HOMEWORK # 5 SOLUTION
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1. Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A^k = \begin{bmatrix} ka^{k-1} & 1 \\ 0 & a^k \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a \neq 0, a \neq 1.$$

(a) Given that $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the control is equal to zero for all k, determine $x(0)$.

(b) Find a general expression for $x(n)$ if the control is given by $u(k) = a^{-k}1^+(k)$ and $x(0) = 0$.

Solutions:

(a) Since $u(k) = 0$, then:

$$x(k+1) = Ax(k) \Rightarrow x(2) = A^2x(0) \Rightarrow x(2) = \begin{bmatrix} 2a^{2-1} & 1 \\ 0 & a^2 \end{bmatrix} x(0)$$

$$\Rightarrow x(0) = \begin{bmatrix} 2a & 1 \\ 0 & a^2 \end{bmatrix}^{-1} x(2) = \frac{1}{2a^3} \begin{bmatrix} a^2 - 1 \\ 2a \end{bmatrix} = \begin{bmatrix} \frac{1}{2a} - \frac{1}{2a^3} \\ \frac{1}{a^2} \end{bmatrix}$$

(b) From the module notes,

$$x(n) = \sum_{k=0}^{n-1} A^{n-1-k}Bu(k) = \sum_{k=0}^{n-1} A^k Bu(n-1-k) = \sum_{k=0}^{n-1} A^k B a^{k-n+1} = \sum_{k=0}^{n-1} \begin{bmatrix} ka^{k-1}a^{k-n+1} \\ 0 \end{bmatrix}.$$

Hence,

$$x(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \begin{bmatrix} a^{-n+2} \sum_{k=0}^{n-1} k(a^2)^k \\ 0 \end{bmatrix} = \begin{bmatrix} a^{-n+2} \frac{d}{da} \left(\frac{1-(a^2)^n}{1-a^2} \right) \\ 0 \end{bmatrix}.$$

2. Consider the discrete-time LTI dynamical system model

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}}_{D} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

(a) Find a general expression for D^k.

(b) Find A^k.

(c) Compute $x(k)$ if the control input is null.

(d) Computer $x(k)$ if the initial conditions are null and the control input is $u(k) = 2^k1^+(k)$ and $\lambda_1 = 4$.

Solutions:

(a) $D^k = \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix}$

(b) $A^k = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$

(c) $x_{\text{zisr}}(k) = A^k x(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} k\lambda_1^{k-1} \\ \lambda_1^k \end{bmatrix}$

(d) The zero-state state response can be written as:

$$x_{\text{zssr}}(n) = \sum_{k=0}^{n-1} A^{n-1-k} B u(k) = \sum_{k=0}^{n-1} A^k B u(n-1-k)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sum_{k=0}^{n-1} \begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} u(n-1-k)$$

$$= 2^n \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sum_{k=0}^{n-1} \begin{bmatrix} 2^k \\ 0 \end{bmatrix} = (2^{2n} - 2^n) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. Consider the following system with two inputs $\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} = u(k)$ and the following dynamics:

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u(k), x(0) = 0.$$

(a) By setting $u_2(k) = 0 \forall k$, and using $u_1(k)$ alone, can the state be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$? If so, find the control $u_1(k)$ that would achieve that for $k = 0, 1, 2$.

(b) By setting $u_1(k) = 0 \forall k$, and using $u_2(k)$ alone, can the state be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$? If so, find the control $u_2(k)$ that would achieve that for $k = 0, 1, 2$.

(c) Assume at $k = 0, 1$, only u_1 can be used and at $k = 2$, only u_2 can be used. Find the input $u(k) \forall k$ such that the state can be steered from $x_0 = 0$ to $x(3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution:

(a) First, note that the system with state space matrices $A, B(:, 1)$ (i.e., the system formed by A and the first control $u_1(k)$ via the first column of matrix B) is full controllable as the rank of controllability matrix is 2. Hence, there should be control actions that steer the system from x_0 to x_f. Let $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We can compute this control via the derivation we discussed in class. It's easy to see that:

$$\begin{bmatrix} A^2 b_1 & A b_1 & b_1 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} = x(3) - x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence, we can use the right inverse (see solutions of problem 5 for the right inverse derivation) and obtain $u(k)$. Note that $\begin{bmatrix} A^2 b_1 & A b_1 & b_1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Then:

$$\begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{\dagger} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{\top} \left(\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{\top} \right)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7/6 \\ -1/3 \\ -11/6 \end{bmatrix}.$$

(b) You cannot obtain solutions to this problem, since the pair A, b_2 (b_2 is the second column of B) yields an inconsistent system of equations that cannot be solved for a control input. You won't be able to obtain a valid pseudo inverse for the rectangular matrix as it does not exist.

(c) Notice that this problem is very similar to the problem in part (a). The only difference is that initially, the system starts from a different B matrix. It is easy to see that:

$$\begin{bmatrix} A^2 b_1 & Ab_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(1) \\ \mathbf{u}_2(2) \end{bmatrix} = x(3) - x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Applying the right inverse (as in (a) above), the solution to the optimal control is:

$$\begin{bmatrix} u_1(0) \\ u_1(1) \\ \mathbf{u}_2(2) \end{bmatrix} = \begin{bmatrix} 2/3 \\ -5/3 \\ 7/3 \end{bmatrix}.$$

4. You are given this system:

$$x(k+1) = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), a \neq 0, b \neq 0.$$

(a) Prove that $A^k = \begin{bmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{bmatrix}$.

(b) If $x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u(k) = 0$, find $x(0)$.

(c) Find $x(k)$ if $u(k) = a^k$ and $x(0) = 0$.

Solutions:

(a) Prove by induction (assume true for k, and prove the result for A^{k+1} by evaluating $A^{k+1} = A^k A$.)

(b) Similar to problem 1, $x(0) = \begin{bmatrix} \frac{1}{a^2} - \frac{2}{a^3} \\ \frac{1}{a^2} \end{bmatrix}$.

(c)

$$x(k) = ka^{k-1} \begin{bmatrix} \frac{k-1}{2a} \\ 1 \end{bmatrix}$$

5. You're given the following DT LTV system:

$$x(k+1) = A(k)x(k) + B(k)u(k).$$

(a) Derive a system of equations whose solution gives the two inputs $u(0), u(1)$ that would drive the system from state $x(0)$ to $x(2)$.

(b) Now assume that

$$A(k) = \begin{bmatrix} 0 & 2-k \\ 0 & 0 \end{bmatrix}, B(k) = \begin{bmatrix} 2-k & 0 \\ 0 & 2-k \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the input sequence $u(0), u(1)$ that would steer the system from $x(0)$ to $x(2)$.

Solutions:

(a) The set of equations can be written as:

$$x(2) - A(1)A(0)x(0) = \begin{bmatrix} B(1) & A(1)B(0) \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

(b) Given the SS matrices, the above equation can be written as:

$$x(2) - A(1)A(0)x(0) = \begin{bmatrix} B(1) & A(1)B(0) \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&2\\0&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1&0&0&2\\0&1&0&0\end{bmatrix}\begin{bmatrix}u(1)\\u(0)\end{bmatrix}$$

Hence,

$$\boxed{\begin{bmatrix}1&0&0&2\\0&1&0&0\end{bmatrix}\begin{bmatrix}u(1)\\u(0)\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}}$$

In the class, we discussed the left pseudo-inverse for matrices that are rectangular, where we have more rows than columns (i.e., tall, *skinny* matrices). In this example, we have a rectangular matrix which is short and *fat* (i.e., more columns than rows). Note the following: If the matrix A has dimensions $n \times m$ and is full rank then use the left inverse if $n > m$ and the right inverse if $n < m$.

- Left inverse is given by

$$A^{\dagger}_{\text{left}} = \left(A^{\mathrm{T}}A\right)^{-1}A^{\mathrm{T}}$$

  where I_m is the $m \times m$ identity matrix
- Right inverse is given by

$$A^{\dagger}_{\text{right}} = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$$

  where I_n is the $n \times n$ identity matrix.

Hence, in this problem, to find the control inputs, we need to use the right inverse, as follows:

$$\begin{bmatrix}u(1)\\u(0)\end{bmatrix} = \begin{bmatrix}u_1(1)\\u_2(1)\\u_1(0)\\u_2(0)\end{bmatrix} = A^{\dagger}_{\text{right}}\begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}u(1)\\u(0)\end{bmatrix} = A^{\dagger}_{\text{right}}\begin{bmatrix}1\\2\end{bmatrix} = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&2\\0&1&0&0\end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix}1&0&0&2\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&2\\0&1&0&0\end{bmatrix}^{\mathrm{T}}\right)^{-1}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0.2\\2\\0\\0.4\end{bmatrix}$$

6. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)(x_1^2(t)-1)\\ \dot{x}_2(t) &= x_2^2(t)+x_1(t)-3\end{aligned}$$

   (a) Find all the equilibrium points of the nonlinear system.
   (b) Determine the stability of the system around each equilibrium point, if possible.

Solutions:

   (a) Setting the state-dynamics to zero, we can find the equilibrium points. There are 5 equilibrium points for the given system, listed as follows:

$$x_e = \begin{bmatrix}x_{e1}\\x_{e2}\end{bmatrix} = \begin{bmatrix}1&1&-1&-1&3\\\sqrt{2}&-\sqrt{2}&2&-2&0\end{bmatrix}.$$

   (b) The stability of the system around an equilibrium point is determined by evaluating the Jacobian matrix $Df(x)$ around each equilibrium point and finding its eigenvalues:

$$Df(x) = \begin{bmatrix}2x_1x_2 & x_1^2-1\\ 1 & 2x_2\end{bmatrix}.$$

   The only equilibrium point that yields a stable $Df(x_e)$ matrix is $x_e^{(2)} = \begin{bmatrix}1\\-\sqrt{2}\end{bmatrix}$, giving $\lambda_1 = \lambda_2 = -2\sqrt{2}$ as the two stable eigenvalues.

Thanks Andres for the solutions for the last problem.

(c) Solve the same problem if the system is in discrete time:

$$\begin{aligned} x_1(k+1) &= x_2(k)(x_1^2(k)-1) \\ x_2(k+1) &= x_2^2(k)+x_1(k)-3. \end{aligned}$$

To obtain the equilibrium points of this system, Matlab's fsolve function will be used:

$$0 = x_2(k)x_1^2(k) - x_2(k) - x_1(k)$$

$$0 = x_2^2(k) + x_1(k) - 3 - x_2(k)$$

A script named "root2d.m" will be created, and the previously mentioned functions will be represented there:

```
function F = root2d(x)
F(1) = x(2)*(x(1))^2-x(2)-x(1);
F(2) = (x(2))^2+x(1)-3-x(2);
```

From here, the following commands will be executed:

```
>> fun = @root2d;
>> x0 = [1, 1];
>> x = fsolve(fun,x0)
```

For initialization $x_1 = 1, x_2 = 1$ the points of equilibrium are:

```
x =

1.2977    1.8973
```

For initialization $x_1 = 100, x_2 = 100$ the points of equilibrium are:

```
x =

0.6471   -1.1133
```

For initialization $x_1 = 1000, x_2 = 1000$ the points of equilibrium are:

```
x =

3.2254    0.3430
```

For initialization $x_1 = -1, x_2 = -1$ the points of equilibrium are:

```
x =

-1.3492   -1.6446
```

To confirm that these results are valid, the following operations will be made to use Matlab's function root:

$$x_2(x_1^2 - 1) = x_1$$

$$x_2 = \frac{x_1}{(x_1^2 - 1)}$$

$$\left(\frac{x_1}{(x_1^2-1)}\right)^2 + x_1 - 3 + \frac{x_1}{(x_1^2-1)} = 0$$

$$x_1^2 + x_1(x_1^2-1) - 3(x_1^2-1)^2 - x_1(x_1^2-1) = 0$$

$$x_1^5 - 3x_1^4 - 3x_1^3 + 7x_1^2 + 2x_1 - 3 = 0$$

```
>> x1 = roots([1 -3 -3 7 2 -3])

x1 =

3.2254
-1.3492
-0.8209
1.2977
0.6471

>> func = @(x1) (x1)/((x1^2)-1);
>> x2 = arrayfun(func,x1)

x2 =

0.3430
-1.6446
2.5177
1.8973
-1.1133
```

To determine the stability of the system, A is obtained as follows:

$$\frac{\partial}{\partial x}x(k+1) = \begin{bmatrix} 2x_2(k)x_1(k) & x_1^2(k)-1 \\ 1 & 2x_2(k) \end{bmatrix} = A$$

For $x(k) = \begin{bmatrix} 1.2977 & 1.8973 \end{bmatrix}^T$, the eigenvalues are 5.3609 and 3.3579, thus the system is unstable:

$$\begin{bmatrix} 2(1.8973)(1.2977) & (1.2977)^2-1 \\ 1 & 2(1.8973) \end{bmatrix} = \begin{bmatrix} 4.9242 & 0.6840 \\ 1 & 3.7946 \end{bmatrix}$$

For $x(k) = \begin{bmatrix} 0.6471 & -1.1133 \end{bmatrix}^T$, the eigenvalues are -1.0767 and -2.5907, for that, the system is unstable:

$$\begin{bmatrix} 2(-1.1133)(0.6471) & (0.6471)^2-1 \\ 1 & 2(-1.1133) \end{bmatrix} = \begin{bmatrix} -1.4408 & 0.4187 \\ 1 & -2.2266 \end{bmatrix}$$

For $x(k) = \begin{bmatrix} 3.2254 & 0.3430 \end{bmatrix}^T$, the eigenvalues are 4.7638 and -1.8652, which means that the system is unstable:

$$\begin{bmatrix} 2(0.3430)(3.2254) & (3.2254)^2-1 \\ 1 & 2(0.3430) \end{bmatrix} = \begin{bmatrix} 2.2126 & 10.4032 \\ 1 & 0.6860 \end{bmatrix}$$

For $x(k) = \begin{bmatrix} -1.3492 & -1.6446 \end{bmatrix}^T$, the eigenvalues are 4.0536 and -2.9051, which means that the system is unstable:

$$\begin{bmatrix} 2(-1.6446)(-1.3492) & (-1.3492)^2-1 \\ 1 & 2(-1.6446) \end{bmatrix} = \begin{bmatrix} 4.4377 & -2.8203 \\ 1 & -3.2892 \end{bmatrix}$$

For the last pair, obtained by using roots $x(k) = \begin{bmatrix} -0.8209 & 2.5177 \end{bmatrix}^T$, the eigenvalues are -3.9472 and 4.8491, meaning that the system is unstable:

$$\begin{bmatrix} 2(2.5177)(-0.8209) & (-0.8209)^2-1 \\ 1 & 2(2.5177) \end{bmatrix} = \begin{bmatrix} -4.1335 & -1.6738 \\ 1 & 5.0354 \end{bmatrix}$$

All the eigenvalues were obtained using Matlab, none of the equilibrium points are stable in discrete time because all the eigenvalues pair are greater than 1 or less than -1.