PREREQUISITES ASSESSMENT
Ahmad F. Taha
August 24, 2015

The objective of this exercise is to assess your knowledge of the course prerequisites. This will give me a good indication of your background. You are not supposed to do so much work for this assignment — it's only assessment, *remember!* You will receive a perfect grade whether you know the answers to the questions or not. I'll read your solutions and try to fill the corresponding gaps in class.

Many of the answers to following problems can be easily found online or can be computed using MATLAB. **Please do not consult with any online references or computational software tools!** If you do not know the answer to any of the questions, leave it blank — you're getting credit anyway.

Linear Algebra

Show your work.

1. Find the eigenvalues, eigenvectors, and inverse of matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

2. Write $\boldsymbol{A}$ in the matrix diagonal transformation, i.e., $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$ — $\boldsymbol{D}$ is the diagonal matrix containing the eigenvalues of $\boldsymbol{A}$.

3. Find the determinant, rank, and null-space set of this matrix:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}.$$

4. Is there a relationship between the determinant and the rank of a matrix?

5. True or False? Prove or give counter examples.
   - $\boldsymbol{AB} = \boldsymbol{BA}$ for all $\boldsymbol{A}$ and $\boldsymbol{B}$
   - $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible $\rightarrow (\boldsymbol{A} + \boldsymbol{B})$ is invertible

Linear Dynamical Systems

A state-space representation of a linear, time-invariant (LTI) dynamical system can be represented as follows:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t), \quad \mathbf{x}_{\text{initial}} = \mathbf{x}_{t_0}, \tag{1}$$

$$\mathbf{y}(t) = \boldsymbol{C}\mathbf{x}(t) + \boldsymbol{D}\mathbf{u}(t), \tag{2}$$

where $\mathbf{x}(t)$ is the dynamic state-vector of the LTI system; $\mathbf{u}(t)$ is the control input-vector; $\mathbf{y}(t)$ is the output, all for the above LTI system; matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are constants of appropriate dimensions.

1. What is the closed-form, state solution to the above differential equation for any time-varying control input, i.e., $\mathbf{x}(t) =$?

2. What are the poles of the above dynamical system? Define asymptotic stability. What's the difference between marginal and asymptotic stability?

3. Under what conditions is the above dynamical system controllable? Observable? Stabilizable? Detectable?

4. What is the transfer function $\left(H(s) = \frac{Y(s)}{U(s)} \right)$ of the above system?

5. What is the state-transition matrix for the above system?

Optimal Control and Dynamic Observers

1. Define the linear quadratic regulator problem for LTI systems in both, words and equations.

2. What is the optimal solution to an optimal control problem? What does it physically mean for CPSs?

3. What is an optimal state-feedback control strategy for linear systems? Define it in words and equations.

4. What is a generic dynamic observer? Luenberger observer?

Convex Optimization

1. Let $f(\mathbf{x})$ be a multi-variable function of three variables, as follows:

$$f(x_1, x_2, x_3) = x_1 x_2 x_3 + 2x_2^3 x_1^2 - 4\cos(x_3 x_2) + \log(\cos(x_2)^2) + 4x_3 - 2x_1.$$

Find the gradient and then the Hessian of $f(\mathbf{x})$.

2. What is an optimization problem? An unfeasible, feasible solution, an optimal solution to a generic optimization problem?

   If you do not know the answer to the above question, you can skip the rest of questions in this section.

3. What is a convex optimization problem? Define it rigorously. Give an example — a numerical example, if possible.

4. What is a semi-definite program (SDP)? Define it rigorously.

5. Given the following optimization problem,

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & c(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{h}(\mathbf{x}) \leq \mathbf{0} \\ & \mathbf{g}(\mathbf{x}) = \mathbf{0}, \end{aligned} \tag{3}$$

what are the corresponding Karush–Kuhn–Tucker (KKT) conditions?

6. What do you know about convex relaxations of non-convex optimization problems? Please be very explicit.