Guest Lecture

Exploiting Linear Matrix Inequalities In Control Systems Design

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Motivation

- Jan Willems (1971): "*The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms...*"
- We live in an era of high-performance computing...
- ... so why not use it?
- Exploiting excellent convex solvers
  - CVX — Link; Reference: [1]
  - YALMIP — Link; Reference: [2]
  - Open-source, efficient, robust, seamless MATLAB integration

Question

How do we use efficient, user-friendly solvers to design modern control systems?

Review: Linear/Bilinear Matrix Inequalities

Example 1

$$\underbrace{A^\top P + PA \prec 0}_{\text{linear in } P} \quad \text{or} \quad \underbrace{A^\top PA - P \prec 0}_{\text{linear in } P}$$

Example 2

$$\left.\begin{bmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & D^\top D - I \end{bmatrix} \prec 0 \right\} \text{linear in } P$$

Example 3

$$\underbrace{A^\top P + PA}_{\text{linear in } P} + \underbrace{2\alpha P}_{?} \prec 0$$

Scenario I: $\alpha > 0$ **fixed** $\implies$ LMI in P

Scenario II: $\alpha > 0$ **variable** $\implies$ BMI in P and α

Review: LMIs/BMIs

Example 4

$$A^\top P + PA + 2\alpha P - PBR^{-1}B^\top P \prec 0$$

Q: For fixed $\alpha > 0$, is this an LMI in P?
A: (Sadly) **no**, it is a **Quadratic Matrix Inequality** (QMI) in P (look at: $\boldsymbol{P}BR^{-1}B^\top\boldsymbol{P}$)

- Q: Why are we hung up on LMIs?
- A: *LMIs are tractable*! (c.f. [3])

Observer Design

CT-LTI System with measurements:

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

Linear observer:

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})}$$

Goal: Design L to ensure global asymptotic stability of error dynamics

- Matrix inequality for observer design:

$$(A - LC)^\top P + P(A - LC) \prec 0, \; P = P^\top \succ 0$$

Observer Design

$$A^\top P + PA - C^\top L^\top P - PLC \prec 0,\ P \succ 0$$

- To-do: Find L, P
- Problem: BMI in L and P
- **Technique #1**: Choose $Y = PL$
- LMIs:

$$\underbrace{A^\top P + PA}_{\text{linear in } P} - \underbrace{C^\top Y^\top - YC}_{\text{linear in } Y} \prec 0,\ P \succ 0$$

- For robustness of solution, rewrite as

$$A^\top P + PA - C^\top Y^\top - YC + 2\alpha P \preceq 0,\ P \succ 0$$

  with fixed $\alpha > 0$
- Get back $L = P^{-1}Y$ ($P \succ 0$, hence invertible)

General Structure of CVX Code in MATLAB

```
cvx_begin sdp quiet
% sdp: semi-definite programming mode
% quiet: no display during computing

% include CVX [variables]
% for example: variable P(3,3) symmetric

minimize([cost]) % convex function
subject to
[affine constraints] % preferably non-strict inequalities

cvx_end
disp(cvx_status) % solution status
```

Snippet in CVX

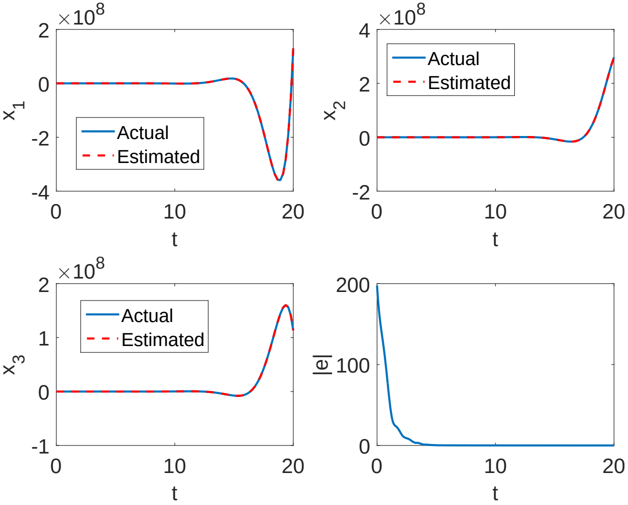
```
cvx_begin sdp

% Variable definition
variable P(n, n) symmetric
variable Y(n, p)

% LMIs
P*sys.A + sys.A'*P - Y*sys.C - sys.C'*Y' + P <= 0
P >= eps*eye(n) % eps is a very small number in MATLAB

cvx_end
sys.L = P\Y; % compute L matrix
```

Simulation



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State/Output Feedback Control

LTI System with output feedback control:

$$\boxed{\begin{aligned} &\dot{x} = Ax + Bu \\ &y = Cx \\ &u = -Ky \end{aligned}}$$

Goal: Design K to ensure global asymptotic stability

- Matrix inequality for output-feedback controller design:

$$(A - BKC)^\top P + P(A - BKC) \prec 0, \; P \succ 0$$

- Simpler case: state-feedback ($C = I$)

$$(A - BK)^\top P + P(A - BK) \prec 0, \; P \succ 0$$

Simpler Case: State-Feedback Control

$$(A - BK)^\top P + P(A - BK) \prec 0, \; P \succ 0$$

- To-do: Find K, P
- Problem: BMI in K and P
- **Technique #2**: Congruence transformation with $S \triangleq P^{-1}$ and $Z \triangleq KS$
- New inequalities

$$SA^\top + AS - SK^\top B^\top - BKS \prec 0$$

- LMIs:

$$\underbrace{SA^\top + AS}_{\text{linear in } S} - \underbrace{Z^\top B^\top - BZ}_{\text{linear in } Z} \prec 0, \; P \succ 0$$

- Get back $P = S^{-1}$, $K = ZS^{-1}$

Snippet in CVX

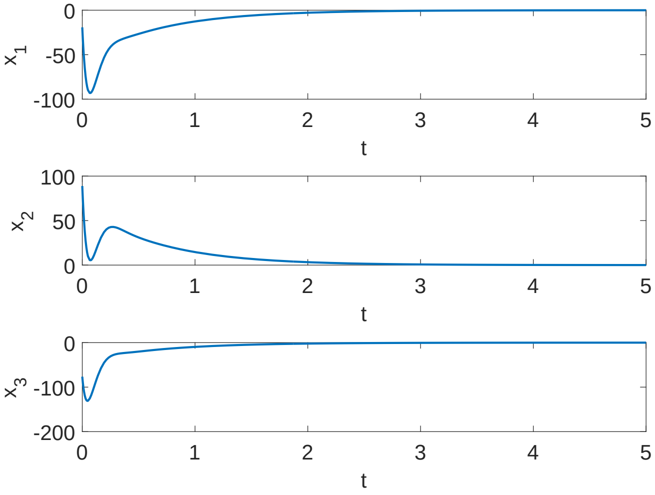
```
cvx_begin sdp

% Variable definition
variable S(n, n) symmetric
variable Z(m, n)

% LMIs
sys.A*S + S*sys.A' - sys.B*Z - Z'*sys.B' <= -eps*eye(n)
S >= eps*eye(n)

cvx_end
sys.K = Z/S; % compute K matrix
```

Simulation



Output-Feedback Control

$$A^\top P + PA - C^\top K^\top B^\top P - PBKC \prec 0,\ P \succ 0$$

- To-do: Find K, P
- Problem: BMI in K and P
- **Technique #3**: Choose M such that $BM = PB$ and $N \triangleq MK$, c.f. [4]
- New inequalities: $A^\top P + PA - C^\top K^\top MB^\top - BMKC \prec 0$
- Linear matrix (in)equalities:

$$\underbrace{A^\top P + PA}_{\text{linear in } P} - \underbrace{C^\top N^\top B^\top - BNC}_{\text{linear in } N} \prec 0,\ BM = PB,\ P \succ 0$$

- Get back $K = M^{-1}N$ (M is invertible if B has full column rank)

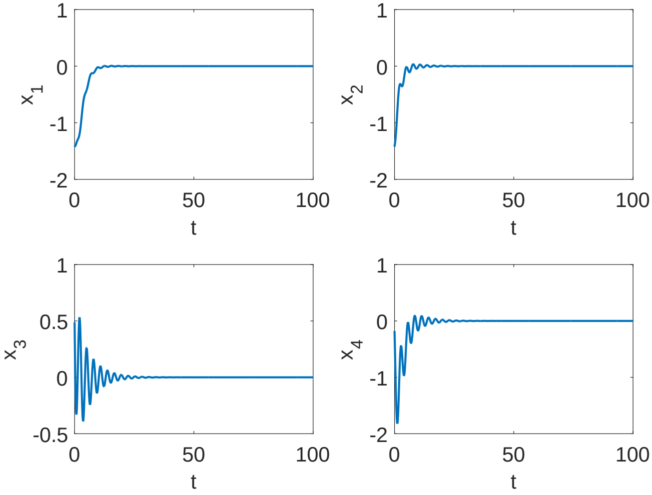
Snippet in CVX

Cool fact: CVX/YALMIP can handle *equality constraints*!

```
cvx_begin sdp quiet
% Variable definition
variable P(n, n) symmetric
variable N(m, p)
variable M(m, m)

% LMIs
P*sys.A + sys.A'*P - sys.B*N*sys.C ...
 - sys.C'*N'*sys.B' <= -eps*eye(n)
sys.B*M == P*sys.B
P >= eps*eye(n);
cvx_end
sys.K = M\N % compute K matrix
```

Simulation



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Technique #4: The Schur Complement Lemma

- QMI:

$$A^\top P + PA + Q - PBR^{-1}B^\top P \prec 0$$

- Very common trick used in control systems
- Block symmetric matrix

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\top & \mathcal{C} \end{bmatrix}$$

Schur Complement

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\top & \mathcal{C} \end{bmatrix} \prec 0 \iff \mathcal{A} \prec 0,\ \mathcal{C} - \mathcal{B}^\top \mathcal{A}^{-1} \mathcal{B} \prec 0$$

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\top & \mathcal{C} \end{bmatrix} \prec 0 \iff \mathcal{C} \prec 0,\ \mathcal{A} - \mathcal{B}\mathcal{C}^{-1} \mathcal{B}^\top \prec 0$$

Application to Optimal Control/LQR

- CT-LTI system, quadratic infinite horizon cost:

$$\mathcal{J} = \int_0^\infty \left(x^\top Q x + u^\top R u \right) dt$$

- Matrices $Q = Q^\top \succ 0$, $R = R^\top \succ 0$
- From Continuous Algebraic Riccati Equation (CARE) [1]:

$$SA^\top + AS + Z^\top B^\top + BZ + SQS + Z^\top RZ \preceq 0$$

- Taking Schur complements:

$$\begin{bmatrix} SA^\top + AS + Z^\top B^\top + BZ & S & Z^\top \\ S & -Q^{-1} & 0 \\ Z & 0 & -R^{-1} \end{bmatrix} \preceq 0$$

- Voilà! LMIs in $S, Z \implies K = ZS^{-1}$

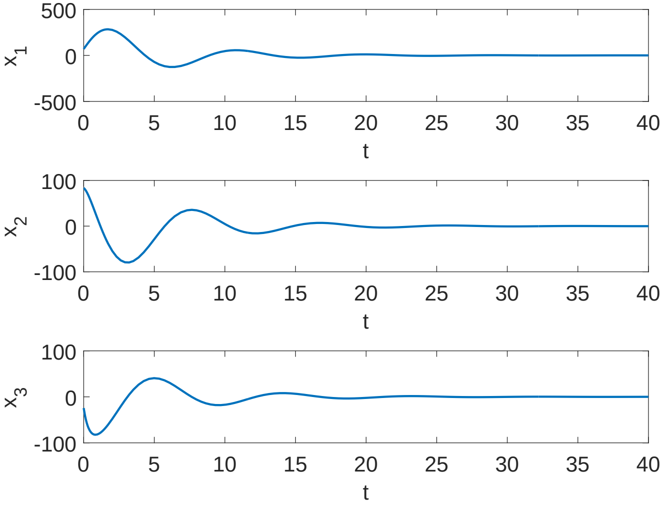
[1] Jing Li Hua O. Wang David Niemann. *Relations Between LMI and ARE with their applications to Absolute Stability Criteria, Robustness Analysis and Optimal Control.*

Snippet in CVX

```
sys.Q = 0.5*eye(n);
sys.R = [0.05, 0; 0 0.1];

cvx_begin sdp quiet
variable S(n, n) symmetric
variable Z(m, n)
% LMIs
[S*sys.A' + sys.A*S + sys.B*Z + Z'*sys.B', S, Z';...
S, -inv(sys.Q), zeros(n,m);...
Z, zeros(m,n), -inv(sys.R)] <= 0
S >= eps*eye(n)
cvx_end
sys.K = Z/S; % compute K matrix
```

Simulation



Discrete-Time LMIs

DT-LTI System with measurements:

$$\boxed{\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] \end{aligned}}$$

Linear observer:

$$\boxed{\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L(y[k] - C\hat{x}[k])}$$

- Discrete-Time Observer Lyapunov Equation:

$$(A - LC)^\top P(A - LC) - P \prec 0, \; P \succ 0$$

- This is a QMI in L

Synthesis of LMIs

- Directly taking Schur complements:

$$\begin{bmatrix} -P & (A-LC)^\top \\ A-LC & -P^{-1} \end{bmatrix} \prec 0 \implies \text{still not an LMI in } P$$

- **Technique #5**: $P = PP^{-1}P$

$$(A-LC)^\top PP^{-1}P(A-LC) - P \prec 0 \Rightarrow \begin{bmatrix} -P & \star \\ PA - YC & -P \end{bmatrix} \prec 0$$

- **Recommend**: Derive for DT-LTI state-feedback controller (you might need $P = P^{-1}PP^{-1}$)

Snippet in CVX

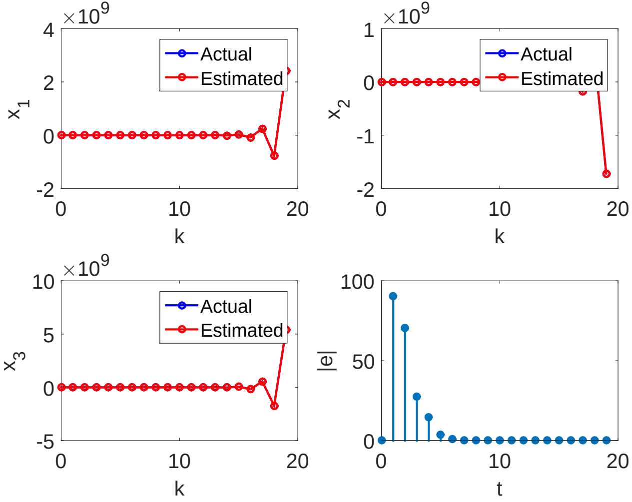
```
cvx_begin sdp quiet

% Variable definition
variable P(n, n) symmetric
variable Y(n, p)

% LMIs
[-P, sys.A'*P - sys.C'*Y'; P*sys.A - Y*sys.C, -P] <= 0
P >= eps*eye(n)

cvx_end
sys.L = P\Y; % compute L matrix
```

Simulation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Technique #6: The S-Procedure

- **Question [2]:** When does:

$$\underbrace{z^\top F_1 z \geq 0}_{z \in \mathbb{R}^n \setminus \{0\}} \implies z^\top F_0 z > 0 \text{ ?}$$

- **Answer:** If there exists a $\kappa \geq 0$ such that $F_0 - \kappa F_1 \succ 0$
- Intuition: If $F_0 - \kappa F_1 \succ 0$ for some $\kappa \geq 0$, then $F_0 \succ \kappa F_1$, so $F_0 \succ 0$ when $F_1 \succeq 0$

[2] http://stanford.edu/class/ee363/lectures/lmi-s-proc.pdf

Application to Globally Lipschitz Nonlinear Systems

Nonlinear system:

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu + B_\phi \phi(x), \\ y &= Cx \end{aligned}}$$

Observer:

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + B_\phi \phi(\hat{x}) + L(y - C\hat{x})}$$

- The nonlinearity ϕ satisfies $\|\phi(x_1) - \phi(x_2)\| \leq \beta \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^n$, (here $\beta > 0$)
- Constraint can be written as:

$$(\phi(x_1) - \phi(x_2))^\top (\phi(x_1) - \phi(x_2)) \leq \beta^2 (x_1 - x_2)^\top (x_1 - x_2)$$

$$\implies \begin{bmatrix} x_1 - x_2 \\ \phi(x_1) - \phi(x_2) \end{bmatrix}^\top \begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x_1 - x_2 \\ \phi(x_1) - \phi(x_2) \end{bmatrix} \geq 0$$

Restatement of Problem

- Ingredient #1: (from Lyapunov stability and Technique #2)
- We need $P \succ 0$ and L such that

$$\begin{bmatrix} x - \hat{x} \\ \phi(x) - \phi(\hat{x}) \end{bmatrix}^\top \begin{bmatrix} * + PA - * - YC & PB_\phi \\ B_\phi^\top P & 0 \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ \phi(x) - \phi(\hat{x}) \end{bmatrix} < 0$$

- Ingredient #2: (from constraint on ϕ)

$$\begin{bmatrix} x - \hat{x} \\ \phi(x) - \phi(\hat{x}) \end{bmatrix}^\top \begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ \phi(x) - \phi(\hat{x}) \end{bmatrix} \geq 0$$

- Compare with S-procedure (choose $z = \begin{bmatrix} x - \hat{x} & \phi(x) - \phi(\hat{x}) \end{bmatrix}^\top$)

$$z^\top F_1 z \geq 0 \implies -z^\top F_0 z > 0 \,? \quad \longrightarrow \exists \kappa \geq 0 : F_0 + \kappa F_1 \prec 0$$

Overall LMI

$$
\begin{bmatrix} A^\top P + PA - C^\top Y^\top - YC + 2\alpha P & PB_\phi \\ B_\phi^\top P & 0 \end{bmatrix} + \kappa \begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix} \preceq 0
$$
$$
P \succ 0
$$
$$
\kappa \geq 0
$$

- Scalars $\alpha > 0$ and $\beta > 0$ are assumed to be known $\implies$ LMIs in P, Y and κ, c.f.
- Referred to as ‘incremental quadratic stability’, c.f. [5]
- Bad estimate of β introduces conservatism

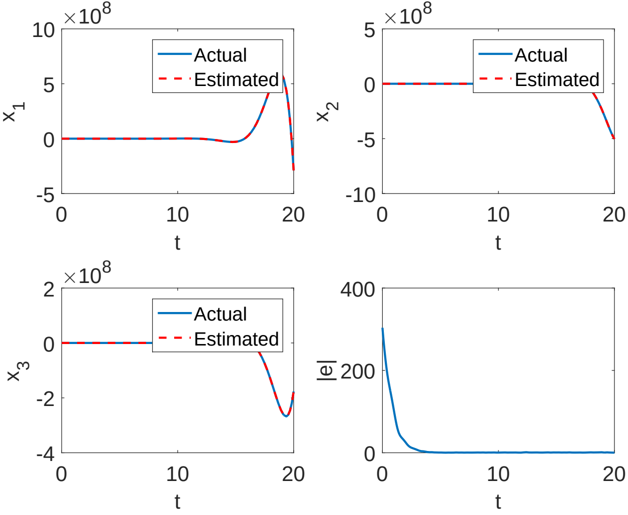
Snippet in CVX

```
cvx_begin sdp quiet
% Variable definition
variable P(n, n) symmetric
variable Y(n, p)
variable kap(1,1)

% LMIs
[P*sys.A + sys.A'*P - Y*sys.C - sys.C'*Y'...
 + 0.1*P + kap*beta^2*eye(n), P*sys.Bf;...
sys.Bf'*P, -kap*eye(1)] <= 0
P >= eps*eye(n)
kap >= 0

cvx_end
sys.L = P\Y; % compute L matrix
```

Simulation



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Technique #6: The Generalized Eigenvalue Problem

$A(x), B(x), C(x) \to$ symmetric matrices

GEVP

$$\begin{aligned} \text{minimize} \quad & \lambda \\ \text{subject to:} \quad & \lambda B(x) - A(x) \succeq 0, \\ & B(x) \succ 0, \\ & C(x) \succ 0 \end{aligned}$$

Bounding Eigenvalues

$$\lambda_1 I \preceq P \preceq \lambda_2 I$$

Application of GEVP in Robust Control

Disturbed LTI System

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bu + G{\color{red}w}\\ z &= Cx + Dw\\ u &= -Kx\end{aligned}}$$

Objective: Choose K to minimize 'peak-gain' effect of w on z, c.f. [6]

$$\begin{aligned}\text{minimize} \quad & \gamma \\ \text{subject to:} \quad & \begin{bmatrix}(A-BK)^\top P + P(A-BK) + 2\alpha P & PG \\ G^\top P & -2\alpha I\end{bmatrix} \preceq 0 \\ & \gamma \begin{bmatrix} P & 0 \\ 0 & I\end{bmatrix} - \begin{bmatrix} C^\top C & C^\top D \\ D^\top C & D^\top D\end{bmatrix} \succeq 0\end{aligned}$$

LMIs for $\mathcal{L}_\infty$ Control

- Use congruence transformation with $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$ on first MI
- Define $S = P^{-1}$, $Z = KS$
- Write $P = SPS$ in second MI and take Schur complements
- LMIs:

$$\begin{aligned} \text{minimize} \quad & \gamma \\ \text{subject to:} \quad & \begin{bmatrix} SA^\top + AS - BZ - Z^\top B^\top + 2\alpha S & G \\ G^\top & -2\alpha I \end{bmatrix} \preceq 0 \\ & \begin{bmatrix} -S & 0 & SC^\top \\ 0 & -I & D^\top \\ CS & D & -\gamma I \end{bmatrix} \preceq 0 \\ & S \succ 0 \end{aligned}$$

Snippet in CVX

```
cvx_begin sdp quiet
variable S(n, n) symmetric
variables Z(m, n) gam(1,1)
minimize(gam)
subject to
[sys.A*S + S*sys.A' - sys.B*Z - Z'*sys.B'...
 + 2*alph*S, sys.G; sys.G', -2*alph*eye(q)] <= 0
[-S, zeros(n, q), S*sys.C';...
     zeros(q,n), -eye(q), sys.D';...
     sys.C*S, sys.D, -gam*eye(p)] <= 0
S >= eps*eye(n) % eps is a very small number in MATLAB
gam >= eps
cvx_end
sys.K = Z/S; % compute K matrix
```

Simulation

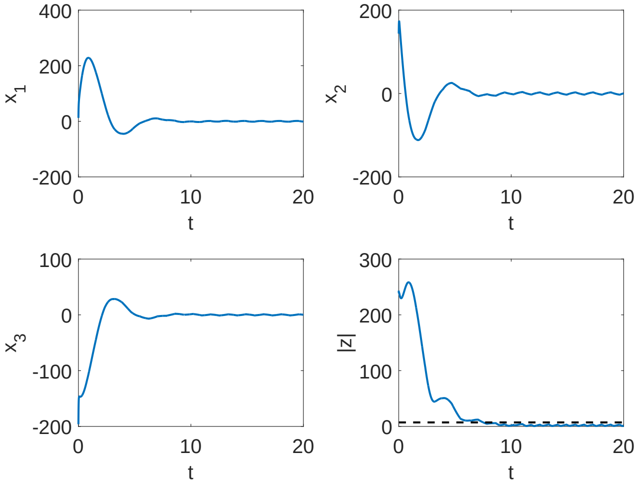

Figure: $\sqrt{\gamma} = 0.781$

Conclusions

- Quadratic stability notions can *generally* be presented as LMIs
- **Key-point**: Convex programming is efficient and solvers are easily available (user-friendly too!)
- Convex relaxations $\implies$ applications galore!
  - Networked/Decentralized/Distributed systems
  - Cybersecurity/Fault-tolerant control
  - Fuzzy control
  - Kalman filtering
  - Information theory
  - Optimal experiment design
  - Advanced control methods (sliding mode, model predictive control)
- Some methods are shown here to get LMIs for controller/observer design (many more available in, c.f. [7, 8])
- Caveat: Could be conservative!

References

- M. Grant, S. Boyd, and Y. Ye, "CVX: Matlab software for disciplined convex programming," 2008.
- J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Computer Aided Control Systems Design, 2004 IEEE International Symposium on.* IEEE, 2004, pp. 284–289.
- M. V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.
- C. A. Crusius and A. Trofino, "Sufficient LMI conditions for output feedback control problems," *Automatic Control, IEEE Transactions on*, vol. 44, no. 5, pp. 1053–1057, 1999.
- B. Açıkmeşe and M. Corless, "Observers for systems with nonlinearities satisfying incremental quadratic constraints," *Automatica*, vol. 47, no. 7, pp. 1339–1348, 2011.
- T. Pancake, M. Corless, and M. Brockman, "Analysis and control of polytopic uncertain/nonlinear systems in the presence of bounded disturbance inputs," in *American Control Conference, 2000. Proceedings of the 2000*, vol. 1, no. 6. IEEE, 2000, pp. 159–163.
- S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory.* SIAM, 1994, vol. 15.
- J. G. VanAntwerp and R. D. Braatz, "A tutorial on linear and bilinear matrix inequalities," *Journal of Process Control*, vol. 10, no. 4, pp. 363–385, 2000.