Credit goes to Andres for this almost perfect solutions. Good job!

THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 2

Andres Rainiero Hernandez Coronado
September 6, 2017

The objective of this homework is to test your understanding of the content of Module 2. Due date of the homework is: Thursday, September 7th, 2017 @ 23:59pm. You have to upload either a **clear** scanned version of your solutions on Blackboard or a typed PDF via LaTeX. For your convenience, I am also attaching the LaTeX source files.

---

1. Go through the following links:
   (a) State space on MATLAB:
   http://www.mathworks.com/videos/state-space-models-part-1-creation-and-analysis-100815.html
   (b) Simulating state space using ODE solver: https://www.mathworks.com/matlabcentral/answers/146782-solve-state-space-equation-by-ode45. Go through the first response only.

2. In this link http://academic.csuohio.edu/richter_h/courses/mce371/mce371_5.pdf, you'll find a quick introduction to state space and its implementation on MATLAB, similar to the one above.
   (a) Go through Pages 3–14 of this PDF presentation. Make sure that you understand the details involved.
   (b) You are now given the following dynamical system (identical to the one given in the PDF):

$$2y^{(4)}(t) + 0.9y^{(3)}(t) + 45.1\ddot{y}(t) + 10\dot{y}(t) + 250y(t) = 250u(t),$$

where $y(t)$ and $u(t)$ are the output and input to the system. Derive **two different** state space representations, i.e., **obtain two sets** of state-space matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ for this fourth order ODE.
By using the State Space Representation equation multiple sets of A, B, C and D matrices can be obtained:
To easily obtain two State Space representations the previous equation will be re-arranged as follows:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} = \frac{0s^4 + 0s^3 + 0s^2 + 0s + 125}{s^4 + 0.45s^3 + 22.55s^2 + 5s + 125}$$

$$\frac{Y(s)}{U(s)} = b_0 + \frac{(b_1 - a_1 b_0)s^3 + (b_2 - a_2 b_0)s^2 + (b_3 - a_3 b_0)s + (b_4 - a_4 b_0)}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

$$\frac{Y(s)}{U(s)} = 0 + \frac{0s^3 + 0s^2 + 0s + 125}{s^4 + 0.45s^3 + 22.55s^2 + 5s + 125}$$

By using the next equation, both Controllable and Observable canonical forms of the State Space representation can be obtained:

$$Y(s) = b_0 U(s) + \hat{Y}(s)$$

$$\hat{Y}(s) = \frac{0s^3 + 0s^2 + 0s + 125}{s^4 + 0.45s^3 + 22.55s^2 + 5s + 125} U(s)$$

$$\frac{\hat{Y}(s)}{0s^3 + 0s^2 + 0s + 125} = \frac{U(s)}{s^4 + 0.45s^3 + 22.55s^2 + 5s + 125} = Q(s)$$

The following assumptions will be made for both state space representations:

$$\begin{matrix} x_1 = Q(s) \\ x_2 = sQ(s) \\ x_3 = s^2Q(s) \\ x_4 = s^3Q(s) \end{matrix} \quad \Rightarrow \quad \begin{matrix} \dot{x}_1 = sQ(s) \\ \dot{x}_2 = s^2Q(s) \\ \dot{x}_3 = s^3Q(s) \\ \dot{x}_4 = s^4Q(s) \end{matrix}$$

For the Controllable canonical form the next operations are performed:

$$U(s) = s^4Q(s) + 0.45s^3Q(s) + 22.55s^2Q(s) + 5sQ(s) + 125Q(s)$$
$$s^4Q(s) = -0.45s^3Q(s) - 22.55s^2Q(s) - 5sQ(s) - 125Q(s) + U(s)$$

$$\hat{Y}(s) = 125Q(s)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -125 & -5 & -22.55 & -0.45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 125 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

For the Observable canonical form the matrix A is equal to the transpose of matrix A from the Controllable form, vector B is made horizontal to become vector C, vector C is made vertical to become vector B, D remains the same:

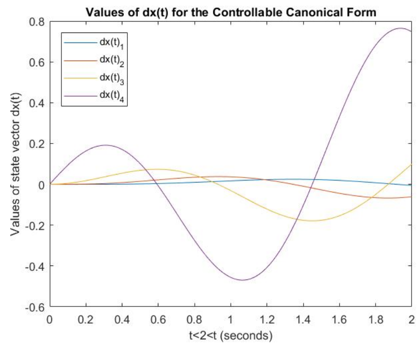
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -125 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -22.55 \\ 0 & 0 & 1 & -0.45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 125 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

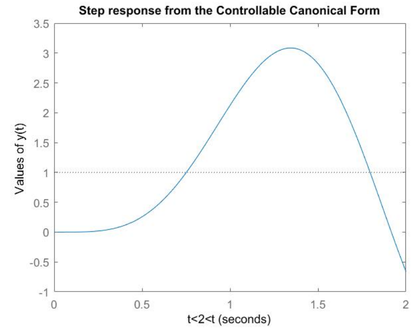
$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

(c) For each set of the derived matrices, and given what you learned about the ODE solvers on MATLAB for state-space systems, simulate the dynamics of this system assuming that the input $u(t)$ is a unit step function ($u(t) = 1$). Consider that the time horizon is equal to 2 seconds. You'll have to plot the states of the system with respect to time, as well as the output $y(t)$. You can assume zero initial conditions.

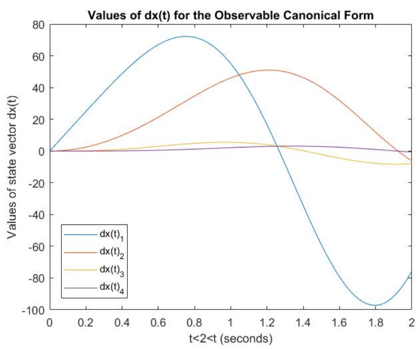
To obtain the corresponding values of the states of the system it is necessary to create a function that defines the values of the vector $\dot{x}(t)$, this can be done by arranging the A and B matrices previously obtained, where the value of $u(t)$ will be equal to 1 to represent a step response. With this, then it is possible to use ode45 to plot the values of the $\dot{x}(t)$ vector through time.
To obtain the value of $y(t)$ through time the step function will be used, the step function uses 1 for the input value as a default.

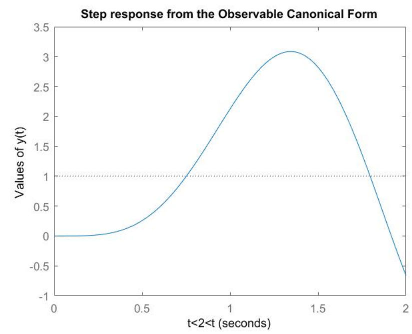
For the Controllable Canonical form set of matrices the next figures are obtained:





For the Observable Canonical form set of matrices the next figures are obtained:





(d) Is there a difference between the output and the states for the two state-space representations? Why/Why Not? Explain your answer.

The output of the system is exactly the same between the two sets because the solution for the system will be the same regardless of the matrices A, B, C and D. The graphs of the both representations change a lot, as the name suggest, the Observable Canonical form yields a high range of values, while the Controllable Canonical form yields values with really low amplitude.

(e) Find the transfer function associated to the two distinct state space representations that you derived in 2-(b). Is the transfer function unique? Why/Why Not?

By using the command ss2tf it is possible to easily find the transfer function for both systems. In both scenarios the transfer function is obtained similar to the one previously derived:

$$\frac{Y(s)}{U(s)} = \frac{125}{s^4 + 0.45s^3 + 22.55s^2 + 5s + 125}$$

As mentioned before, the transfer function is the same because the two different sets of matrices A, B, C and D represent the exact same system, thus the transfer does not change.

3. Assume that two systems, N_1 and N_2, are cascaded in series. System N_1 is defined by the derivative operator (i.e., the output to N_1 is the derivative of its input), and system N_2 is defined by a function $\gamma(t)$ (i.e., the output of N_2 is the input of N_2 multiplied by $\gamma(t)$). Is the overall, cascaded system linear? Nonlinear? Time-varying? Time-invariant? Prove it.

   The overall system is linear because:

$$y(t) = N2(N1(u(t))) = \gamma(t)\frac{d}{dt}u(t)$$

$$y_1(t) = \gamma(t)\frac{d}{dt}u_1(t)$$
$$y_2(t) = \gamma(t)\frac{d}{dt}u_2(t)$$
$$y(t) = \gamma(t)\frac{d}{dt}(\alpha_1 u_1(t) + \alpha_2 u_2(t))$$
$$\gamma(t)\frac{d}{dt}(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \alpha_1\gamma(t)\frac{d}{dt}u_1(t) + \alpha_2\gamma(t)\frac{d}{dt}u_2(t)$$

   On the other hand, the system is time varying because a shifted input produces a different output of that when the output is shifted:

$$\gamma(t)\frac{d}{dt}(u(t-T)) \neq \gamma(t-T)\frac{d}{dt}(u(t-T))$$

4. What happens when two systems, out of which one is LTI while the other is LTV, are connected together in series? Would the cascaded overall system still remain linear?

   Yes, since both sub-systems are Linear, the Commutative property states that the overall system will remain linear, it can be demonstrated by the following:

   Supose N_1 is the LTI system and N_2 is the LTV system:

$$x(t) = N_1(u(t))$$
$$y(t) = N_2(x(t))$$

   The following equations are true since both systems are linear:

$$x(t) = N_1(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \alpha_1 N_1(u_1(t)) + \alpha_2 N_1(u_2(t))$$
$$y(t) = N_2(\alpha_1 x_1(t) + \alpha_2 x_2(t)) = \alpha_1 N_2(x_1(t)) + \alpha_2 N_2(x_2(t))$$

   For x:

$$x_1(t) = N_1(u_1(t))$$
$$x_2(t) = N_1(u_2(t))$$
$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

   For y:

$$y_1(t) = N_2(N_1(u_1(t)))$$
$$y_2(t) = N_2(N_1(u_2(t)))$$
$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

   When analyzing the linearity of the overall system the next equations are obtained and the linearity equation is satisfied:

$$y(t) = N_2(N_1(\alpha_1 u_1(t) + \alpha_2 u_2(t)))$$
$$y(t) = N_2(\alpha_1 N_1(u_1(t)) + \alpha_2 N_1(u_2(t)))$$
$$y(t) = \alpha_1 N_2(N_1(u_1(t))) + \alpha_2 N_2(N_1(u_2(t))) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Now suppose that the order of the cascading is reversed, does that change the overall system output given the same input? Yes/No answers do not suffice. You have to prove your result.

The overall output will remain the same if the time conditions are equal for both systems because the overall system will be LTV. Supose that:

$$N_1(u(t)) = \beta(t)u(t)$$
$$N_2(u(t)) = \gamma(-t)u(t)$$

Where $\gamma(-t)$ represents an LTV. For the both combinations of the system (y_1 and y_2) the next equations are obtained:

$$y_1 = \gamma(-t)\beta(t)u(t)$$
$$y_2 = \beta(t)\gamma(-t)u(t)$$

This means that y_1 and y_2 are equal due to the Commutative property.

5. Assume that a system N is linear, with an output defined as $y(t) = N(u(t))$. Prove that if the input to the system is zero for all $t \geq 0$, then the output must be also 0 for all $t \geq 0$.

   The additivity property cane be used to demonstrate that, since the system is linear, the next equation is true:

$$N(\alpha_1 u(t) + \alpha_2 u(t) + \cdots + \alpha_n u_n(t)) = \alpha_1 N(u_1(t)) + \alpha_2 N(u_2(t)) + \cdots + \alpha_n N(u_n(t))$$

   The problem states that $y(t) = N(u(t)) = 0$, then it is possible to substitute the values of $N(u_{1\ldots n}(t))$ from the previous equation to obtain:

$$N(\alpha_1 u(t) + \alpha_2 u(t) + \cdots + \alpha_n u_n(t)) = \alpha_1(0) + \alpha_2(0) + \cdots + \alpha_n(0)$$
$$N(\alpha_1 u(t) + \alpha_2 u(t) + \cdots + \alpha_n u_n(t)) = 0$$

6. A Trump-Obama dynamical system that exists nowhere follows these two differential equations:

$$\ddot{T}(t) + \alpha_1(t)\dot{T}(t) - \alpha_2(t)\dot{C}(t) = \alpha_3(t)u(t) \quad (1)$$
$$\dot{C}(t) = \alpha_4(t)u(t) - C(t) - \alpha_5(t)T(t), \quad (2)$$

   where $T(t)$ and $C(t)$ are the two mental **states** of Jalyooka Trump and Palyooka Obama, $u(t)$ is the control input, and $\alpha_i(t)$ functions are all time-varying functions. Derive the state-space representation of this surely non-existent dynamical system. You should be able to obtain an equation similar to this:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}(t)\mathbf{x}(t) + \boldsymbol{B}(t)u(t),$$

   where $\mathbf{x}(t)$ is the state-vector of the system (minimum of size 3) and $\boldsymbol{A}, \boldsymbol{B}$ are state-space matrices that you should derive (in terms of $\alpha(t)$ functions).

   Hint: let $x_1(t) = T(t)$ an $x_3(t) = C(t)$.

   The vector $x(t)$ will have the form:

$$x_1(t) = T(t)$$
$$x_2(t) = \dot{T}(t)$$
$$x_3(t) = C(t)$$

Thus, it can be concluded that the vector $\dot{x}(t)$ has the following form:

$$\dot{x}_1(t) = \dot{T}(t)$$

$$\dot{x}_2(t) = \ddot{T}(t)$$

$$\dot{x}_3(t) = \dot{C}(t)$$

Since equation 1 can't be represented because it includes a $\dot{C}$ expression, it can substituted with the equivalence stated in equation 2, obtaining:

$$\ddot{T}(t) + \alpha_1(t)\dot{T}(t) - \alpha_2(t) * (\alpha_4(t)u(t) - C(t) - \alpha_5(t)T(t)) = \alpha_3(t)u(t)$$

By doing some re-arrangement the following equation is obtained:

$$\ddot{T}(t) = -\alpha_2(t)C(t) - \alpha_1(t)\dot{T}(t) - \alpha_2(t)\alpha_5(t)T(t) + (\alpha_2(t)\alpha_4(t) + \alpha_3(t)u(t)$$

And it can be arranged as an State-Space equation as follows:

$$\begin{bmatrix} \dot{T}(t) \\ \ddot{T}(t) \\ \dot{C}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha_2(t)\alpha_5(t) & -\alpha_1(t) & -\alpha_2(t) \\ -\alpha_5(t) & 0 & -1 \end{bmatrix} \begin{bmatrix} T(t) \\ \dot{T}(t) \\ C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2(t)\alpha_4(t) + \alpha_3(t) \\ \alpha_4(t) \end{bmatrix} u(t)$$

7. The simplified dynamics of the vertical ascent of a Space X rocket can be modeled as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g\left(\dfrac{D}{x_1(t)+D}\right)^2 + \dfrac{\ln(u)}{m} \end{bmatrix},$$

where D is the distance from earth to the surface of the rocket (assumed to be constant), m is the actual mass of the rocket, g is the gravity constant, and u is the thrust that is assumed to be constant.

Find the equilibrium states (x_1^*, x_2^*) of the above dynamic system.

It is necessary to identify that the equilibrium of the system is achieved when $\dot{x}_1(t) = 0$ and $\dot{x}_2(t) = 0$. Given that $\dot{x}_1(t) = x_2(t)$ then $x_2(t) = 0$.

From $\dot{x}_2(t) = 0$ it is necessary to determine the value of $x_1(t)$:

$$0 = -g\left(\frac{D}{x_1(t)+D}\right)^2 + \frac{\ln(u)}{m}$$

$$g\left(\frac{D}{x_1(t)+D}\right)^2 = \frac{\ln(u)}{m}$$

$$\frac{D}{x_1(t)+D} = \sqrt{\frac{\ln(u)}{gm}}$$

$$\frac{D}{\sqrt{\dfrac{\ln(u)}{gm}}} - D = x_1(t)$$

Thus, the equilibrium states are:

$$x_1(t) = \frac{D}{\sqrt{\dfrac{\ln(u)}{gm}}} - D$$

$$x_2(t) = 0$$

8. A transfer function of a linear system is given by:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2s^3 + 4s + 0.5}{0.5s^3 + 8s^2 + 16s + 22}.$$

Derive the state-space controllable canonical form for this system. You should derive the canonical form and state space matrices, rather than listing them.

It is necessary to re-arrange the given equation:

$$\frac{Y(s)}{U(s)} = b_0 + \frac{(b_1 - a_1 b_0)s^2 + (b_2 - a_2 b_0)s + (b_3 - a_3 b0)}{s^3 + a_1 s^2 + a_2 s + a_3} = 2 + \frac{-32s^2 + -60s - 87.5}{s^3 + 16s^2 + 32s + 44}$$

The process to obtain the State Space equations begins with the next equation:

$$Y(s) = b_0 U(s) + \hat{Y}(s)$$

$$\hat{Y}(s) = \frac{-32s^2 + -60s - 87.5}{s^3 + 16s^2 + 32s + 44} U(s)$$

$$\frac{\hat{Y}(s)}{-32s^2 + -60s - 87.5} = \frac{U(s)}{s^3 + 16s^2 + 32s + 44} = Q(s)$$

$$U(s) = s^3 Q(s) + 16s^2 Q(s) + 32sQ(s) + 44Q(s) \Rightarrow s^3 Q(s) = -16s^2 Q(s) - 32sQ(s) - 44Q(s) + U(s)$$

$$\hat{Y}(s) = -32s^2 Q(s) - 60Q(s) - 87.5Q(s) \Rightarrow Y(s) = 2U(s) + (-32s^2 Q(s) - 60Q(s) - 87.5Q(s))$$

By doing the following maping the last set of equations is obtained:

$$\begin{array}{rcl} X_1(s) = Q(s) & & sX_1(s) = X_2(s) \\ X_2(s) = sQ(s) & \Rightarrow & sX_2(s) = X_3(s) \\ X_3(s) = s^2 Q(s) & & sX_3(s) = s^3 Q(s) \end{array}$$

After replacing the $Q(s)$ terms with $X(s)$ and obtaining the Inverse Laplace transform of the equations, the next set of equations is obtained:

$$\dot{x}_3 = -16x_3 - 32x_2 - 44x_1 + u$$

$$y = -32x_3 - 60x_2 - 87.5x_1 + 2u$$

Then, the Controllable Canonical form state space representation is obtained:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -44 & -32 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -87.5 & -60 & -32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix} u$$

9. Is the system defined by

$$y(t) = N(u(t)) = |u(t)| + \alpha(t)u(t)$$

linear or nonlinear? Time varying or time-invariant? Prove your answers.

Assuming that $Ns(u(t)) = |u(t)|$ is a sub-system of the overall system, then the overall system is non-linear because N_1 is non-linear:

$$Ns_1(u(t)) = |u_1(t)|$$
$$Ns_2(u(t)) = |u_2(t)|$$
$$Ns(u(t)) = |\beta_1 u_1(t) + \beta_2 u_2(t)|$$

Then:

$$|\beta_1 u_1(t) + \beta_2 u_2(t)| \neq \beta_1 |u_1(t)| + \beta_2 |u_2(t)|$$

On the other hand, the system is time-varying because there is an internal parameter represented by $\gamma(t)$, a shifted output of the overall system is not equal to the output of the system given shifted inputs:

$$|u(t-T)| + \alpha(t-T)u(t-T) \neq |u(t-T)| + \alpha(t)u(t-T)$$

10. In this problem, we will study the equilibrium of Susceptible-Infectious-Susceptible (SIS) in epidemics—similar to what we discussed in class. The SIS model is appropraite to model viral diseases such as influenza, since recovered people do not grant permanent immunity from being infected again. In other words, you are always susceptible to getting a cold—sad truth, but that's a first world problem.

Anyway, the dynamics of a simplified SIS model can be written as

$$\frac{dS}{dt} = -\frac{\beta S I}{N} + \gamma I \quad (3)$$

$$\frac{dI}{dt} = \frac{\beta S I}{N} - \gamma I \quad (4)$$

where $S(t)$ is the number of people that are susceptible at time t and $I(t)$ is the number of infected people at time t, where N is the total number of people.

Assume that the number of people is fixed, that is $S(t) + I(t) = N$.

(a) Given the above assumption, reduce the above dynamical system from 2 states $(S(t), I(t))$ to a dynamic system with only one state $I(t)$. You should obtain something like

$$I(t) = f(\beta, N, \gamma, I).$$

The two metioned states are redundant, $\frac{dS}{dt} = -\frac{dI}{dt}$, it is possible to know the rate of change of Susceptible people by analyzing the rate of change of Infected people. With the mentioned assumption, it can be concluded that:

$$\frac{dI}{dt} = \frac{\beta(N-I)I}{N} - \gamma I$$

This is partially correct, but mathematically incomplete. Check the next page.

(b) What is the equilibrium of the system? Analyze the stability of the solution of the first oder ODE of $I(t)$. In other words, explain what happens as $t \to$ as any of these parameters β, N, γ change. Be clear and concise.

The equilibrium of the system is represented when th rates of change of either Infected or susceptible people are equal to 0.

If β increases, the amount of susceptible people that interact with infected people increases, meaning that the rate of change increases.

If N changes, it means that the overall amount of people changes but the proportion of susceptible and infected people remains the, represented by $N - I$ on the numerator and N on the denominator.

If γ changes, it means that more people are infected, i.e. less people are susceptible to get infected. If γ increases it means that a lot of people are infected and the rate of change of infected people will slow down, on the other hand, if γ decreases, it means that less people are infected and more people are susceptible.

This is a complete solution for this problem. Credit goes to Sebastian for this analysis.

10\. a. Substituting $S(t) = N - I(t)$ to equation $\frac{dI}{dt} = \frac{\beta S I}{N} - \gamma I$ yields

$$\frac{dI}{dt} = \frac{\beta(N-I)I}{N} - \gamma I = \frac{\beta N}{N} I - \frac{\beta I^2}{N} - \gamma I = (\beta-\gamma)I - \frac{\beta}{N} I^2 = \dot{I}(t)$$

b\. Setting $\dot{I}(t) = 0$, we get

$$0 = (\beta-\gamma)I - \frac{\beta}{N} I^2 \iff 0 = I\left((\beta-\gamma) - \frac{\beta}{N} I\right) \iff I = 0 \text{ and } I = \frac{(\beta-\gamma)N}{\beta}$$

the equilibrium points are (0) and $((\beta-\gamma)N/\beta)$.

The jacobian of $\dot{I}(t)$ is $\frac{\partial \dot{I}(t)}{\partial I(t)} = \frac{\partial f(\beta, N, \gamma, I)}{\partial I} = (\beta-\gamma) - 2\frac{\beta}{N} I = J(I(t))$

at $I_{e_1} = 0$, we have

$$J(I_{e_1}) = \beta - \gamma$$

the linearized dynamics is

$$\dot{I}(t) = (\beta-\gamma)I(t)$$

the solution is

$$\frac{dI(t)}{dt} = (\beta-\gamma)I(t) \iff \frac{1}{I(t)} dI(t) = (\beta-\gamma)\, dt$$

$$\int_{t_0}^{t} \frac{1}{I(t)} dI(t) = \int_{t_0}^{t} (\beta-\gamma)\, dt$$

$$\ln\left(\frac{I(t)}{I(t_0)}\right) = (\beta-\gamma)(t-t_0)$$

$$I(t) = e^{(\beta-\gamma)(t-t_0)} I(t_0)$$

which is stable as $t \to \infty$ if $\beta - \gamma < 0$ or $\beta < \gamma$.

at $I_{e_2} = \frac{(\beta-\gamma)N}{\beta}$, we have

$$J(I_{e_2}) = (\beta-\gamma) - 2\frac{\beta}{N}\left(\frac{(\beta-\gamma)N}{\beta}\right)$$

$$= (\beta-\gamma) - 2(\beta-\gamma)$$

$$= -(\beta-\gamma)$$

$$= \gamma - \beta$$

the linearized dynamics is

$$\dot{I}(t) = (\gamma-\beta)I(t)$$

by using the previous result, we get the solution

$I(t) = e^{(\gamma-\beta)(t-t_0)} I(t_0)$, which is stable as $t \to \infty$ if $\gamma - \beta < 0$ or $\gamma < \beta$