$

  - **Only valid for matrices with distinct, real eigenvalues**

Rank of a Matrix

- Rank of a matrix: $\text{rank}(\boldsymbol{A})$ is equal to the number of linearly independent rows or columns
  - **Example 1:** $\text{rank}\left(\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix}\right) = ?$
  - **Example 2:** $\text{rank}\left(\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}\right) = ?$
- Rank computation: reduce the matrix to a simpler form, generally row echelon form, by elementary row operations
  - **Example 2 Solution:**

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \rightarrow 2r_1 + r_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \rightarrow -3r_1 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\rightarrow r_2 + r_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow -2r_2 + r_1 \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(\boldsymbol{A}) = 2$$

- **For a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$:** $\text{rank}(\boldsymbol{A}) \leq \min(m, n)$

Null Space of a Matrix

- The Null Space of any matrix $\boldsymbol{A}$ is the subspace $\mathcal{K}$ defined as follows:

$$\mathrm{N}(\boldsymbol{A}) = \mathrm{Null}(\boldsymbol{A}) = \ker(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathcal{K} | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$$

- $\mathrm{Null}(\boldsymbol{A})$ has the following three properties:
  - $\mathrm{Null}(\boldsymbol{A})$ always contains the zero vector, since $\boldsymbol{A}\boldsymbol{0} = \boldsymbol{0}$
  - If $\boldsymbol{x} \in \mathrm{Null}(\boldsymbol{A})$ and $\boldsymbol{y} \in \mathrm{Null}(\boldsymbol{A})$, then $\boldsymbol{x} + \boldsymbol{y} \in \mathrm{Null}(\boldsymbol{A})$
  - If $\boldsymbol{x} \in \mathrm{Null}(\boldsymbol{A})$ and c is a scalar, then $c\boldsymbol{x} \in \mathrm{Null}(\boldsymbol{A})$
- **Example:** Find $\mathrm{N}(\boldsymbol{A})$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ -4 & 2 & 3 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/16 & 0 \\ 0 & 1 & 13/8 & 0 \end{array}\right] \Rightarrow a = -\frac{1}{16}c, b = -\frac{13}{8}c \quad \Rightarrow \boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \begin{bmatrix} -1/16 \\ -13/8 \\ 1 \end{bmatrix} = \tilde{\alpha} \begin{bmatrix} -1 \\ -26 \\ 16 \end{bmatrix}}$$

Linear Algebra — Example 2

- Find the determinant, rank, and null-space set of this matrix:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

  - $\det(\boldsymbol{B}) = 0$
  - $\text{rank}(\boldsymbol{B}) = 2$
  - $\text{null}(\boldsymbol{B}) = \alpha \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \forall\ \alpha \in \mathbb{R}$
- Is there a relationship between the determinant and the rank of a matrix?
  - Yes! Matrix drops rank if determinant $=$ zero $\Rightarrow$ 1 zero evalue
- True or False?
  - $\boldsymbol{AB} = \boldsymbol{BA}$ for all $\boldsymbol{A}$ and $\boldsymbol{B}$—**FALSE!**
  - $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible $\rightarrow (\boldsymbol{A} + \boldsymbol{B})$ is invertible—**FALSE!**

Matrix Exponential — 1

- Exponential of scalar variable:

$$e^a = \sum_{i=0}^{\infty} \frac{a^i}{i!} = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \cdots$$

- Power series converges $\forall\ a \in \mathbb{R}$
- How about matrices? For $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, matrix exponential:

$$e^{\boldsymbol{A}} = \sum_{i=0}^{\infty} \frac{\boldsymbol{A}^i}{i!} = \boldsymbol{I}_n + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \frac{\boldsymbol{A}^4}{4!} + \cdots$$

- What if we have a time-variable?

$$e^{t\boldsymbol{A}} = \sum_{i=0}^{\infty} \frac{(t\boldsymbol{A})^i}{i!} = \boldsymbol{I}_n + t\boldsymbol{A} + \frac{(t\boldsymbol{A})^2}{2!} + \frac{(t\boldsymbol{A})^3}{3!} + \frac{(t\boldsymbol{A})^4}{4!} + \cdots$$

Matrix Exponential Properties

For a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and a constant $t \in \mathbb{R}$:

1. $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \;\; \Rightarrow \;\; e^{\boldsymbol{A}t}\boldsymbol{v} = e^{\lambda t}\boldsymbol{v}$
2. [2]$\det(e^{\boldsymbol{A}t}) = e^{(\text{trace}(\boldsymbol{A}))t}$
3. $(e^{\boldsymbol{A}t})^{-1} = e^{-\boldsymbol{A}t}$
4. $e^{\boldsymbol{A}^\top t} = (e^{\boldsymbol{A}t})^\top$
5. If $\boldsymbol{A}, \boldsymbol{B}$ commute, then: $e^{(\boldsymbol{A}+\boldsymbol{B})t} = e^{\boldsymbol{A}t}e^{\boldsymbol{B}t} = e^{\boldsymbol{B}t}e^{\boldsymbol{A}t}$
6. $e^{\boldsymbol{A}(t_1+t_2)} = e^{\boldsymbol{A}t_1}e^{\boldsymbol{A}t_2} = e^{\boldsymbol{A}t_2}e^{\boldsymbol{A}t_1}$

[2] Trace of a matrix is the sum of its diagonal entries.

When Is It Easy to Find $e^{\boldsymbol{A}}$? Method 1

Well...Obviously if we can directly use $e^{\boldsymbol{A}} = \boldsymbol{I}_n + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \cdots$

Three cases for Method 1

Case 1 $\boldsymbol{A}$ is nilpotent[3], i.e., $\boldsymbol{A}^k = 0$ for some k. Example:

$$\boldsymbol{A} = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$$

Case 2 $\boldsymbol{A}$ is idempotent, i.e., $\boldsymbol{A}^2 = \boldsymbol{A}$. Example:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Case 3 $\boldsymbol{A}$ is of rank one: $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$

$$\boldsymbol{A}^k = (\boldsymbol{v}^T\boldsymbol{u})^{k-1}\boldsymbol{A}, \; k = 1, 2, \ldots$$

[3] Any triangular matrix with 0s along the main diagonal is nilpotent

Method 2 — Jordan Canonical Form

- All matrices, whether diagonalizable or not, have a Jordan canonical form: $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^{-1}$, then $e^{\boldsymbol{A}t} = \boldsymbol{T}e^{\boldsymbol{J}t}\boldsymbol{T}^{-1}$
- Generally, $\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & & \\ & \ddots & \\ & & \boldsymbol{J}_p \end{bmatrix} \boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \Rightarrow$

$$e^{\boldsymbol{J}_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \dots & \frac{t^{n_i-1}e^{\lambda_i t}}{(n_i-1)!} \\ 0 & e^{\lambda_i t} & \ddots & \frac{t^{n_i-2}e^{\lambda_i t}}{(n_i-2)!} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_i t} \end{bmatrix} \Rightarrow e^{\boldsymbol{A}t} = \boldsymbol{T} \begin{bmatrix} e^{\boldsymbol{J}_1 t} & & \\ & \ddots & \\ & & e^{\boldsymbol{J}_o t} \end{bmatrix} \boldsymbol{T}^{-1}$$

- Jordan blocks and marginal stability

Examples

- Find $e^{\boldsymbol{A}(t-t_0)}$ for matrix A given by:

$$\boldsymbol{A} = \boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \end{bmatrix}^{-1}$$

- **Solution:**

$$e^{\boldsymbol{A}(t-t_0)} = \boldsymbol{T} e^{\boldsymbol{J}(t-t_0)} \boldsymbol{T}^{-1}$$

$$= \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \end{bmatrix} \begin{bmatrix} e^{-(t-t_0)} & 0 & 0 & 0 \\ 0 & 1 & t-t_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-(t-t_0)} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \end{bmatrix}^{-1}$$

- Find $e^{\boldsymbol{A}(t-t_0)}$ for matrix A given by:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Jordan Canonical Form

Theorem (Jordan Canonical Form)

For any $A \in \mathbb{R}^{n\times n}$, there exists a nonsingular $T \in \mathbb{C}^{n\times n}$ such that

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}, \qquad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}$$

- Unique up to permutation of Jordan blocks
- Diagonalizable matrices are special cases with all $n_i = 1$

Definition (Algebraic and Geometric Multiplicity)

The *algebraic multiplicity* of an eigenvalue λ_i is the sum of the sizes of all Jordan blocks corresponding to it; its *geometric multiplicity* is the number of all such Jordan blocks.

Finding Jordan Canonical Form

1. The objective here is to show how to find $A = TJT^{-1}$ for a nondiagonalizable matrix A
2. Assume that matrix A has n eigenvalues
   - k evalues are distinct AND not repeated (multiplicity $= 1$, $\lambda_1, \lambda_2, \ldots, \lambda_k$)
   - Hence, there are $n-k$ evalues that are repeated (multiplicity ≥ 2)
3. First, Find the k eigenvectors relating to these eigenvalues and list the first k eigenvalues on the first k diagonal entries of J. Also, group the first k eigenvectors in the first k columns of T
4. What's left now: $n-k$ generalized evectors of the other evalues that are repeated at least twice, and the Jordan blocks corresponding to these evalues
5. Assume that out of the $n-k$ evalues, there are m distinct ones
6. Find the evectors that correspond to the m distinct ones—you should obtain at least m evectors
7. What's left now: find the other generalized evectors (i.e., $n-k-m$ evectors) and Jordan blocks (number of Jordan blocks corresponding to the repeated evalues is equal to the number of linearly independent evectors)

- Example: find the Jordan canonical form of this matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}, \pi_A(\lambda) = \lambda^4(\lambda - 1) = 0$$

- Two eigenvalues: $\lambda_1 = 1$ (not repeated), $\lambda_2 = 0$ (repeated 4 times)
- First: find evector for $\lambda_1 = 1$

$$(A - \lambda_1 I_5)v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix} v_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \end{bmatrix}^\top$$

- Now, let's find the generalized evectors for $\lambda_2 = 0$ and the associated Jordan block. Note that the A matrix is of rank 3
- First, find the LI evectors of λ_2:

$$(A - \lambda_2 I_5)v_2 = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} v_2 = 0 \Rightarrow v_2 \in \mathcal{N}(A)$$

- You can see that v_2 actually spans two column vectors since A is of rank 3
- The two LI evectors generated from $Av_2 = 0$ are:

$$v_2^1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^\top, v_2^2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}^\top$$

- Therefore, we have two Jordan blocks corresponding to λ_2
- We have to alternatives for the sizes these two Jordan blocks: either (3,1) or (2,2)
- How do we know the correct size?
- The number of Jordan blocks of size exactly j is

$$2 \dim \ker(A - \lambda_i I)^j - \dim \ker(A - \lambda_i I)^{j+1} - \dim \ker(A - \lambda_i I)^{j-1}$$

- Hence, the number of Jordan blocks of size 1 is: $2 * 2 - 3 - 0 = 1$, hence the size the Jordan blocks of size 3 is also one, which means $(3, 1)$ is a legit Jordan block sizes

$$\Rightarrow J = ?$$

- Now that we have the Jordan blocks, we need to find the two other generalized evectors corresponding to v_2^2

Examples

- Find $e^{\boldsymbol{A}(t-t_0)}$ for matrix A given by:

$$\boldsymbol{A} = \boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^{-1} = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix}^{-1}$$

- **Solution:**

$$e^{\boldsymbol{A}(t-t_0)} = \boldsymbol{T}e^{\boldsymbol{J}(t-t_0)}\boldsymbol{T}^{-1}$$

$$= \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix}\begin{bmatrix}e^{-(t-t_0)} & 0 & 0 & 0\\ 0 & 1 & t-t_0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & e^{-(t-t_0)}\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix}^{-1}$$

- Find $e^{\boldsymbol{A}(t-t_0)}$ for matrix A given by:

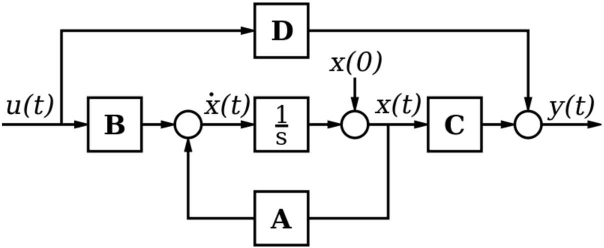
$$\boldsymbol{A}_1 = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix} \text{ and } \boldsymbol{A}_2 = \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}$$

Solution to the State-Space Equation

- In the next few slides, we'll answer this question: what is a solution to this vector-matrix first order ODE:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

- By solution, we mean a closed-form solution for $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ given:
  - An initial condition for the system, i.e., $\boldsymbol{x}(t_{initial}) = \boldsymbol{x}(0)$
  - A given control input signal, $\boldsymbol{u}(t)$, such as a step-input ($u(t) = 1$), ramp ($u(t) = t$), or anything else



The Curious Case of Autonomous Systems—Case 1

- Let's assume that we seek solution to this system first:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t), \boldsymbol{x}(0) = \boldsymbol{x}_0 = \text{given} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned}$$

- This means that the system operates without any control input—**autonomous system** (e.g., autonomous vehicles)
- First, let's look at $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$—what's the solution to this first order ODE?
  - First case: $\boldsymbol{A} = a$ is a scalar $\Rightarrow x(t) = e^{at}x_0$
  - Second case: $\boldsymbol{A}$ is a matrix

$$\boxed{\Rightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 \Rightarrow \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}_0}$$

- Exponential of scalars is very easy, but exponentials of matrices can be very challenging
- Hence, for an nth order system, where $n \geq 2$, we need to compute the matrix exponential in order to get a solution for the above system—we learned that in the linear algebra revision section

Example (Case 1)

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0, \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}_0}$$

- Find the solution for these two autonomous systems separately:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \boldsymbol{C}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \boldsymbol{x}_0^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \boldsymbol{C}_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}, \boldsymbol{x}_0^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Note that this system is diagonalizable **(Case A)**
- If the system is not diagonalizable, we have to look for other methods to find the matrix exponential
- In particular, we have to find the Jordan form
- Anyway, let's find the state and output solutions now for this diagonalizable system
- **Solution:**

Case 2—Systems with Inputs

- MIMO (or SISO) LTI dynamical system:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \boldsymbol{x}(t_0) = \boldsymbol{x}_{t_0} = \text{given} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

- The to the above ODE is given by:

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau}$$

- Clearly the output solution is:

$$\boxed{\boldsymbol{y}(t) = \underbrace{\boldsymbol{C}\left(e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0}\right)}_{\text{zero input response}} + \underbrace{\boldsymbol{C}\left(\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau\right) + \boldsymbol{D}\boldsymbol{u}(t)}_{\text{zero state response}}}$$

- **Question:** how do I analytically compute $\boldsymbol{y}(t)$ and $\boldsymbol{x}(t)$?
- **Answer:** you need to (a) **integrate** and (b) **compute matrix exponentials** (given $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{x}_{t_0}, \boldsymbol{u}(t)$)

Example (Case 2)

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\, d\tau$$

$$\boldsymbol{y}(t) = \underbrace{\boldsymbol{C}\left(e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0}\right)}_{\text{zero input response}} + \underbrace{\boldsymbol{C}\left(\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\, d\tau\right) + \boldsymbol{D}\boldsymbol{u}(t)}_{\text{zero state response}}$$

- Find the solution for these two LTI systems with inputs:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{C}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \boldsymbol{x}_0^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, D_1 = 0, u_1(t) = 1$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{C}_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}, \boldsymbol{x}_0^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, D_2 = 1, u_2(t) = 2e^{-2t}$$

- **Solution:**

Questions And Suggestions?

Thank You!

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IFF you want to know more ☺