THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5143
LINEAR SYSTEMS AND CONTROL

HOMEWORK # 6
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These are all practice problems for you to go through (if you want). As I mentioned before, you are only responsible for the homeworks, quizzes, modules. So you don't have to solve, but I'm hoping these problems will help you practice better (and understand the material better). Unfortunately, I won't be able to provide solutions for these problems, but students who solve them and document the solutions properly

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1. A dynamic system modeled by a nonlinear ODE is given as follows

$$\ddot{y} + 2y\dot{y} + uy = 2u,$$

where $u(t)$ is the input and $y(t)$ is the output.

(a) Construct the nonlinear state space model of the system.

(b) Assuming that $u(t) = u = 2$ is a constant input input signal, find all equilibrium points of the system.

(c) Determine the stability of all of the equilibrium points you found in the previous part.

2. Consider the following DT LTI system

$$x(k+1) = Ax(k) = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} x(k), \;\; y(k) = Cx(k) = \begin{bmatrix} -1 & 1 \end{bmatrix} x(k).$$

(a) Is A nilpotent? Of what order?

(b) Suppose $y(0) = 1$ and $y(1) = 0$. Can we **uniquely find** $x(0)$? If yes, find it. If not, explain why you cannot.

(c) Suppose $y(1) = 1$ and $y(2) = 0$. Can we **uniquely find** $x(0)$? If yes, find it. If not, explain why you cannot.

3. Consider the following system

$$\dot{x}(t) = \frac{1}{t+1} \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$

(a) Find $x(t)$ if $u(t) = 1 + e^{-t}$.

(b) Is $x(t)$ asymptotically stable if $u(t) = 0$ for all $t \geq 0$?

4. Consider an LTI CT system

$$\dot{x}(t) = \begin{bmatrix} -2-6t & 4+8t \\ -1-2t & 2+2t \end{bmatrix} x(t).$$

Find the state transition matrix $\phi(t, t_0)$ of $A(t)$.

5. Consider an LTI CT system

$$\dot{x}(t) = \begin{bmatrix} -\cos(t) & \cos(t) \\ 0 & -2\cos(t) \end{bmatrix} x(t).$$

Find the state transition matrix $\phi(t, t_0)$ of $A(t)$.

6. A DT LTI system is described as follows

$$x(k+1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k).$$

Find the control inputs $u(0), u(1)$ and $u(2)$ that can steer the system from $x(0) = [1\ 1]^\top$ to $x(3) = [0\ 0]^\top$ with the least control energy. In other words, you have to ensure that the control inputs produce the least control energy

$$u(0)^2 + u(1)^2 + u(2)^2.$$

This is a least squares kind of problem. Think carefully.

7. Consider the discrete-time LTI dynamical system:

$$x(k+1) = Ax(k) + Bu(k), \;\; y(k) = Cx(k),$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}.$$

(a) Is the system controllable?

(b) What is the set of reachable space in 3 time-steps, assuming that the initial condition is $x(0) = 0$? In other words, what is a set that contains all possible values of $x(3)$ given some control function $u(k)$ for $k = 0, 1, 2$?

(c) Is the system observable?

(d) Find the unobservable subspace, if any.

(e) Is the system asymptotically stable?

(f) The system is stabilizable. True or False?

(g) The system is detectable. True or False?

(h) The transfer function of a DTLTI system is given by: $H(z) = C(zI - A)^{-1}B$. Compute the transfer function.

Solutions:

(a) Controllability matrix of the given system is:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{C}) = 3 \Rightarrow \textbf{system is not controllable}$$

(b) Set of states that can be reached from a zero initial state condition is given by the subspace of $\mathbb{R}^4$ spanned by only the first three columns of the controllability matrix $\mathcal{C}$.

(c) Observability matrix of the given system is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{O}) = 2 \Rightarrow \textbf{system is not observable}$$

(d) As introduced in class, the set of unobservable subspace is the *null-space* of $\mathcal{O}$, that is:

$$\text{null}(\mathcal{O}) := \left\{ x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \in \mathbb{R}^4 \mid x_2 = x_3 = 0 \right\}$$

(e) System is not asymptotically stable; A has two eigenvalues at $-1, 1$, and for discrete systems the eigenvalues at the borders of the unit disk makes the system marginally stable (given that the size of Jordan block is not greater than 1), not asymptotically stable.

(f) False — the unstable eigenvalue -1 is uncontrollable (PBH test).

(g) False — the unstable eigenvalue -1 is unobservable (PBH test).

(h) $H(z) = C(zI - A)^{-1}B = \dfrac{1}{z}$.

8. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)(x_1^2(t) - 1)\\ \dot{x}_2(t) &= x_2^2(t) + x_1(t) - 3\end{aligned}$$

(a) Find all the equilibrium points of the nonlinear system.

(b) Determine the stability of the system around each equilibrium point, if possible.

Solutions:

(a) Setting the state-dynamics to zero, we can find the equilibrium points. There are 5 equilibrium points for the given system, listed as follows:

$$x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 & 3 \\ \sqrt{2} & -\sqrt{2} & 2 & -2 & 0 \end{bmatrix}.$$

(b) The stability of the system around an equilibrium point is determined by evaluating the Jacobian matrix $Df(x)$ around each equilibrium point and finding its eigenvalues:

$$Df(x) = \begin{bmatrix} 2x_1x_2 & x_1^2 - 1 \\ 1 & 2x_2 \end{bmatrix}.$$

The only equilibrium point that yields a stable $Df(x_e)$ matrix is $x_e^{(2)} = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$, giving $\lambda_1 = \lambda_2 = -2\sqrt{2}$ as the two stable eigenvalues.

9. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)x_2(t) - 2x_1(t)\\ \dot{x}_2(t) &= x_1(t) - x_2(t) - 1.\end{aligned}$$

(a) Find all the equilibrium points of the nonlinear system.

(b) Determine the stability of the system around each equilibrium point, if possible.

10. The following nonlinear system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)(2 - x_2(t))\\ \dot{x}_2(t) &= -x_2(t)(1 - x_1(t)),\end{aligned}$$

has two equilibrium point solutions:

$$x_{e,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_{e,2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(a) Obtain the two linearized state space models corresponding to the two equilibrium points.

(b) Determine the stability of the system around each equilibrium point, if possible.

11. Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system. For each of the following statements, determine if it is true or false. If it is true, **prove** why; if it is false, find a counter example.

(a) The system $\dot{x}(t) = -Ax(t)$ is asymptotically stable.
Solutions: False. Eigenvalues of $-A$ are $-\lambda_i, \forall i = 1, \ldots, n$, which are all in the RHP.

(b) The system $\dot{x}(t) = A^\top x(t)$ is asymptotically stable.
Solutions: True. Eigenvalues of $A^\top$ are the same as A.

(c) The system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable (assume A^{-1} exists).
Solutions: True. Eigenvalues of A^{-1} are the same as $\frac{1}{\lambda_i}$ and they're all in the open LHP.

(d) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.
Solutions: False. Counter example: $A = \begin{bmatrix} -1 & 10 \\ 0 & -1 \end{bmatrix}$.

(e) The system $\dot{x}(t) = A^2 x(t)$ is asymptotically stable.
Solutions: False. Counter example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

12. Assume that $x(k+1) = Ax(k)$ is an asymptotically stable discrete-time LTI system. For each of the following statements, determine if it is true or false. If it is true, explain why; if it is false, find a counter example.

(a) The system $x(k+1) = -Ax(k)$ is asymptotically stable.
Solutions: True. Eigenvalues remain in the unit disk.

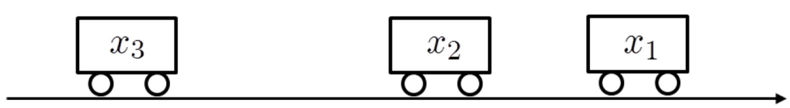
(b) The system $x(k+1) = A^\top x(k)$ is asymptotically stable.
Solutions: True. Eigenvalues do not change.

(c) The system $x(k+1) = A^{-1}x(k)$ is asymptotically stable (assume A^{-1} exists).
Solutions: False. Eigenvalues becomes larger than 1.

(d) The system $x(k+1) = (A + A^\top)x(k)$ is asymptotically stable.
Solutions: False. Counter example: $A = 0.9$.

(e) The system $x(k+1) = A^2 x(k)$ is asymptotically stable.
Solutions: True. If $-1 < \lambda_i < 1, \Rightarrow 0 < \lambda_i^2 < 1$.

13. Consider three cars moving on the same lane, whose initial locations at time $t = 0$ are $x_1(0) = x_2(0) = x_3(0) = 0$. The above figure exemplifies the movement of cars in 1-D. Suppose the goal is for all three cars to meet at the same location (it does not matter where this meet-up location is). To achieve this goal, the following system dynamics can be designed, where $u(t)$ is an input control for the leading car:



$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) - x_1(t) + u(t) && (1) \\
\dot{x}_2(t) &= \frac{x_1(t) + x_3(t)}{2} - x_2(t) && (2) \\
\dot{x}_3(t) &= x_2(t) - x_3(t) && (3)
\end{aligned}
$$

In other words, the leading and trailing cars will both move toward the middle car instantaneously; while the middle car will move towards the center of the leading and the trailing cars.

(a) Represent the above dynamics an CT-LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A, B are matrices that you should determine.

Solutions: Clearly, $A = \begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) Find e^{At} for all $t \in \mathbb{R}$.

Hint: $\begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & -2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix}$

Solutions: $e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & e^{-2t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix}$

(c) Suppose $u(t) = 1, t \geq 0$. Find the expression of $x(t)$ for $t \geq 0$. Also, find $x(t)$ if $t \to \infty$.
Solution: Given that $u(t) = 1 \; \forall t \geq 0$, we obtain:

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & & \\ & e^{-2(t-\tau)} & \\ & & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5(1-e^{-2t}) & \\ & & 1-e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} \end{aligned}$$

(d) Describe the steady-state behaviors of $x_i(t), i = 1, 2, 3$. Your description must have physical, applied meaning. **Solutions:** If $t \to \infty$, we obtain:

$$x(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.25t + 0.625 \\ 0.25t - 0.125 \\ 0.25t - 0.375 \end{bmatrix}$$

In the steady state, all three cars move at the constant velocity 0.25 to the right, with car 1 still in the lead and car 3 still trailing, and the distance between car 1 and car 2 is 0.75 while the distance between car 2 and car 3 is 0.25.

(e) What is the constant velocity for the three cars?
Solutions: 0.25.

(f) Assess the following CT-LTI system properties: stability, controllability, stabilizability, observability, and detectability, given that the outputs for the system are the first two states, i.e., $x_1(t)$ and $x_2(t)$. You'll have to obtain the C-matrix.
Solutions :

i. *Stability*: system is **not** asymptotically stable, as one eigenvalue is equal to 0—on the $j\omega$-axis.
ii. *Controllability:* the controllability matrix of the given LTI system is $\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$ and it's full rank, hence the system is fully controllable.
iii. *Stabilizability:* an LTI system that is fully controllable is stabilizable.
iv. *Observability:* the observability matrix of the given LTI system is $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ and it's full rank, hence the system is fully observable.
v. *Detectability:* an LTI system that is fully observable is by definition detectable.

14. You are given the following LTI dynamical system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x_{\text{initial}} = x_{t_0} &(4)\\ y(t) &= Cx(t) &(5) \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x(t_0) = x(1) = [0\ 1\ 1]^\top.$$

Recall that the closed-form to the above differential equation for any time-varying control input is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

(a) Is A nilpotent of order 2? (i.e., is $A^2 = 0$?)

(b) Determine $e^{At}, e^{A(t-\tau)}, e^{A(t-t_0)}$, $t_0 = 1$. Recall that

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = \boldsymbol{I}_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \frac{(At)^4}{4!} + \cdots$$

(c) If $u(t) = 0$, determine $x(t)$ (or the zero-input state state-response) given the provided initial conditions.

(d) If $x(t_0) = x(1) = 0$ and $u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, determine $x(t)$ (or the zero-state, state-response).

(e) Determine $y(t)$ if $u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x(t_0) = x(1) = [0\ 1\ 1]^\top$.

15. A dynamical CTLTI system is characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$.

(a) Find a linear state-observer gain $L = [l_1\ l_2]^\top$ such that the poles of the estimation error are -5 and -7.

(b) Can you place both poles at -6? If yes, what is the corresponding observer gain?

Solutions:

(a) First, we find $A - LC$ in terms of l_1 and l_2:

$$A - LC = \begin{bmatrix} 1 - l_1/2 & 3 - l_1 \\ 3 - l_2/2 & 1 - l_2 \end{bmatrix}.$$

Since the roots of the designed observer are -5 and -7, the desired characteristic polynomial is:

$$\pi_{A-LC} = (\lambda + 5)(\lambda + 7) = \lambda^2 + 12\lambda + 35.$$

The characteristic polynomial in terms of l_1 and l_2 can be written as:

$$+\lambda^2 + \lambda \underbrace{(-2 + \frac{l_1}{2} + l_2)}_{=12} \underbrace{-8 + \frac{5l_1}{2} + \frac{l_2}{2}}_{=35} = 0.$$

Solving the following linear system of equations,

$$\begin{aligned} 35 &= -8 + \frac{5l_1}{2} + \frac{l_2}{2} \\ 12 &= -2 + \frac{l_1}{2} + l_2, \end{aligned}$$

we obtain $l_1 = 16$ and $l_2 = 6$.

(b) Placing poles at $\lambda = -6$ means that

$$\pi_{A-LC} = (\lambda + 6)^2 = \lambda^2 + 12\lambda + 36$$

or,

$$
\begin{aligned}
44 &= \frac{5l_1}{2} + \frac{l_2}{2} \\
14 &= \frac{l_1}{2} + l_2,
\end{aligned}
$$

A solution to the above system of equations is $l_1 = 16.44$ and $l_2 = 5.77$.

16. Given the following LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x_{\text{initial}} = x_{t_0}. \tag{6}$$

The closed-form to the above differential equation for any time-varying control input is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

Show that the above solution is in fact a solution to the system dynamics.

Hint — Leibniz Differentiation Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,\mathrm{d}x\right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x,\theta)\,\mathrm{d}x + f\big(b(\theta),\theta\big)\cdot b'(\theta) - f\big(a(\theta),\theta\big)\cdot a'(\theta)$$

17. Determine whether the following system is controllable, observable, detectable, and stabilizable. You have to justify your answer.

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & -4 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t).$$

18. Find the state transition matrix for the following system:

$$\dot{x}(t) = \begin{bmatrix} -t & 0 \\ -\cos(t) & -t \end{bmatrix} x(t).$$

19. Find the state transition matrix for the following system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t),$$

followed by the STM to this system:

$$\dot{x}(t) = \begin{bmatrix} -\dfrac{1}{1+e^t} & \dfrac{1}{1+e^t} \\ 0 & \dfrac{-2}{1+e^t} \end{bmatrix} x(t).$$

Hint: $\int \frac{1}{1+e^t}dt = \int \frac{1+e^t-e^t}{1+e^t}dt = \int \frac{1+e^t}{1+e^t} - \frac{e^t}{1+e^t}dt = \ldots$

20. Find the state transition matrix for the following system:

$$\dot{x}(t) = \begin{bmatrix} -2t & t \\ 0 & -t \end{bmatrix} x(t).$$

Solution:

$$A(t) = \begin{bmatrix} -2t & t \\ 0 & -t \end{bmatrix} = t\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} = t\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2t & 0 \\ 0 & -t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = TD(t)T^{-1} \Rightarrow \phi(t,t_0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{-(t^2-t_0^2)} & 0 \\ 0 & e^{-0.5(t^2-t_0^2)} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

For this problem, you'll have to find the evectors again.

21. You are given the following LTI dynamical system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x_{\text{initial}} = x_{t_0} && (7) \\ y(t) &= Cx(t) && (8) \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

(a) Is the system stable?

(b) Is the above system controllable or not? Justify your answer.

(c) What is the controllability subspace?

(d) Is the above system observable or not? Justify your answer.

(e) Obtain the unobservable subspace of the system.

(f) Is there a state feedback controller $u(t) = -Kx(t)$ such that $A - BK$ has eigenvalues $\{-2, -2, -2\}$? If yes, find this state feedback gain K. Justify why if your answer is no.

(g) Is there a state observer such that $A - LC$ has eigenvalues $\{1, -1, -2\}$? If yes, find this state feedback gain L. Justify why if your answer is no.

(h) If $u(t) = 0$, and $x(0)$ is any non-zero initial condition, would $y(t)$ be bounded? In other words, can you find a positive constant M such that $|y(t)| \leq M \quad \forall t > t_0$?

22. Obtain the linearized state space representation of the following nonlinear system around $x_e = \begin{bmatrix} x_{e1} \\ x_{e2} \end{bmatrix}$ and $u_e = u^*$ (i.e., these quantities are known and given):

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\sin(x_2(t)) + x_2(t)u(t) && (9) \\ \dot{x}_2(t) &= x_1(t)e^{-x_2(t)} + u^2(t) && (10) \\ y(t) &= 2x_1(t)x_2(t) + x_2^2(t). && (11) \end{aligned}$$

Solution:

$$\begin{aligned} \begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} \sin(x_{e2}) & x_{1e}\cos(x_{2e}) + u_e \\ e^{-x_{2e}} & -x_{1e}e^{-x_{2e}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} x_{2e} \\ 2u_e \end{bmatrix} \Delta u(t) && (12) \\ \Delta\dot{x}(t) &= A\Delta x(t) + B\Delta u(t) && (13) \end{aligned}$$

and

$$\Delta y(t) = \begin{bmatrix} 2x_{e2} & 2x_{e1} + 2x_{e2} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

where $\Delta x(t) = x(t) - x_e$ and $\Delta u(t) = u(t) - u_e$.

23. Design an OBC (i.e., $u(t) = -K\hat{x}(t)$) for the following SISO system [Problem from Module 8]

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Solution:

(a) Before doing anything, check whether system is cont. (or stab.) and obs. (or det.): **system is cont. AND obs.**

(b) First, design a stabilizing state feedback control, i.e., find K s.t.

$$eig(A - BK) < 0, A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \Rightarrow K = \begin{bmatrix} 4 & 2 \end{bmatrix} \text{ does the job}$$

(c) Second, design a stabilizing observer (estimator), i.e., find L s.t.

$$eig(A - LC) < 0, A - LC = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 10 & 100 \end{bmatrix}^\top \text{ does the job}$$

(d) Finally, overall system design:

$$\begin{aligned} u(t) &= -K\hat{x}(t) = -4\hat{x}_1(t) - 2\hat{x}_2(t) \\ \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + 10(y(t) - \hat{x}_1(t)) \\ \dot{\hat{x}}_2(t) &= u(t) + 100(y(t) - \hat{x}_1(t)) \end{aligned}$$