Module 03
Linear Systems Theory: Necessary Background

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EE 5243: Introduction to Cyber-Physical Systems

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Module 03 Outline

We will review in couple lectures the necessary background needed in linear systems theory and design. Outline of this module is as follows:

1. Computation of solution for an ODE, LTI systems
2. Stability of linear systems and Jordan blocks
3. Discrete dynamical systems
4. Controllability, observability, stabilizability, detectability
5. Design of controllers and observers
6. Linearization of nonlinear systems

LTI Systems

- LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x_{\text{initial}} = x_{t_0}, \tag{1}$$
$$y(t) = Cx(t) + Du(t), \tag{2}$$

- We now know that the solution is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

- Clearly the output solution is:

$$y(t) = \underbrace{C\left(e^{A(t-t_0)}x_{t_0}\right)}_{\text{zero input response}} + \underbrace{C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)}_{\text{zero state response}}$$

- **Question:** how do I analytically compute the solution to (1)?
- **Answer:** you need to (a) integrate and (b) compute matrix exponentials (given $A, B, C, D, x_{t_0}, u(t)$)

Matrix Exponential — 1

- Exponential of scalar variable:

$$e^a = \sum_{i=0}^{\infty} \frac{a^i}{i!} = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \cdots$$

- Power series converges $\forall\ a \in \mathbb{R}$
- How about matrices? For $A \in \mathbb{R}^{n \times n}$, matrix exponential:

$$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots$$

- What if we have a time-variable?

$$e^{tA} = \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} = I_n + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \frac{(tA)^4}{4!} + \cdots$$

Matrix Exponential Properties

For a matrix $A \in \mathbb{R}^{n \times n}$ and a constant $t \in \mathbb{R}$:

1. $Av = \lambda v \;\; \Rightarrow \;\; e^{At}v = e^{\lambda t}v$
2. [1] $\det(e^{At}) = e^{(\text{trace}(A))t}$
3. $(e^{At})^{-1} = e^{-At}$
4. $e^{A^\top t} = (e^{At})^\top$
5. If A, B commute, then: $e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$
6. $e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = e^{At_2}e^{At_1}$

[1] Trace of a matrix is the sum of its diagonal entries.

When Is It Easy to Find e^A? Method 1

Well...Obviously if we can directly use $e^A = I_n + A + \frac{A^2}{2!} + \cdots$

Three cases:

- A is nilpotent[2], i.e., $A^k = 0$ for some k. Example: $A = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$
- A is idempotent, i.e., $A^2 = A$. Example: $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$
- A is of rank one: $A = uv^T$ for $u, v \in \mathbb{R}^n$

$$A^k = (v^T u)^{k-1} A, \; k = 1, 2, \ldots$$

[2] Any triangular matrix with 0s along the main diagonal is nilpotent

Method 2 — Jordan Canonical Form

- All matrices, whether diagonalizable or not, have a Jordan canonical form: $A = TJT^{-1}$, then $e^{At} = Te^{Jt}T^{-1}$
- Generally, $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$, $J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \Rightarrow$,

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \dots & \frac{t^{n_i-1}e^{\lambda_i t}}{(n_i-1)!} \\ 0 & e^{\lambda_i t} & \ddots & \frac{t^{n_i-2}e^{\lambda_i t}}{(n_i-2)!} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_i t} \end{bmatrix} \Rightarrow e^{At} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_o t} \end{bmatrix} T^{-1}$$

- Jordan blocks and marginal stability

Example 1

- Find $e^{A(t-t_0)}$ for matrix A given by:

$$A = TJT^{-1} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1}$$

- **Solution:**

$$e^{A(t-t_0)} = Te^{J(t-t_0)}T^{-1}$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} e^{-(t-t_0)} & 0 & 0 & 0 \\ 0 & 1 & t-t_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-(t-t_0)} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1}$$

Example 2 — Quiz Time

- Consider a dynamical system defined by:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = 0,\ x(t_0) = x(1) = [0\ \ 1\ \ 1]^\top$$

- Determine $y(t)$ if $u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^+(t)$.
- **Solution:** $y(t) = \begin{bmatrix} 0.5(t-1)^2 \\ t-1 \end{bmatrix}$
- MATLAB demo

From CTLTI to DTLTI Systems

- Given A, B, C, D for a continuous time, LTI system, what are the the equivalent matrices for the discrete dynamics?
- For a sampling period T, the equivalent representation is:

$$\tilde{A} = e^{AT}, \tilde{B} = \left[\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B \, d\tau\right], \tilde{C} = C, \tilde{D} = D$$

- Dynamics: $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$, $y(k) = \tilde{C}x(k) + \tilde{D}u(k)$
- MATLAB command: `[Ad,Bd]=c2d(A,B,T)`

LTI Discrete Systems

- LTI dynamical system:

$$x(k+1) = Ax(k) + Bu(k), \quad x_{\text{initial}} = x_{k_0}, \tag{3}$$
$$y(k) = Cx(k) + Du(k), \tag{4}$$

- We now know that the solution is given by:

$$x(k) = A^k x_{k_0} + \sum_{i=k_0}^{k} A^{k-1-i} Bu(i) = A^k x_{k_0} + \sum_{i=k_0}^{k} A^i Bu(k-1-i)$$

- Clearly the output solution is:

$$y(k) = \underbrace{C\left(A^k x_{k_0}\right)}_{\text{zero input response}} + \underbrace{C \sum_{i=k_0}^{k} A^{k-1-i} Bu(i) + Du(k)}_{\text{zero state response}}$$

- **Question:** how do I analytically compute the solution to (3)?
- **Answer:** you need to (a) evaluate summations and (b) compute matrix powers

Discrete LTI System Example

- Consider the following time-invariant discrete dynamics:

$$x(k+1) = T\begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix} T^{-1}x(k) + T\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

- Determine A^k. **Solution:** $A^k = T\begin{bmatrix} 0^k & 0 \\ 0 & 0.25^k \end{bmatrix} T^{-1}$
- Find the zero-state state-response and $x(9)$ given that $u(k) = 0.5^k 1^+(k)$
- **Solution?** Work it out and show it to me next time...

Controllability — 1

A CTLTI system is defined as follows:

$$\dot{x} = Ax + Bu, x(0) = x_0$$

- Over the time interval $[0, t_f]$, control input $u(t)\ \forall\ t \in [0, t_f]$ steers the state from x_0 to x_{t_f}:

$$x(t_f) = e^{At_f}x_0 + \int_0^{t_f} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Controllability Definition

CTLTI system is controllable at time $t_f > 0$ if for any initial state and for any target state (x_{t_f}), a control input $u(t)$ exists that can steer the system states from $x(0)$ to $x(t_f)$ over the defined interval.

- Reachable subspace: space of all reachable states
- DTLTI Controllability

Controllability — 2

Controllability Test

For a system with n states and m control inputs, the test for controllability is that matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times nm}$$

has full row rank (i.e., $\text{rank}(\mathcal{C}) = n$).

- The test is equivalent for DTLTI and CTLTI systems

Theorem

The following statements are equivalent:

1. $\mathcal{C}$ is full rank
2. PBH Test: for any $\lambda \in \mathbb{C}$, $\text{rank}\,[\lambda I - A \;\; B] = n$
3. Eigenvector Test: for any evector $v \in \mathbb{C}$ of A, $v^T B \neq 0$
4. For any $t_f > 0$, the so-called Gramian matrix is nonsingular:

$$W(t_f) = \int_0^{t_f} e^{A\tau} B B^\top e^{A^\top \tau}\, d\tau$$

Observability — 1

DTLTI system (n states, m inputs, p outputs):

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \tag{5}$$

$$y(k) = Cx(k) + Du(k), \tag{6}$$

- **Application**: **given that** A, B, C, D**, and** $u(k), y(k)$ **are known** $\forall k = 0 : 1 : k-1$, **can we determine** $x(0)$?
- **Solution:**

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

Observability — 2

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{k-2}B & \dots & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

- Or:

$$Y(k-1) = \mathcal{O}_k x(0) + \mathcal{T}_k U(k-1) \Rightarrow$$
$$\mathcal{O}_k x(0) = Y(k-1) - \mathcal{T}_k U(k-1)$$

- Since $\mathcal{O}_k, \mathcal{T}_k, Y(k-1), U(k-1)$ are all known quantities, then we can find a unique $x(0)$ iff $\mathcal{O}_k$ is full rank

Observability Definition

DTLTI system **is observable at time** k if the initial state $x(0)$ can be uniquely determined from any given $u(0), \dots, u(k-1), y(0), \dots, y(k-1)$.

- Unobservable subspace: null-space of $\mathcal{O}_k$

Observability — 3

Observability Test

For a system with n states and p outputs, the test for observability is that matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np \times n}$ has full column rank (i.e., $\text{rank}(\mathcal{C}) = n$).

- The test is equivalent for DTLTI and CTLTI systems

Theorem

The following statements are equivalent:

1. $\mathcal{O}$ is full rank, system is observable
2. PBH Test: for any $\lambda \in \mathbb{C}$, $\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$
3. Eigenvector Test: for any evector $v \in \mathbb{C}$ of A, $Cv \neq 0$
4. The matrices $\sum_{i=0}^{n-1} (A^\top)^i C^\top C A^i$ for the DTLIT and $\int_0^t e^{A^\top \tau} C^\top C e^{A\tau} \, d\tau$ for the CTLTI are nonsingular

Example

- Consider a dynamical system defined by:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is this system controllable?
- Is this system observable?
- **Answers:** Yes, Yes!
- MATLAB commands: `ctrb, obsv`

Controller Design

- Open-loop control: design $u(t)$ directly, i.e., through optimization or learning
- Closed-loop control: design $u(t)$ as a function of state, i.e., $u(t) = g(x, t)$
- Linear state-feedback (LSF) control: design matrix K such that the control $u(t) = -Kx(t) + v(t)$ yields a desirable state-response
- Dynamics under LSF:

$$\dot{x}(t) = (A - BK)x(t) + Bv(t), \;\; v(t) \text{ is a reference signal}$$

- **Objective:** design K such that eigenvalues of $A - BK$ are stable or at a certain location
- **Fact:** if the system **is controllable,** $\texttt{eig}(A - BK)$ **can be arbitrarily reassigned**

Stabilizability

Stabilizability Definition

DTLTI or CTLIT system, defined by (A, B), is stabilizable if there exists a matrix K such that $A - BK$ is stable.

Stabilizability Theorem

DTLTI or CTLIT system, defined by (A, B) is stabilizable if all its uncontrollable modes correspond to stable eigenvalues of A.

Facts:

- A is stable $\Rightarrow (A, B)$ is stabilizable
- (A, B) is controllable $\Rightarrow (A, B)$ is stabilizable as well
- (A, B) is not controllable $\Rightarrow$ it could still be stabilizable

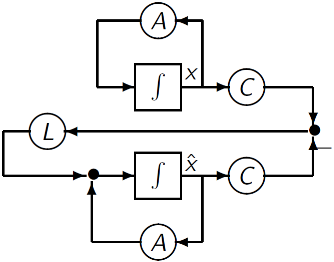
Observer Design

Original system with unknown $x(0)$:

$$\dot{x} = Ax,$$
$$y = Cx$$

Simulator with linear feedback:

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}), \quad \hat{x}(0) = 0$$
$$\hat{y} = C\hat{x}$$



- Define dynamic estimation error: $e(t) = x(t) - \hat{x}(t)$
- Error dynamics:

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - LC)(x(t) - \hat{x}(t)) = (A - LC)e(t)$$

- Hence, $e(t) \to 0$, as $t \to \infty$ if $\texttt{eig}(A - LC) < 0$
- **Objective:** design observer/estimator gain L such that $\texttt{eig}(A - LC) < 0$ or at a certain location

Detectability

Detectability Definition

DTLTI or CTLIT system, defined by (A, C), is detectable if there exists a matrix L such that $A - LC$ is stable.

Detectability Theorem

DTLTI or CTLIT system, defined by (A, C) is detectable if all its unobservable modes correspond to stable eigenvalues of A.

Facts:

- A is stable $\Rightarrow (A, C)$ is detectable
- (A, C) is observable $\Rightarrow (A, C)$ is detectable as well
- (A, B) is not observable $\Rightarrow$ it could still be detectable
- If system has some unobservable modes that are unstable, then no gain L can make $A - LC$ stable
- $\Rightarrow$ Observer will fail to track system state

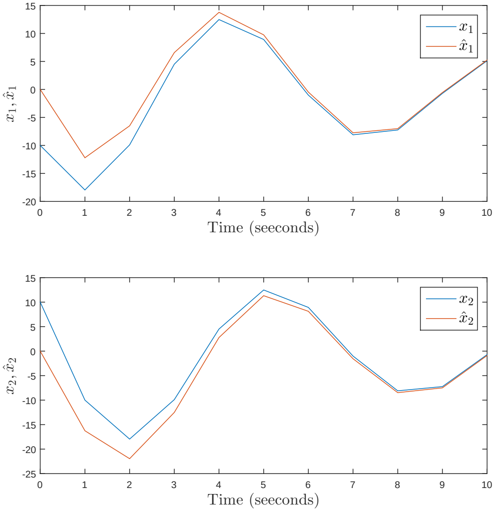
Example — Controller Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Is the system stable? What are the eigenvalues?
- **Solution:** unstable, $\texttt{eig}(A) = 4, -2$
- Find linear state-feedback gain K (i.e., $u = -Kx$), such that the poles of the closed-loop controlled system are -3 and -5
- Characteristic polynomial: $\lambda^2 + (k_1 - 2)\lambda + (3k_2 - k_1 - 8) = 0$
- **Solution:** $u = -Kx = -[10 \;\; 11] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -10x_1 - 11x_2$
- MATLAB command: $K = $ `place(A,B,eig_desired)`
- What if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, can we stabilize the system?

Example — Observer Design

- Given a system characterized by $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$
- Find linear state-observer gain $L = [l_1 \; l_2]^\top$ such that the poles of the estimation error are -5 and -3
- Characteristic polynomial:
  $\lambda^2 + (-2 + l_2 + 0.5l_1)\lambda + (-8 + 0.5l_2 + 2.5l_1) = 0$
- **Solution:** $L = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$
- MATLAB command: $L =$ `place(A',C',eig_desired)`

MATLAB Example

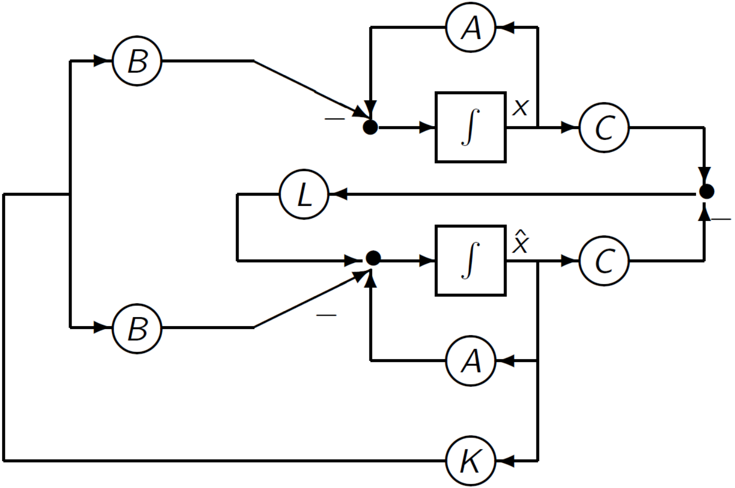



```
A=[1 -0.8; 1 0];
B=[0.5; 0];
C=[1 -1];
% Selecting desired poles
eig_desired=[.5 .7];
L=place(A',C',eig_desired)';
% Initial state
x=[-10;10];
% Initial estimate
xhat=[0;0];
% Dynamic Simulation
XX=x;
XXhat=xhat;
T=10;
% Constant Input Signal
UU=.1*ones(1,T);
for k=0:T-1,
u=UU(k+1);
y=C*x;
yhat=C*xhat;
x=A*x+B*u;
xhat=A*xhat+B*u+L*(y-yhat);
XX=[XX,x];
XXhat=[XXhat,xhat];
end
% Plotting Results
subplot(2,1,1)
plot(0:T,[XX(1,:);XXhat(1,:)]);
subplot(2,1,2)
plot(0:T,[XX(2,:);XXhat(2,:)]);
```

Observer-Based Control — 1

- Recall that for LSF control: $u(t) = -Kx(t)$
- What if $x(t)$ is not available, i.e., it can only be estimated?
- **Solution:** get $\hat{x}$ by designing L
- Apply LSF control using $\hat{x}$ with a LSF matrix K to both the original system and estimator
- **Question:** how to design K and L simultaneously? Poles of the closed-loop system?

Observer-Based Control — 2



Observer-Based Control — 3

- Closed-loop dynamics:

$$\dot{x}(t) = Ax(t) - BK\hat{x}(t) \quad (7)$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - \hat{y}(t)) - BK\hat{x}(t) \quad (8)$$

- Or

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

- Transformation: $\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$

- Hence, we can write:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{A_{\text{cl}}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

- If the system is controllable & observable $\Rightarrow \texttt{eig}(A_{textcl})$ can be arbitrarily assigned by proper K and L

- What if the system is stabilizable and detectable?

Nonlinear Systems

- Many dynamical systems are not originally linear
- To analyze the system (stability, observability, controllability), we need a linearized representation of the system
- For a nonlinear dynamical system $\dot{x}(t) = f(x)$, follow this procedure to linearize:
  1. Put the ODE in state-space vector form
  2. Find all equilibrium points by solving $f(x) = 0$
  3. List all the possible solutions: $x_e^1, x_e^2, x_e^3, \ldots$
  4. Find the Jacobian matrix of the nonlinear dynamics, $Df(x)$
  5. Linearize using Taylor series expansion:

$$\dot{x}(t) = f(x_e^i) + Df(x)\Big|_{x=x_e^i}(x - x_e^i)$$

  6. Determine which equilibrium points are stable. If $Df(x)\Big|_{x=x_e^i} \prec 0$, the equilibrium point is locally stable.

Linearization Example

$$\begin{aligned}\dot{x}_1 &= -x_1^2 + x_2 \\ \dot{x}_2 &= 3 - x_2 - x_3 \\ \dot{x}_1 &= 2 - x_3\end{aligned}$$

- Find all the equilibrium points for the given nonlinear system and determine their corresponding local stability
- **Solution:**

$$x_e^1 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \text{asymptotically stable}$$

$$x_e^2 = \begin{bmatrix}-1\\1\\2\end{bmatrix}, \text{unstable}$$

Questions And Suggestions?

Thank You!

Please visit

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IFF you want to know more ☺