Module 07
Controllability and Controller Design of Dynamical LTI Systems

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October 18, 2017

Controllability Introduction

A CT LTI system with m inputs and n states is defined as follows:

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$$

- **Controllability**: the ability to move a system (i.e., its states $x(t)$) from one point in space to another via certain control signals $u(t)$
- **Rigorous definition**: Over the time interval $[0, t_f]$, control input $u(t)\ \forall\ t \in [0, t_f]$ steers the state from x_0 to x_{t_f}:

$$x(t_f) = e^{At_f}x_0 + \int_0^{t_f} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Controllability Definition

LTI system **is controllable at time** $t_f > 0$ if for any initial state and for any target state (x_{t_f}), a control input $u(t)$ exists that can steer the system states from $x(0)$ to $x(t_f)$ over the defined interval.

LTI system **is called controllable** if it is controllable at a large enough t_f.

Example

Consider this dynamical system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), x_0 = 0$$

Is this system controllable?
Well, clearly

$$x_1(t) = x_2(t) = \int_{t_0}^{t} u(\tau)d\tau$$

Hence, no control input can steer the system to $x_1(t) \neq x_2(t)$, i.e., to distinct x_1 and x_2. Hence, the system is NOT controllable.

Controllability Questions

Four main questions are asked when solving controllability-related problems:

1. Where can we transfer x_0 for a time horizon $[0, t_f]$?
2. If answer to 1) above is doable, how do we choose the control $u(t)$ for the specified time horizon?
3. How quickly can x_0 be transferred to x_f?
4. What is a low-cost $u(t)$ that does this operation?

– We'll try answer some of these questions

Reachability vs. Controllability

- Reachability is a concept similar to controllability
- Consider an LTI system with zero initial condition $x_0 = 0$:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- **Reachability:** The above system is called **reachable at time** t_f if the system can be steered from $x_0 = 0$ to **any** x_f over the time interval $[0, t_f]$
- **Theorem:** At any $t_f > 0$, the system is controllable **if and only if it is reachable**

Reachable Set, Subspace

Definition of Reachable Set

The **reachable set** at time $t_f > 0$ is the set of states the system can be steered to using arbitrary control inputs over $[0, t_f]$:

$$\mathcal{R}_{t_f} = \left\{ \int_0^{t_f} e^{A(t_f-\tau)} Bu(\tau) d\tau \,\middle|\, u(t), 0 \le t \le t_f \right\}$$

- System is reachable at t_f if $\mathcal{R}_{t_f} = \mathbb{R}^n$
- Hence, we can say that $\mathcal{R}_{t_f}$ is a subspace of $\mathbb{R}^n$
- $\mathcal{R}_{t_f}$ is the image of the linear map taking $u(t)$ as input and producing x_{t_f} as output
- Also, note that $\mathcal{R}_{t_f} \subset \mathcal{R}_{t_{f2}}$, $t_f < t_{f2}$
- – What does that mean? It means that if you give the system more time, it'll be able to reach more states in $\mathbb{R}^n$
- **Reachable subspace**—set of all reachable states, i.e., the union of all reachable sets:

$$\mathcal{R} = \bigcup_{t_f > 0} \mathcal{R}_{t_f}$$

Controllability of DT LTI Systems

- Consider the following DT LTI system:

$$x(k+1) = Ax(k) + Bu(k), x(0) = x_0$$

- Recall that given a final time k_f and corresponding $u(k)$, $x(k_f)$ can be written as:

$$x_f = x(k_f) = A^{k_f} x_0 + \sum_{j=0}^{k_f-1} A^{k_f-1-j} Bu(j)$$

DT Controllability Definition

The above system is **controllable at time** k_f if **for any** $x_0, x_f \in \mathbb{R}^n$, a control $u(k), \forall k = 0, \ldots, k_f - 1$ exists that can steer the system from x_0 to x_f at time k_f

The system is called **controllable if it is controllable at a large** k_f

DT Reachable Set, Subspace

Definition of Reachable Set

The **reachable set** at time $k_f > 0$ is the set of states the system can be steered to using arbitrary control inputs over $[0, t_f]$:

$$\mathcal{R}_{t_f} = \left\{ \sum_{j=0}^{k_f-1} A^{k_f-1-j} B u(j) \,\middle|\, u(k), k = 0, 1, \ldots, k_f - 1 \right\}$$

- System is reachable at k_f if $\mathcal{R}_{k_f} = \mathbb{R}^n$
- Hence, we can say that $\mathcal{R}_{k_f}$ is a subspace of $\mathbb{R}^n$
- $\mathcal{R}_{k_f}$ is the image of the linear map taking $u(t)$ as input and producing x_{k_f} as output
- Also, note that $\mathcal{R}_{k_f} \subset \mathcal{R}_{k_{f2}}$, $k_f < k_{f2}$
  - What does that mean? It means that if you give the system more time, it'll be able to reach more states in $\mathbb{R}^n$
- **Reachable subspace**—set of all reachable states:

$$\mathcal{R} = \bigcup_{k_f > 0} \mathcal{R}_{k_f}$$

Characterizing Controllability

- So, we defined controllability for both CT and DT LTI systems
- But how do we figure out whether a system is controllable/reachable or not?
- Is there a litmus test given state-space matrices? Yes!
- Consider this DT system $x(k+1) = Ax(k) + Bu(k)$
- Let's answer the question of controllability and try find $u(k)$ that would steer $x_0 = 0$ to a predefined $x(k_f) = x(n) = x_n$
- If we can find this $u(k)$ for all $k = 0, 1, \ldots, k_f - 1$, then the system is controllable/reachable
- n here is also the size of the state x, i.e., we have n controls to reach our desired state

Controllability Test Derivation

$$x(k+1) = Ax(k) + Bu(k)$$

Notice that:

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= A^2x(0) + ABu(0) + Bu(1) \\ \vdots &= \vdots \\ x(n) &= A^n x(0) + A^{n-1}Bu(0) + A^{n-2}Bu(1) + \ldots Bu(n-1) \end{aligned}$$

Since x_n, x_0 are both predefined (or predetermined), we want to find a control sequence $u(0), u(1), \ldots, u(n-1)$ such that the system is controllable. We can write the above system of equations as:

$$\boxed{x(n) - A^n x(0) = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = \mathcal{C} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}}$$

Controllability Test Derivation (Cont'd)

If this matrix $\mathcal{C}$ defined in the previous slide is full rank, the previous equation can be written as:

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = \mathcal{C}^{\dagger}(x(n) - A^n x(0))$$

- Matrix $\mathcal{C}$: **controllability matrix**
- If the system is single input, $\mathcal{C}$ would be square and the $\dagger$ sign would be replaced by -1 (the inverse of square matrix)
- If the system is multi input (m inputs), $\mathcal{C}$ would be rectangular of dimension $n \times (m \cdot n)$ (hence, we need a right inverse to find the pseudo inverse)
- Hence: **the LTI system $x(k+1) = Ax(k) + Bu(k)$ is controllable if the matrix $\mathcal{C}$ is full rank**

Cayley-Hamilton Theorem and Controllability

For a general $n \times n$ matrix A, the Cayley-Hamilton theorem states that

$$p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n = 0$$

- This means that the n-th power of A can be written as a linear combination of the lower powers of A, where c_i's are constants
- **This also means that A satisfies the characteristic polynomial**
  - For a matrix A, the evalue equation is

$$\pi_A(\lambda) = |\lambda I_n - A| = 0 \Rightarrow \lambda^n + c_{n-1}\lambda^{n-1} + \ldots c_0 = 0$$

  - Replacing λ with A, you'll obtain the Cayley-Hamilton theorem
- How does that relate to controllability?
- It implies that for $k \geq n$, you don't get more information from the system

Controllability Tests

Controllability Test

For a system with n states and m inputs, controllability test:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times nm}$$

has full row rank (i.e., $\text{rank}(\mathcal{C}) = n$).

The following statements and tests for controllability are equivalent:

- T1 $\mathcal{C}$ is full rank
- T2 PBH Test: for all $\lambda_i \in eig(A)$, $\text{rank}\,[\lambda_i I - A \;\; B] = n$
- T3 Eigenvector Test: for any **left evector** w_i of A, $w_i^\top B \neq 0$
- T4 For any $t_f > 0$, the so-called Gramian matrix is nonsingular:

$$W(t_f) = \int_0^{t_f} e^{A\tau} BB^\top e^{A^\top \tau}\, d\tau = \int_0^{t_f} e^{A(t_f-\tau)} BB^\top e^{A^\top (t_f-\tau)}\, d\tau$$

- T4' For DT systems, for any $n > 0$, the Gramian is nonsingular:

$$W(n-1) = \sum_{m=0}^{n-1} A^m BB^\top (A^\top)^m$$

Left and Right Eigenvectors

- We know how to solve for eigenvectors of any square matrix:

$$(A - \lambda_i I)v_i = 0$$

- This definition for eigenvector above is called the *right eigenvector*
- The left evector of a matrix is defined as:

$$w_i^\top (A - \lambda_i I) = 0$$

- They are of course related:

$$A = TDT^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \vdots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^\top & \text{———} \\ w_2^\top & \text{———} \\ & \vdots \\ w_n^\top & \text{———} \end{bmatrix}$$

- Controllability Test 1 uses the left eigenvectors instead of the right ones:
  - For any **left evector** w_i of A, $w_i^\top B \neq 0$

Example 1 — Tests 1, 2, and 3

- Now that we've talked about the idea of controllability, let's see how that relates to examples of LTI systems
- Are these systems controllable? Uncontrollable?
- Example 1: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- This the above system controllable? Use Test 1
- **Solution:** find the controllability matrix

$$\mathcal{C} = [B\ AB\ A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

- This matrix is full rank $\Rightarrow$ system is controllable via Test 1
- Try Test 2: find all the evalues of A, and check that $\lambda_i \in eig(A)$, rank $[\lambda_i I - A \;\; B] = 3$ for $\lambda_{1,2,3}$
- Try Test 3 for the three evectors of A, we have $v_i^T B \neq 0$ for $v_{1,2,3}$

Example 2 — Test 1

Investigate the controllability of this system

$$A = diag(\lambda_1, \lambda_2, \lambda_3), B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 \\ b_3 & \lambda_3 b_3 & \lambda_3^2 b_3 \end{bmatrix}$$

- If $b_i = 0$ for some i, rank $\mathcal{C} < 3$
- We should investigate $\mathcal{C}$ further. Notice that:

$$\mathcal{C} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 \\ b_3 & \lambda_3 b_3 & \lambda_3^2 b_3 \end{bmatrix} \rightarrow \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ 0 & (\lambda_2 - \lambda_1) b_2 & (\lambda_2^2 - \lambda_1^2) b_2 \\ 0 & 0 & (\lambda_3^2 - \lambda_1^2) b_3 - (\lambda_2^2 - \lambda_1^2) \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1} b_3 = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) b_3 \end{bmatrix}$$

- **Final conclusion**: The system is controllable if and only if $b_i \neq 0 \forall i$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$

Example 3 — Test 4

- Via the controllability Gramian test, prove that this CT LTI system is controllable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

- Recall that the system is controllable if for any $t_f > 0$, the so-called Gramian matrix is nonsingular:

$$W(t_f) = \int_0^{t_f} e^{A\tau} BB^\top e^{A^\top \tau}\, d\tau$$

- In this example:

$$W(t_f) = \int_0^{t_f} e^{A\tau} BB^\top e^{A^\top \tau}\, d\tau = \int_0^{tf} \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} d\tau =$$

$$\int_0^{t_f} \begin{bmatrix} \tau \\ 1 \end{bmatrix} \begin{bmatrix} \tau & 1 \end{bmatrix} d\tau = \int_0^{t_f} \begin{bmatrix} \tau^2 & \tau \\ \tau & 1 \end{bmatrix} d\tau = \begin{bmatrix} t_f^3 \cdot 1/3 & 0.5t_f^2 \\ 0.5t_f^2 & t_f \end{bmatrix} = W(t_f)$$

- For $W(t_f)$ to be nonsingular, we need $t_f^3 > 0$, and $t_f^4 \cdot (1/3) - (0.5t_f^2)(0.5t_f^2) > 0$ which is always true for $t_f > 0$

Design of Controllers

- We talked about controllability
- We also know whether any LTI system is controllable or not
- But what we don't know is how to design a controller that would move my $x(t_0)$ to an $x(t_f)$ of my choice
- How can we do that?
  - For DT control systems, we saw how we can do that
  - The answer is more complicated

Design of Controllers via Controllability Gramian

- So, say that the system is controllable (DT or CT LTI system)
- How can I find the control law $u(t)$ that would take me from any $x(t_0)$ to $x(t_f)$?
- The controllability Gramian allows you to achieve that

Control Design Via the Gramian

For any $x(0) = x_0$, and $x(t_f) = x_{t_f}$, the control input law:

$$u(t) = -B^\top e^{A^\top (t_f - t)} W^{-1}(t_f) \left[e^{At_f} x_0 - x_{t_f} \right]$$

$$= -B^\top e^{A^\top (t_f - t)} \left(\int_0^{t_f} e^{A\tau} B B^\top e^{A^\top \tau} \, d\tau \right)^{-1} \left[e^{At_f} x_0 - x_{t_f} \right], \quad \forall t = [0, t_f]$$

will transfer x_0 to x_{t_f} at $t = t_f$.

You can prove the above theorem by simply substituting $u(t)$ into

$$x(t) = e^{At} x(0) + \int_0^{t_f} e^{A(t-\tau)} B u(\tau) \, d\tau$$

Design of State Feedback Controllers

- In the previous slide, we addressed the question of transfer states from one location to another
- Another question can be to simply stabilize the system
- Assume that the system has some +ve evalues $\Rightarrow$ unstable system
- **Solution:** design a controller that stabilizes the system
- A question that pertains to controllability is **to design a state feedback controller** for the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

- A state feedback control problem takes the following form:

  find matrix $K \in \mathbb{R}^{m\times n}$ such that $u(t) = -Kx(t)$ is the control law

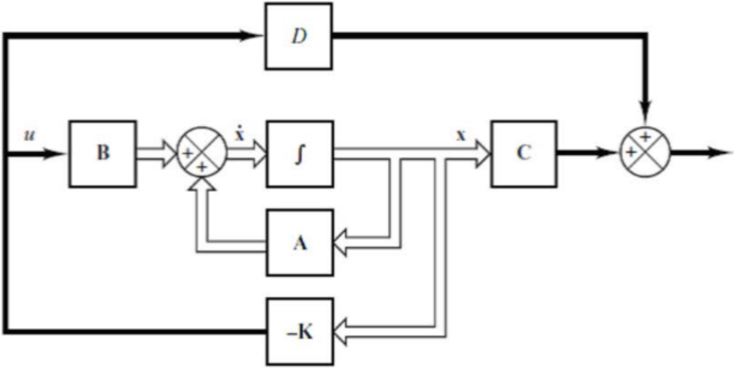
- Hence, system dynamics become

$$\dot{x}(t) = (A - BK)x(t)$$

- **State feedback control objective:**

  find K such that $(A - BK)$ has eigenvalues in the LHP

Schematic for State Feedback Control



Since $u(t) = -Kx(t)$, the updated dynamics of the system become:

$$\dot{x}(t) = Ax(t) + Bu(t) = (A - BK)x(t) = A_{cl}x(t)$$
$$y(t) = Cx(t) + Du(t) = (C - DK)x(t) = C_{cl}x(t)$$

where 'cl' denotes the "closed-loop" system.

Example — Controller Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Is the system stable? What are the eigenvalues?
- **Solution:** unstable, $\texttt{eig}(A) = 4, -2$
- Find linear state-feedback gain K (i.e., $u = -Kx$), such that the poles of the closed-loop controlled system are -3 and -5
- **Solution:** $n = 2$ and $m = 1 \Rightarrow K \in \mathbb{R}^{2\times 1} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$
- What is $A_{cl} = A - BK$?

$$A - BK = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 - k_1 & 3 - k_2 \\ 3 & 1 \end{bmatrix}$$

- Characteristic polynomial: $\lambda^2 + (k_1 - 2)\lambda + (3k_2 - k_1 - 8) = 0$
- What to do next? Say that you want the desired closed loop poles to be at $\lambda_1 = -3$ and $\lambda_2 = -5$
- Then the characteristic polynomial for A_{cl} should be $(\lambda+3)(\lambda+5) = \lambda^2+8\lambda+15 = 0 \quad \equiv \quad \lambda^2+(k_1-2)\lambda+(3k_2-k_1-8) = 0$
- **Solution:** $u(t) = -Kx(t) = -[10 \;\; 11] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -10x_1(t) - 11x_2(t)$

Example 2

- In the previous example, we designed a state feedback controller that stabilizes the initially unstable system
- That system was a CT system, but the analysis remains the same for DT systems (you want the eigenvalues to be: $|\lambda_{cl} < 1|$)
- Example 2—What if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, can we stabilize the system?
- The answer is: NO!
- Why? Because the system is not controllable, as $\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which has rank 1, and A has two unstable evalues (both at $\lambda = 1$), hence I can only stabilize one eigenvalue, but cannot stabilize both
- That means there is no way I can find $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ such that A_{cl} is asymptotically stable with stable evalues

Example 3

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

- Problem: this system is unstable ($\lambda_{1,2,3} = 0$)
- Solution: design a state feedback controller $u(t) = -[k_1 \ k_2 \ k_3]x(t)$ that would shift the eigenvalues of the system to $\lambda_{1,2,3} = -1$ (i.e., stable location)
- First, find A_{cl}:

$$A_{cl} = A - BK = \begin{bmatrix} -k_1 & -k_2 & -k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Second, find the characteristic polynomial of A_{cl} and equate it with the desired location of evalues:

$$\lambda^3 + k_1\lambda^2 + k_2\lambda + k_3 \quad \equiv \quad (\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

- Hence, $K = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$ solves this problem and ensures that all the closed loop system evalues are at -1 (check evalues($A - BK$))

Stabilizability

- Controllability is a very strong property for LTI systems to satisfy
- Imagine a huge system with 1000s of states
- Controllability would mean that all of the 1000s of states can be arbitrarily reached by certain controls
- Most large systems are not controllable—you cannot simply place all the poles in a location of your own
- **Stabilizability**—a key property of dynamical systems—is a relaxation from the often not satisfied controllability condition

Stabilizability Theorem

A system with state-space matrices (A, B) is called stabilizable if there exist a state feedback matrix K such that the closed-loop system $A - BK$ is stable

Stabilizability — 2

Stabilizability Theorem

A system with state-space matrices (A, B) is called stabilizable if there exist a state feedback matrix K such that the closed-loop system $A - BK$ is stable

- Difference between pole placement and stabilizability theorems is that the former assigns any locations for the eigenvalues, whereas stabilizability only guarantees that the closed loop system is stable
- If A is stable $\Rightarrow$ (A, B) is stabilizable
- If (A, B) is controllable $\Rightarrow$ it is stabilizable
- If (A, B) is not controllable, it could still be stabilizable
- Stabilizability means the following:
  - A system has n evalues: k are stable and $n - k$ are unstable
  - **Stabilizability implies that the $n - k$ unstable evalues can be placed in a stable location**
  - What about the other k stable ones? Well, some of them **can** be placed in a different location, but stabilizability does not guarantee that—it only ensures that unstable evalues can be stabilized

Example

- Is the pair $A = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ controllable? Stabilizable?
- The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  which has rank 1. Hence, the system is not controllable.
- Stabilizability: Notice that we have two distinct eigenvalues ($\lambda_{1,2} = -2$ and $\lambda_3 = 0$)
- PBH test tells us $\lambda_3 = 0$ which is the not asymptotically stable evalue fails the rank condition:

$$rank([\lambda_3 I - A \quad , \quad B]) = 2 < 3$$

- Therefore, eigenvalue 0 cannot be placed in a stable location in the LHP
- Hence, the system is not stabilizable

Example

- Consider the following system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\\ \dot{x}_2(t) &= u(t)\end{aligned}$$

- Is the system controllable? Stabilizable?
- Clearly, the system is not controllable—is it stabilizable? Let's see
- There are many ways to do that
- You can check if the unstable eigenvalues satisfy PBH test, or you can find $A - BK$ and see if such gain matrix exists
- So: $A_{cl} = A - BK = \begin{bmatrix} 1 & 0 \\ -k_1 & -k_2 \end{bmatrix} \Rightarrow \text{eig}(A_{cl}) = \lambda_{1,2} = 1, -k2$
- Hence, for any gain matrix K (k_1, k_2), one of the eigenvalues of A will always be equal to 1, which is unstable
- Therefore, the system is not stabilizable and not controllable
- What if $\dot{x}_1(t) = -x_1(t)$? Would the system be controllable? Stabilizable?

Remarks about Stabilizability

Uncontrollable Modes

If λ is an uncontrollable eigenvalue of (A, B), then λ will also be an eigenvalue of $A + BK$ for any gain matrix K.

Stabilizability Theorem (2)

A pair (A, B) is stabilizable if and only if $\text{rank}([\lambda I - A \;\; B]) = n$ for every eigenvalue λ of A with nonnegative real part.

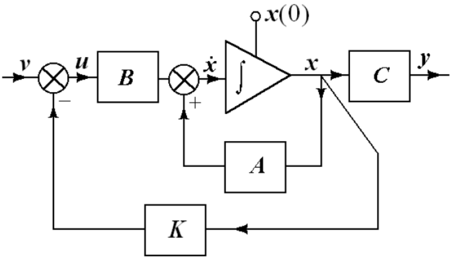
Pole Placement Problem for CT LTI Systems

- Suppose that A has some positive eigenvalues problem
- **Objective:** find a control $u(t) = -Kx(t)$, i.e., find K such that matrix $A - BK$ has only strictly -ve evalues in predefined locations

Pole Placement Theorem

Assuming that the pair (A, B) is controllable ($\mathcal{C}$ is full rank), then there exists a feedback matrix K such that the closed-loop system eigenvalues (evalues of $A - BK$) can be placed in arbitrary locations.

$$u(t) = -Kx(t) + v(t) \Rightarrow \dot{x}(t) = (A - BK)x(t) + v(t), \; v(t) = \text{reference signal}$$



Controllable Canonical Form

- Recall the controllable canonical form for a single input system:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}}_{\boldsymbol{A}\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)}_{\boldsymbol{B}u(t)}$$

- No matter what the values of a_i's are, the above system is ALWAYS controllable
- How can you prove this? Derive the controllability matrix—you'll see that it's always full rank!
- Does that mean all systems are controllable? No it doesn't!
- We can only reach the controllable canonical form if there exists a transformation that would transform a controllable system into the controllable canonical form

CCF — 2

CCF Transformation Theorem

If $\dot{x}(t) = Ax(t) + Bu(t)$ has only one input ($m = 1$) and is controllable, there exists a state-coordinate change, defined as $z(t) = Tx(t)$, such that

$$z(t) = (TAT^{-1})z(t) + TBu(t)$$

is in controllable canonical form.

- Eigenvalues of LTI systems do not change after applying linear transformation
- For systems that are initially uncontrollable, no transformation exists that would put the system in its controllable canonical form
- Transformations for LTI systems preserve properties (stabilizability, controllability)
- Transformations only shape the state-space dynamics in a nice, compact form

Important Notes

- In this module, we studied controllability, stabilizability, pole placement problems, and the design of state feedback controller to stabilize a potentially nonlinear system
- These results give theoretical guarantees to stabilize **linear systems**
- What happens if you apply a state feedback controller on a nonlinear system?
- This might stabilize the nonlinear system, and it might not
- You should try that in your project
- Discussion on that...

Questions And Suggestions?

Thank You!

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