Your Name: Solutions

Your Signature:

- **Exam duration:** 2 hours and 30 minutes.
- This exam is closed book, closed notes, closed laptops, closed phones, closed tablets, closed pretty much everything.
- **No calculators** of any kind are allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, **even if your answer is correct.**
- Place a box around your final answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- You can ask as many questions as you want.
- This exam has 16 pages, plus this cover sheet. Please make sure that your exam is complete, that you read all the exam directions and rules.

| Question Number | Maximum Points | Your Score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 50 | |
| 4 | 15 | |
| 5 | 25 | |
| 6 | 15 | |
| 7 | 45 | |
| 8 | 25 | |
| 9 | 25 | |
| 10 | 20 | |
| Total | 250 | |

1. (15 total points) Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system.

   For each of the following statements, determine if it is true or false. If it is true, **prove** why; if it is false, find a counter example.

   (a) (3 points) The system $\dot{x}(t) = -Ax(t)$ is asymptotically stable.

   **Solutions: False.** Eigenvalues of $-A$ are $-\lambda_i, \forall i = 1, \ldots, n$, which are all in the RHP.

   (b) (3 points) The system $\dot{x}(t) = A^\top x(t)$ is asymptotically stable.

   **Solutions: True.** Eigenvalues of $A^\top$ are the same as $A$.

   (c) (3 points) The system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable (assume $A^{-1}$ exists).

   **Solutions: True.** Eigenvalues of $A^{-1}$ are the same as $\frac{1}{\lambda_i}$ and they're all in the open LHP.

   (d) (3 points) The system $\dot{x}(t) = (A + A^\top)x(t)$ is asymptotically stable.

   **Solutions: False.** Counter example: $A = \begin{bmatrix} -1 & 10 \\ 0 & -1 \end{bmatrix}$.

   (e) (3 points) The system $\dot{x}(t) = A^2 x(t)$ is asymptotically stable.

   **Solutions: False.** Counter example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. (15 total points) Assume that $x(k+1) = Ax(k)$ is an asymptotically stable discrete-time LTI system.

   For each of the following statements, determine if it is true or false. If it is true, explain why; if it is false, find a counter example.

   (a) (3 points) The system $x(k+1) = -Ax(k)$ is asymptotically stable.

   **Solutions: True.** Eigenvalues remain in the unit disk.

   (b) (3 points) The system $x(k+1) = A^\top x(k)$ is asymptotically stable.

   **Solutions: True.** Eigenvalues do not change.

   (c) (3 points) The system $x(k+1) = A^{-1}x(k)$ is asymptotically stable (assume $A^{-1}$ exists).

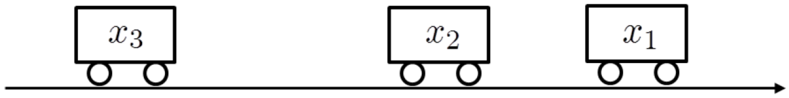
   **Solutions: False.** Eigenvalues becomes larger than 1.

   (d) (3 points) The system $x(k+1) = (A + A^\top)x(k)$ is asymptotically stable.

   **Solutions: False.** Counter example: $A = 0.9$.

   (e) (3 points) The system $x(k+1) = A^2x(k)$ is asymptotically stable.

   **Solutions: True.** If $-1 < \lambda_i < 1$, $Rightarrow$ $0 < \lambda_i^2 < 1$.




3. (50 total points) Consider three cars moving on the same lane, whose initial locations at time $t = 0$ are $x_1(0) = x_2(0) = x_3(0) = 0$. The above figure exemplifies the movement of cars in 1-D. Suppose the goal is for all three cars to meet at the same location (it does not matter where this meet-up location is). To achieve this goal, the following system dynamics can be designed, where $u(t)$ is an input control for the leading car:

$$\dot{x}_1(t) = x_2(t) - x_1(t) + u(t) \quad (1)$$

$$\dot{x}_2(t) = \frac{x_1(t) + x_3(t)}{2} - x_2(t) \quad (2)$$

$$\dot{x}_3(t) = x_2(t) - x_3(t) \quad (3)$$

In other words, the leading and trailing cars will both move toward the middle car instantaneously; while the middle car will move towards the center of the leading and the trailing cars.

(a) (5 points) Represent the above dynamics an CT-LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $A, B$ are matrices that you should determine.

> **Solutions:** Clearly, $A = \begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) (5 points) Find $e^{At}$ for all $t \in \mathbb{R}$.

*Hint:* $\begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & -2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix}$

> **Solutions:** $e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & e^{-2t} & \\ & & e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix}$

(c) (20 points) Suppose $u(t) = 1, t \geq 0$. Find the expression of $x(t)$ for $t \geq 0$. Also, find $x(t)$ if $t \to \infty$.

**Solution:** Given that $u(t) = 1\ \forall t \geq 0$, we obtain:

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & & \\ & e^{-2(t-\tau)} & \\ & & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5(1-e^{-2t}) & \\ & & 1-e^{-t} \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} \end{aligned}$$

(d) (3 points) Describe the steady-state behaviors of $x_i(t), i = 1,2,3$. Your description must have physical, applied meaning.

**Solutions:** If $t \to \infty$, we obtain:

$$x(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} t & & \\ & 0.5 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.25t + 0.625 \\ 0.25t - 0.125 \\ 0.25t - 0.375 \end{bmatrix}$$

In the steady state, all three cars move at the constant velocity 0.25 to the right, with car 1 still in the lead and car 3 still trailing, and the distance between car 1 and car 2 is 0.75 while the distance between car 2 and car 3 is 0.25.

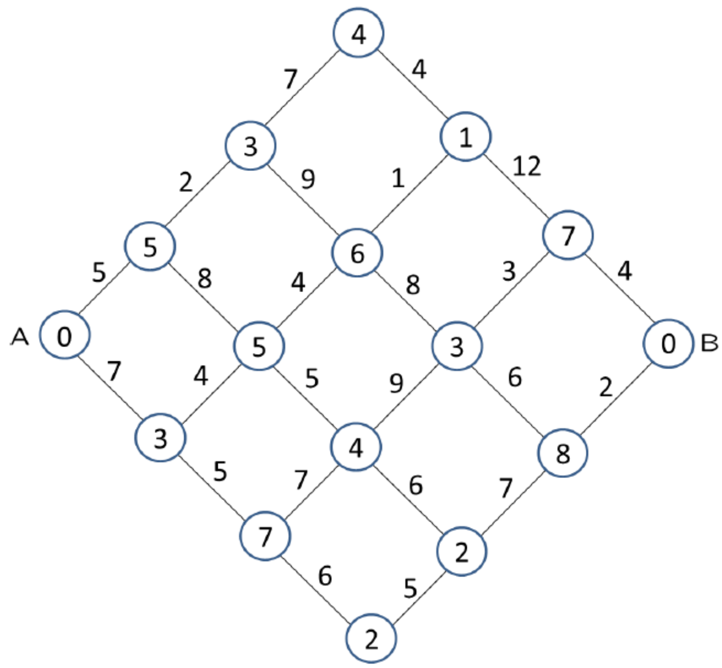
(e) (2 points) What is the constant velocity for the three cars?

**Solutions:** 0.25.

(f) (15 points) Assess the following CT-LTI system properties: stability, controllability, stabilizability, observability, and detectability, given that the outputs for the system are the first two states, i.e., $x_1(t)$ and $x_2(t)$. You'll have to obtain the $C$-matrix.

**Solutions (3pts each):**

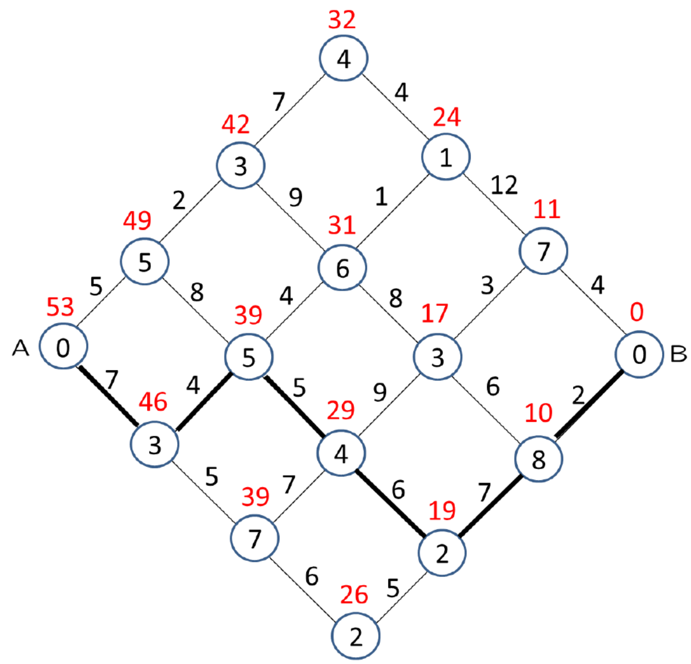
1. *Stability*: system is **not** asymptotically stable, as one eigenvalue is equal to 0—on the $j\omega$-axis.
2. *Controllability:* the controllability matrix of the given LTI system is $\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$ and it's full rank, hence the system is fully controllable.
3. *Stabilizability:* an LTI system that is fully controllable is stabilizable.
4. *Observability:* the observability matrix of the given LTI system is $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ and it's full rank, hence the system is fully observable.
5. *Detectability:* an LTI system that is fully observable is by definition detectable.




4. (15 total points) Suppose in the above graph, one starts from node A on the left and tries to reach node B on the right by only moving to the right at each step. The cost of any path is the sum of the following:

   - the cost of all the edges it passes through as indicated by the numbers above the edges
   - the cost of all the intermediate nodes it visits as indicated by the numbers inside the circles.

   (a) (15 points) Use the dynamic programming method to find the path from A to B with the smallest cost. You should only use dynamic programming to solve the above problem. You receive **no credit** if you don't show how you applied DP. Once you're done, draw a diagram/map that shows the DP-solution that exemplifies your numerical solution.




**Solutions:**

5. (25 total points) Consider the following CT-LTI model of a dynamical system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \ \ x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

with the following cost function:

$$J = \int_0^\infty \left( x^\top(t)x(t) + u^2(t) \right) dt.$$

(a) (20 points) Find the linear state-feedback control law that minimizes $J$.

**Solutions:** To find the optimal controller for this problem, we first solve the algebraic Riccati (ARE) equation we discussed in Module 05. This equation defines the infinite horizon, steady-state solution for CARE—as the final time, $t_f$, is equal to infinity.

The problem given is as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = 1.$$

The optimal linear state-feedback controller is given by:

$$u^*(t) = -Kx(t) = -R^{-1}B^\top P^* x(t),$$

where $P^*$ is a 2by2 positive definite matrix solution to ARE:

$$A^\top P + PA + Q - PBR^{-1}B^\top = 0.$$

Letting $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, we obtain:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving the above equation, we obtain:

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Other solutions for $P$ are invalid as $P$ has to be a positive definite matrix. Therefore, the optimal state-feedback control law is:

$$u^*(t) = -Kx(t) = -R^{-1}B^\top P^* x(t) = -x_1(t) - 3x_2(t).$$

(b) (5 points) Find the value of the performance index for the closed-loop system.

**Solutions:** In homework 5, we derived the value for the performance index for the optimal control problem driven by the state-feedback controller. The performance index value is:

$$J = x_0^\top P x_0 = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 15.$$

*You are almost halfway through the exam. It might be a good idea to rehydrate and walk for a minute or two...*

*...Unless you're superman/superwoman...*

*...You're not superman/superwoman.*

6. (15 total points) Compute $x = x(t)$ that solves the following optimal control problem:

$$\text{minimize } J(u) = \int_1^2 \sqrt{1+u^2(t)}dt$$

$$\text{subject to} \quad \dot{x}(t) = u(t),\ x(1) = 0,\ x(2) = 5.$$

(a) (15 points) Using the HJB-equation, find the linear state-feedback control law that minimizes $J$.

**Solutions:** First, we construct the Hamiltonian:

$$\mathcal{H}(x, u, J_x, t) = \sqrt{1+u^2(t)} + \lambda(x,t)u(t).$$

Since there are no constraints on $u(t)$, the optimal controller candidate is:

$$0 = \frac{\partial \mathcal{H}}{\partial u} = \lambda(x,t) + \frac{u}{\sqrt{1+u^2}} \Rightarrow u^*(t) = \frac{\lambda(x,t)}{\sqrt{1-\lambda^2(x,t)}}.$$

Now, we formulate the HJB equation:

$$-V_t(x,t) = \left(\frac{\partial \mathcal{H}}{\partial x}\right)^\top = 0 \Rightarrow V(t,x) = v \textbf{ is constant.}$$

Therefore, $u^*(t) = \dfrac{\lambda(x,t)}{\sqrt{1-\lambda^2(x,t)}} = \dfrac{\lambda}{\sqrt{1-\lambda^2}} = c$ is also constant. Since $x(1)$ and $x(2)$ are given, we can determine $u^*(t) = c$, as follows:

$$x(2) = x(1) + \int_1^2 c\, d\tau, \Rightarrow u^*(t) = c = 5.$$

Therefore,

$$x(t) = \int_1^t 5\, d\tau = 5t - 5.$$

7. (45 total points) For the following dynamical system under unknown inputs,

$$\begin{aligned} \dot{x}_p(t) &= A_p(t)x_p(t) + B_p^{(1)}u_1(t) + B_p^{(2)}u_2(t) \\ y(t) &= C_p x_p(t), \end{aligned}$$

a sliding-mode observer (SMO) can be designed with the following dynamics:

$$\begin{aligned} \dot{\hat{x}}_p(t) &= A_p\hat{x}_p(t) + B_p^{(1)}u_1(t) + L(y(t) - \hat{y}(t)) - B_p^{(2)}E(\hat{y}, y, \eta) \\ \hat{y}(t) &= C_p\hat{x}_p(t), \end{aligned}$$

where $E(\cdot)$ is defined as ($\eta$ is SMO gain):

$$E(\hat{y}, y, \eta) = \begin{cases} \eta \dfrac{F(\hat{y} - y)}{\|F(\hat{y} - y)\|_2}, & \text{if } F(\hat{y} - y) \neq 0 \\ 0, & \text{if } F(\hat{y} - y) = 0. \end{cases}$$

(a) (15 points) The SMO design objective is to find matrices $P = P^\top \succ 0, F$ and $L$ that satisfy the following equations for a predefined $Q = Q^\top \succ 0$:

$$FC_p = (B_p^{(2)})^\top P$$

$$(A_p - LC_p)^\top P + P(A_p - LC_p) = -Q$$

Are the above two equations linear matrix inequalities? If no, formulate the above equations as a set of linear matrix inequalities.

**Solutions:** The We have two (nonlinear) matrix equations in terms of $P, F, L$:

$$\begin{aligned} (A_p - LC_p)^\top P + P(A_p - LC_p) &= -Q \\ P &= P^\top \\ FC_p &= (B_p^{(2)})^\top P \end{aligned}$$

We use a simple LMI trick. Set $Y = PL$, rewrite above system of **linear** matrix equations as:

$$\begin{aligned} A_p^\top P + PA_p - C_p^\top Y^\top - YC_p &= -Q \\ P &= P^\top \\ FC_p &= (B_p^{(2)})^\top P \end{aligned}$$

From the solutions of $P, Y$, we can solve for $L$: $L = P^{-1}Y$. Hence, the SMO design problem can be written as an LMI, albeit we have more variables to solve for.

(b) (15 points) Write a CVX script to solve the SMO design problem, written as an LMI.

**Solutions:** Sample CVX code—

```
cvx_clear
cvx_begin sdp quiet

variable P(n, n) symmetric
variable Y(n, p)
variable F(m2, p)


minimize(0)

subject to

Ap'*P + P*Ap - Y*Cp - Cp'*Y' <= 0
F*Cp-Bp2'*P==0;
P >= 0

cvx_end

L = P\Y;
```

(c) (15 points) If your state-estimates are converging, how can you reconstruct (or estimate) the vector of unknown inputs from the system dynamics? You only need this equation. Think about it–the solution is very simple.

$$\dot{x}_p(t) = A_p x_p(t) + B_p^{(1)} u_1(t) + B_p^{(2)} u_2(t)$$

*Hint:* You should first discretize the above dynamical system and then use $\hat{x}(t)$ to obtain $u_2(t)$.

**Solutions:** Since the state-estimates are converging, then $\hat{x}(t) \to x(t)$ or $\hat{x}(k) \to x(k)$. Leveraging the given hint, we can obtain the discretized version of the system dynamics, as follows:

$$x(k+1) \approx \hat{x}(k+1) = A_d \hat{x}(k) + B_p d^{(1d)} u_1(k) + B_p d^{(2)} \hat{u}_2(k).$$

Since the estimation is accurate, and since we have $u_1(k)$, $A_d, B_p d$ we can obtain an estimate of the unknown input vector $u_2(k)$ as follows:

$$\hat{u}_2(k) = \left(B_p d^{(2)}\right)^{\dagger} \left(\hat{x}(k+1) - A_d \hat{x}(k) - B_p d^{(1d)} u_1(k)\right),$$

where $\left(B_p d^{(2)}\right)^{\dagger}$ is the pseudo-inverse of $B_p d^{(2)}$, which means we need $B_p d^{(2)}$ to be full-column rank.

8. (25 total points) The plant (p) and controller (c) dynamics of a networked control system (NCS) are given as follows:

$$\dot{x}_p = A_p x_p + B_p \hat{u} \quad (4)$$
$$y = C_p x_p \quad (5)$$
$$\dot{x}_c = A_c x_c + B_c \hat{y} \quad (6)$$
$$u = C_c x_c + D_c \hat{y}, \quad (7)$$

where the state-space matrices are constant with appropriate dimensions **and zero-order hold (ZOH) is considered to the exchanged signals through the network.** The NCS architecture is shown in the below figure.

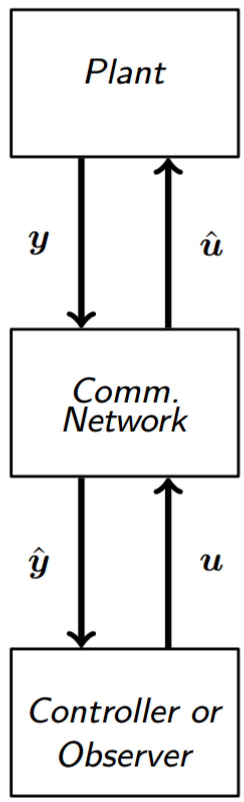



Figure 1: NCS architecture.

The network-induced error state is defined as follows:

$$e(t) = \begin{bmatrix} e_y(t) \\ e_u(t) \end{bmatrix} = \begin{bmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{bmatrix}.$$

(a) (25 points) Derive the dynamics of the combined NCS state:

$$z(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \\ e_y(t) \\ e_u(t) \end{bmatrix},$$

i.e., derive

$$\boxed{\dot{z}(t) = Az(t)}$$

where $A$ is a matrix you should determine in terms of $A_p, B_p, C_p, A_c, B_c, C_c, D_c$ only, given that ZOH is considered for signals $\hat{y}$ and $\hat{u}$.

**Solutions:** Check Module 08. The overall state of the NCS is:

$$x(t)^\top = [x_p(t)^\top x_c(t)^\top]$$

Without a network, $e(t) = 0$, dynamics are reduced to:

$$\dot{x}(t) = A_{11}x(t),\ A_{11} = \begin{bmatrix} A_p + B_pD_cC_p & B_pC_c \\ B_cC_p & A_c \end{bmatrix}.$$

With the network, $e(t) \neq 0$, define network-induced error state:

$$e(t)^\top = [\hat{y}(t)^\top \hat{u}(t)^\top] - [y(t)^\top u(t)^\top].$$

Hence, the combined NCS state can be written as:

$$z(t)^\top = [x(t)^\top e(t)^\top].$$

Given the plant and controller dynamics, the NCS overall state dynamics are obtained as follows:

$$\dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = Az(t).$$

$$A_{12} = \begin{bmatrix} B_pD_c & B_p \\ B_c & 0 \end{bmatrix}, A_{21} = -\begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix} A_{11}, A_{22} = -\begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix} A_{12}.$$

9. (25) The following optimization problem is given:

$$\underset{x}{\text{minimize}} \frac{x^\top Q x}{x^\top P x},$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

*Hint: set the denominator to 1, and make this a constraint to the optimization problem. Then, you'll get an equality-constrained optimization problem, and hence you can use the KKT conditions we discussed in class.*

(a) (25 points) Solve the above optimization problem.

**Solutions:** Given the hint, the problem becomes:

$$\begin{array}{ll} \text{minimize} & x^\top Q x \\ \text{subject to} & 1 - x^\top P x = 0. \end{array}$$

We now formulate the Lagrangian of the optimization problem:

$$\mathcal{L}(x, \lambda) = x^\top Q x + \lambda(1 - x^\top P x).$$

The first-order necessary conditions of optimality can be written produces:

$$\nabla_x \mathcal{L}(x, \lambda) = 2Qx - 2\lambda P x = 0,$$

or

$$(\lambda I_2 - P^{-1} Q)x = 0 \rightarrow \left( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) x \rightarrow \begin{bmatrix} \lambda - 1 & \\ & \lambda - 2 \end{bmatrix} x = 0.$$

Therefore, $\lambda_1 = 1, \lambda_2 = 2$ and the corresponding solutions are:

$$x^{(1)} = \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x^{(2)} = \pm \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

By simply comparing $\lambda_1$ and $\lambda_2$ and their corresponding solutions, we find that $\lambda_1 = 1$ is optimal in comparison with $\lambda_2 = 2$. Checking the second-order conditions, we obtain the Hessian of $\mathcal{L}(x, \lambda)$:

$$\nabla_x^2 \mathcal{L}(x^{(1)}, \lambda_1) = \begin{bmatrix} 0 & \\ & 4 \end{bmatrix},$$

which is positive semi-definite. Finding the tangent space and checking the positive-definiteness of the Hessian on the tangent space, we conclude that $x^{(1)} = \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a strict minimizer for the given optimization problem.

10. (20) The following optimization problem is given:

$$\text{maximize} \quad x_1^2 + 4x_2^2$$

$$\text{subject to} \quad x_1^2 + 2x_2^2 \le 2$$

(a) (10 points) Derive the KKT conditions and find the set of points satisfying these conditions.

**Solutions:** The Lagrangian can be written as:

$$\mathcal{L}(x,\mu) = -(x_1^2 + 4x_2^2) + \mu(x_1^2 + 2x_2^2 - 2).$$

Applying the KKT conditions, we obtain the following candidates:

- $x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mu^{(1)} = 0$
- $x^{(2)} = \pm \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \mu^{(2)} = 1$
- $x^{(3)} = \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mu^{(3)} = 2$

(b) (10 points) Apply the second order necessary conditions to determine whether the KKT conditions-satisfying points are minimizers or not.

**Solutions:** Applying the SONC, we obtain the following:

- $x^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a strict local minimizer of the problem;
- $x^{(2)} = \pm \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$ doesn't satisfy the SONC;
- $x^{(3)} = \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a strict local maximizer of the problem.