# Module 02
# CPS Background: Linear Systems Preliminaries

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**EE 5243: Introduction to Cyber-Physical Systems**

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# In Class Assessment Exam

- **Objective:** assess students' knowledge of the course prerequisites
- You are not supposed to do so much work for this exam — it's only assessment, *remember!*
- You will receive a perfect grade whether you know the answers or not

# Linear Algebra — 1

- Find the eigenvalues, eigenvectors, and inverse of matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

- **Answers**
  - Eigenvalues: $\lambda_{1,2} = 5, -2$
  - Eigenvectors: $\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, \boldsymbol{v}_2 = \begin{bmatrix} -\frac{4}{3} & 1 \end{bmatrix}^\top$
  - Inverse: $\boldsymbol{A}^{-1} = -\frac{1}{10}\begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}$
- Write $\boldsymbol{A}$ in the matrix diagonal transformation, i.e., $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{-1}$ — $\boldsymbol{D}$ is the diagonal matrix containing the eigenvalues of $\boldsymbol{A}$.
- **Answers**
  - Diagonal Transformation: $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_1 \end{bmatrix}^{-1}$

# Linear Algebra — 2

- Find the determinant, rank, and null-space set of this matrix:

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

- **Answers**
  - $\det(\boldsymbol{B}) = 0$
  - $\text{rank}(\boldsymbol{B}) = 2$
  - $\text{null}(\boldsymbol{B}) = \alpha \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \forall\ \alpha \in \mathbb{R}$
- Is there a relationship between the determinant and the rank of a matrix?
- **Answer**
  - Yes! Matrix drops rank if determinant = zero → minimal of 1 zero evalue
- True or False?
  - $\boldsymbol{AB} = \boldsymbol{BA}$ for all $\boldsymbol{A}$ and $\boldsymbol{B}$
  - $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible → $(\boldsymbol{A} + \boldsymbol{B})$ is invertible

# LTI Systems — 1

- LTI dynamical system can be represented as follows:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t), \quad \mathbf{x}_{\text{initial}} = \mathbf{x}_{t_0}, \tag{1}$$

$$\mathbf{y}(t) = \boldsymbol{C}\mathbf{x}(t) + \boldsymbol{D}\mathbf{u}(t), \tag{2}$$

  - $\mathbf{x}(t)$: dynamic state-vector of the LTI system, $\mathbf{u}(t)$: control input-vector
  - $\mathbf{y}(t)$: output-vector and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are constant matrices

- What is the closed-form, state solution to the above differential equation for any time-varying control input, i.e., $\mathbf{x}(t) = ?$

- **Answer:**

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau$$

*Can you verify that the above solution is actually correct?* Hint:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,\mathrm{d}x\right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x,\theta)\,\mathrm{d}x + f\big(b(\theta),\theta\big)\cdot b'(\theta) - f\big(a(\theta),\theta\big)\cdot a'(\theta)}_{\text{Leibniz Differentiation Theorem}}$$

# LTI Systems — 2

- What are the poles of the above dynamical system? Define asymptotic stability. What's the difference between marginal and asymptotic stability?
- **Answers:**
  - Poles: $\texttt{eig}(\boldsymbol{A})$
  - Asymptotic stability: poles **strictly** in the LHP
  - Marginal stability: some poles can be on the imaginary axis
- What is the transfer function $\left(H(s) = \dfrac{Y(s)}{U(s)}\right)$ of the above system?
- **Answers:**
  - TF:

$$H(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

  - Above TF valid for MIMO systems — it becomes a TF matrix, rather than a scalar quantity
- See Chen [1995] for more details

# LTI Systems — 3

- What is the state-transition matrix for the above system?
- **Answer:**
  - $\phi(t, t_0) = e^{\boldsymbol{A}(t-t_0)}$
  - How about linear, **time-varying** systems?
- Under what conditions is the above dynamical system controllable? Observable? Stabilizable? Detectable?
- **Answers:**
  - Controllability
  - Observability
  - Stabilizability
  - Detectability

# Optimal Control and Dynamic Observers — 1

- Define the linear quadratic regulator problem for LTI systems in both, words and equations
- **Answers:**
  - Objective: minimize the total cost of state-deviation and consumed control (i.e., taking control actions)
  - Constraints: state-dynamics, control inputs, initial conditions
  - Equations:

$$\begin{aligned}
\underset{\boldsymbol{u}(t),\boldsymbol{x}(t)}{\text{minimize}} \quad & J = \boldsymbol{x}^\top(t_1)\boldsymbol{F}(t_1)\boldsymbol{x}(t_1) + \int_{t_0}^{t_1} \left(\boldsymbol{x}^\top\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top\boldsymbol{R}\boldsymbol{u} + 2\boldsymbol{x}^\top\boldsymbol{N}\boldsymbol{u}\right) dt \\
\text{subject to} \quad & \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\
& \boldsymbol{x}(t_i) = \boldsymbol{x}_{t_i} \\
& \boldsymbol{u} \in \mathcal{U}, \ \boldsymbol{x} \in \mathcal{X}
\end{aligned} \tag{3}$$

# Optimal Control and Dynamic Observers — 2

- What is the optimal solution to an optimal control problem? What does it physically mean for CPSs?
- **Answers:**
  - It's the optimal trajectory of the control input and the corresponding state-trajectory
  - Physical meaning: *you're better off selecting this control input, among all other — possibly infinite — control input alternatives*
- What is a generic dynamic observer? Luenberger observer?
- **Answer:**
  - An estimator for internal states of the system

# Optimization — 1

- Let $f(x)$ be a multi-variable function of three variables, as follows:

$$f(x_1, x_2, x_3) = x_1 x_2 x_3 + 2x_2^3 x_1^2 - 4\cos(x_3 x_2) + \log(\cos(x_2)^2) + 4x_3 - 2x_1$$

- Find the Jacobian and Hessian of $f(\mathbf{x})$
- **Answers:**

$$\nabla f = \begin{bmatrix} 4x_1 x_2^3 + x_3 x_2 - 2 \\ 6x_1^2 x_2^2 + x_1 x_3 - (2\sin(x_2))/\cos(x_2) + 4x_3 \sin(x_2 x_3) \\ x_1 x_2 + 4x_2 \sin(x_2 x_3) + 4 \end{bmatrix},$$

$$\nabla^2 f =$$

$$\begin{bmatrix} 4x_2^3 & 12x_1 x_2^2 + x_3 & x_2 \\ 12x_1 x_2^2 + x_3 & 12x_1^2 x_2 - (2\sin(x_2)^2)/\cos(x_2)^2 + 4x_3^2 \cos(x_2 x_3) - 2 & x_1 + 4\sin(x_2 x_3) + 4x_2 x_3 \cos(x_2 x_3) \\ x_2 & x_1 + 4\sin(x_2 x_3) + 4x_2 x_3 \cos(x_2 x_3) & 4x_2^2 \cos(x_2 x_3) \end{bmatrix}$$

# Optimization — 2

- What is an optimization problem? An unfeasible, feasible solution, an optimal solution to a generic optimization problem?
- What is a convex optimization problem? Define it rigorously
- **Answers:**
  - An optimization problem of finding some $\boldsymbol{x}^* \in \mathcal{X}$ such that:

$$f(\boldsymbol{x}^*) = \min\{f(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{X}\}$$

    * $\mathcal{X} \subset \mathbb{R}^n$ is the feasible set and $f(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ is the objective, is called convex if $\mathcal{X}$ is a closed convex set and $f(\boldsymbol{x})$ is convex on $\mathbb{R}^n$
  - Alternatively, convex optimization problems can be written as:

$$\begin{aligned} &\text{minimize} && f(\boldsymbol{x}) && (4)\\ &\text{subject to} && g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \ldots, m && (5)\end{aligned}$$

    * $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ are convex
- See [Boyd & Vandenberghe, 2004] for more

# Optimization — 3

- What is a semi-definite program (SDP)? Define it rigorously.
- **Answers:**
  - A semidefinite program minimizes a linear cost function of the optimization variable $\boldsymbol{z} \in \mathbb{R}^n$ subject to a matrix inequality condition
  - An SDP can formulated as follows:

$$\underset{\boldsymbol{z}\in\mathbb{R}^n}{\text{minimize}} \quad f(\boldsymbol{z}) \tag{6}$$

$$\text{subject to} \quad \boldsymbol{F}(\boldsymbol{z}) \succeq \boldsymbol{O}, \tag{7}$$

* Where

$$\boldsymbol{F}(\boldsymbol{z}) \triangleq \boldsymbol{F}_0 + \sum_{i=1}^{n} z_i \boldsymbol{F}_i$$

- Given the following optimization problem,

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & c(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{h}(\boldsymbol{x}) \leq \boldsymbol{0} \\ & \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}, \end{aligned} \tag{8}$$

what are the corresponding Karush–Kuhn–Tucker (KKT) conditions?

# Questions And Suggestions?


**Thank You!**

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**IFF** you want to know more ☺

# References I

Boyd, S., & Vandenberghe, L. (2004). *Convex Optimization*. New York, NY, USA: Cambridge University Press.

Chen, C.-T. (1995). *Linear System Theory and Design*. New York, NY, USA: Oxford University Press, Inc., 2nd ed.