Module 07
Dynamic State Estimation for Dynamical Systems

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EE 5243: Introduction to Cyber-Physical Systems

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Module 07 Outline

In this module, we will:

1. Introduce dynamic state estimation (DSE)
2. Discuss classes of observers/estimators + Applications
3. Briefly discuss stochastic estimators — Kalman filter & Co.
4. Deterministic observers
5. Unknown input observers for linear & nonlinear systems
6. Examples

CPSs & Dynamic State Estimation

- What is *dynamic state estimation* (DSE)?
  - Accurately depicting what's happening inside a system
- Precisely: estimating internal system states
  - In circuits: *voltages and currents*
  - Water networks: *amount of water flowing*
  - Chemical plants: *concentrations*
  - Robots and UAVs: *location & speed*
  - Humans: *temperature, blood pressure, glucose level*
- *So how does having estimates help me?*
  - Well, if you have estimates, you can do control
  - And if you do good control, you become better off!
- In power systems: DSE can tell me what's happening to generators & lines

$\Rightarrow$ PREVENTING/PREDICTING BLACKOUTS!

Observers vs. State Estimators — What's the Difference?

- Dynamic observer: dynamical system that *observes* the internal system state, given a set of input & output measurements
- State estimator: estimates the system's states under different assumptions
- Estimators: utilized for state estimation and parametric identification
- Observers: used for deterministic systems, Estimators: for stochastic dynamical systems
- If statistical information on process and measurement is available, stochastic estimators can be utilized
- This assumption is strict for many dynamical systems
- Quantifying distributions of measurement and process noise is very challenging

Current DSE Methods — Stochastic Estimators

- Stochastic estimators:
  - Extended Kalman Filter (EKF)
  - Unscented Kalman filter (UKF)
  - Square-root Unscented Kalman filter (SRUKF)
  - Cubature Kalman Filter (CKF)

* Stochastic estimators used if distributions of measurement & process noise are available

- System dynamics:

$$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1}$$
$$y_k = h(x_k, u_k) + v_k$$

  - $w_{k-1} \sim N(0, Q_{k-1})$ and $v_k \sim N(0, R_k)$: process & measurement noise
  - Q_{k-1} and R_k: covariance of q_{k-1} & r_k

Stochastic Estimator: The Extended Kalman Filter

- Most stochastic estimators have two main steps: predictions & updates
- EKF (=KF+Nonlinearities) algorithm:

(1) Prediction:

State esimate prediction: $\hat{\boldsymbol{x}}_{k|k-1} = f(\hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{u}_{k-1})$

Predicted covariance estimate: $\boldsymbol{P}_{k|k-1} = \boldsymbol{F}_{k-1}\boldsymbol{P}_{k-1|k-1}\boldsymbol{F}_{k-1}^{\top} + \boldsymbol{Q}_{k-1}$

(2) Update:

Innovation or measurement residual: $\tilde{\boldsymbol{y}}_k = \boldsymbol{z}_k - h(\hat{\boldsymbol{x}}_{k|k-1})$

Innovation (or residual) covariance: $\boldsymbol{S}_k = \boldsymbol{H}_k\boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^{\top} + \boldsymbol{R}_k$

Near-optimal Kalman gain: $\boldsymbol{K}_k = \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^{\top}\boldsymbol{S}_k^{-1}$

Updated covariance estimate: $\boldsymbol{P}_{k|k} = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H}_k)\boldsymbol{P}_{k|k-1}$

Updated state estimate: $\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}_k\tilde{\boldsymbol{y}}_k$

$$\boldsymbol{F}_{k-1} = \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{u}_{k-1}}, \quad \boldsymbol{H}_k = \left.\frac{\partial h}{\partial \boldsymbol{x}}\right|_{\hat{\boldsymbol{x}}_{k|k-1}}$$

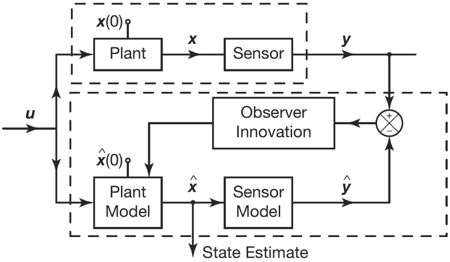
Current DSE Methods — Determinstic Estimators (Observers)

- Deterministic observers for:
  - LTI systems
  - LTI systems + Unknown Inputs
  - LTI systems + Unknown Inputs + Measurement Noise / Attack Vectors
  - Nonlinear systems (bounded nonlinearity)
  - Nonlinear systems + Unknown Inputs
  - Nonlinear systems + Unknown Inputs + Measurement Noise / Attack Vectors
  - LTI delayed systems
  - LTI delayed systems + Unknown Inputs
  - Hybrid systems
  - ... and many more

* Deterministic estimators used if measurement and process noise distributions are not available

What are Dynamical State Observers?

- Controllers often need values for the full state-vector of the plant
- This is nearly impossible in most complex systems
- *Why?* You simply can't put sensors everywhere, and some states are unaccessible
- Observer: a dynamical system that **estimates** the states of the system based on the plant's *inputs* and *outputs* [1]
- Who introduced observers? David Luenberger in 1963, Ph.D. dissertation



[1] Figure from the 2013 ACC Workshop on: *Robust State and Unknown Input Estimation: A Practical Guide to Design and Applications*, by Stefen Hui and Stanislaw H. Żak.

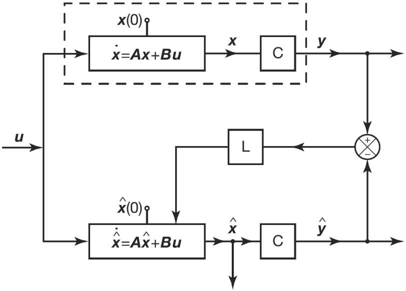
Luenberger Observer and Plant Dynamics

- Plant Dynamics: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \quad x(0) \text{ not given} \end{cases}$

Observers Dynamics: $\begin{cases} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \leftarrow Innovation \\ \dot{\hat{x}} &= A\hat{x} + Bu + LC(x - \hat{x}) \end{cases}$

- Error dynamics [2]:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)(x - \hat{x}) \to 0, \text{ as } t \to \infty, \textbf{iff } \lambda_i(A - LC) < 0$$

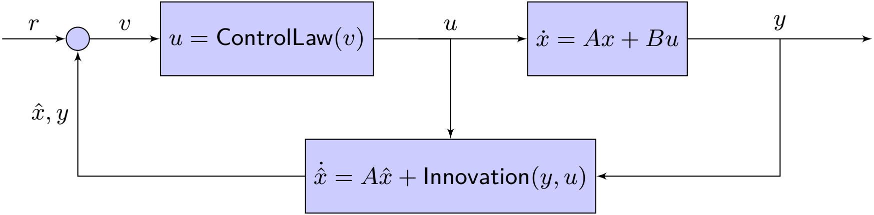


[2] Figure from the 2013 ACC Workshop on: *Robust State and Unknown Input Estimation: A Practical Guide to Design and Applications*, by Stefen Hui and Stanislaw H. Żak.

Observer-Based Control (OBC)

- After designing an observer for an LTI system, obtain state estimates $(\hat{x}(t))$
- What to do with $\hat{x}(t)$? Well, use it for control $\Rightarrow$ *Observer-Based Control!*
- OBC dynamics:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \text{Innovation}(y,u) \\ u = \text{ControlLaw}(v), \quad v = \begin{bmatrix} \hat{x} & y & r \end{bmatrix} \end{cases}$$



Observer-Based Control — The Equations

- Closed-loop dynamics:

$$\begin{aligned}\dot{x} &= Ax - BK\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + L(y-\hat{y}) - BK\hat{x}\end{aligned}$$

- Or

$$\begin{bmatrix}\dot{x} \\ \dot{\hat{x}}\end{bmatrix} = \begin{bmatrix}A & -BK \\ LC & A-LC-BK\end{bmatrix}\begin{bmatrix}x \\ \hat{x}\end{bmatrix}$$

- Transformation: $\begin{bmatrix}x \\ e\end{bmatrix} = \begin{bmatrix}x \\ x-\hat{x}\end{bmatrix} = \begin{bmatrix}I & 0 \\ I & -I\end{bmatrix}\begin{bmatrix}x \\ \hat{x}\end{bmatrix}$

- Hence, we can write:

$$\begin{bmatrix}\dot{x} \\ \dot{e}\end{bmatrix} = \underbrace{\begin{bmatrix}A-BK & BK \\ 0 & A-LC\end{bmatrix}}_{A_{\text{cl}}}\begin{bmatrix}x \\ e\end{bmatrix}$$

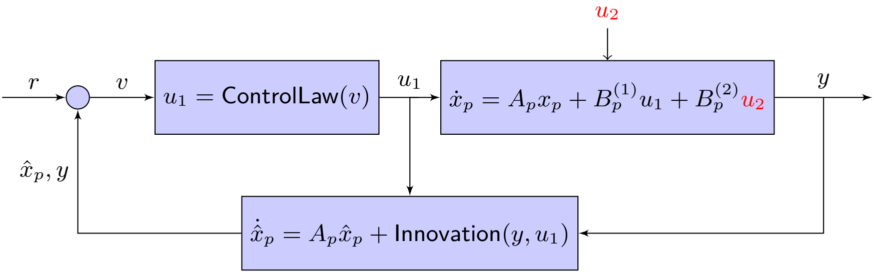
- If the system is controllable & observable $\Rightarrow \texttt{eig}(A_{\text{cl}})$ can be arbitrarily assigned by proper K and L

- What if the system is stabilizable and detectable?

Unknown Input Observers (UIO) — Why?

- Deterministic observers work well without uncertainties
- Fail to accurately estimate the plant state under uncertainties
- ***Solution?*** Design of Unknown Input Observers (UIO)
- Unknown input u_2 models uncertainties, disturbances or nonlinearities
- **Main idea:** come up with a clever innovation term that *nullifies* that effect of unknown u_2

$$\begin{cases} \dot{\hat{x}}_p = A_p\hat{x}_p + \text{Innovation}(y, u_1) \\ u_1 = \text{ControlLaw}(v), \quad v = \begin{bmatrix}\hat{x}_p & y & r\end{bmatrix} \end{cases}$$



Most Well-Known UIOs

- Different UIOs have been developed:
  - UIOs for LTI systems [Bhattacharyya, 1978]
  - Hui and Żak [Hui & Żak, 2005]
  - Sliding-mode differentiator UIO [Floquet et al., 2006]
  - Hou and Müller observer [Zhang et al., 2012]
  - Observers for Lipschitz nonlinear systems [Chen & Saif, 2006]
  - Walcott-Żak sliding mode observer [Walcott & Żak, 1987]
  - Utkin's sliding mode observer [Utkin, 1992]
- Some observers have performance guarantees
- Most UIOs have assumptions related to the LTI SS matrices
- We will discuss some UIOs

System and UIO Dynamics — One UIO Architecture

- Plant Dynamics:

$$\dot{x}_p = A_p x_p + B_p^{(1)} u_1 + B_p^{(2)} u_2$$
$$y = C_p x_p, \;\; x_p(0) \text{ not given}$$

- n states, m_1 known inputs, m_2 unknown inputs, p measurable outputs
- UIO Dynamics:

$$\begin{aligned} \dot{x}_c =& \; A_c x_c + B_c^{(1)} y + B_c^{(2)} u_1, \\ \hat{x}_p =& \; x_c + My, \end{aligned}$$

- Error dynamics:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (I - MC)(A - LC)e$$

- **Objective:** design M, L, $A_c, B_c^{(1)}$ and $B_c^{(2)}$ such that $e(t) \to 0$ as $t \to \infty$
- **Assumptions:**
  1. Pair (A_p, C_p) is detectable
  2. $\text{rank}(C_p B_p^{(2)}) = \text{rank}(B_p^{(2)})$ — *rank matching condition* implies that there must be at least as many independent outputs as unknown inputs
  3. $x_c(0) = (I - MC_p)v$, v is arbitrary vector

UIO Design

- We want to estimate x_p
- The presented observer assumes unknown initial plant conditions
- UIO is motivated by writing x_p as:

$$x_p = (I - MC_p)x_p + MC_px_p = \underbrace{(I - MC_p)x_p}_{\text{Unknown}} + \underbrace{My}_{\text{Known}}$$

- **Objective**: analyze the unknown portion of x_p, that is $x_c = (I - MC_p)x_p$
- We then have: $\dot{x}_c = (I - MC_p)\dot{x}_p + \texttt{AddedConvergenceTerm}$
- Then, obtain $\hat{x}_p = x_c + My$
- Design matrix parameters such that unknown input u_2 is nullified [Hui & Żak, 2005]

UIO Design — 2

- UIO Dynamics [Hui & Żak, 2005] (recall that $x_p = x_c + My$):

$$
\begin{aligned}
\dot{x}_c =& (I - MC_p)\dot{x}_p + \texttt{AddedConvergenceTerm} \\
=& (I - MC_p)\left(A_p x_p + B_p^{(1)} u_1 + B_p^{(2)} u_2\right) + \texttt{AddedConvergenceTerm} \\
=& (I - MC_p)\left(A_p x_c + A_p M y + B_p^{(1)} u_1 + \underbrace{L(y - C_p x_c - C_p M y)}_{\texttt{AddedConvergenceTerm}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_c =& A_c x_c + B_c^{(1)} y + B_c^{(2)} u_1, \\
\hat{x}_p =& x_c + My,
\end{aligned}
$$

where:

* $(I - MC_p)B_p^{(2)} = 0$
* $A_c = (I - MC_p)(A_p - LC_p), B_c^{(2)} = (I - MC_p)B_P^{(1)}$
* $B_c^{(1)} = (I - MC_p)(A_p M + L - LC_p M)$

UIO Design Parameters

- Given $A_p, B_p^{(1)}, B_p^{(2)}, C_p$, find M, L such that $e(t) \to 0$ as $t \to \infty$
- Precisely, $M \in \mathbb{R}^{n \times p}$ is chosen such that $\boxed{(I - MC_p)B_p^{(2)} = 0}$
- Solution: $\boxed{M = B_p^{(2)} \left(\left(C_p B_p^{(2)} \right)^{\dagger} + H_0 \left(I_p - \left(C_p B_p^{(2)} \right) \left(C_p B_p^{(2)} \right)^{\dagger} \right) \right)}$
- H_0 is a design matrix
- L is an added gain to improve the convergence of the estimated state $(\hat{x}_p)$

* **Note:** the above solution encapsulates the *projection* nature of MC_p: $(MC_p)^2 = MC_p$ and hence $I - MC_p$ is also a projection

- Basically, nullifying the unknown input by $(I - MC_p)$

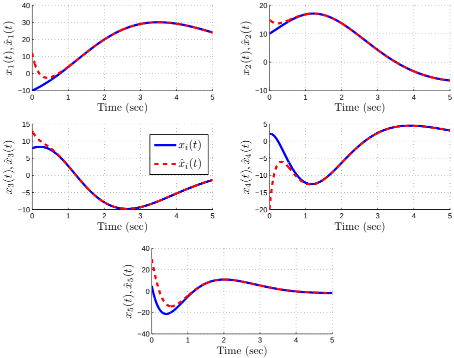
* This UIO design can be easily extended to reduced-order designs; read [Hui & Żak, 2005] for more

Numerical Results for the UIO

- Given a stable LTI MIMO system with 2 known, 2 unknown inputs, 4 outputs.
- Unknown inputs are all $u_2(t) = 0.5\sin(t)$, SS matrices:

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -5 & -10 & -10 & -5 \end{bmatrix}, \; B_p^{(1)} = B_p^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \; C_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- UIO state estimates converge to the actual states



Sliding Mode Observers — Introduction

- Sliding model control: nonlinear control method whose structure depends on the current state of the system
- Sliding mode observers (SMO): nonlinear observers driving state trajectories of estimation error to zero or to a bounded neighborhood
- SMOs have high resilience to measurement noise
- See [Utkin, 1992] for more on SMOs

System and SMO Dynamics — Second UIO Architecture

- Plant Dynamics:

$$\dot{x}_p = A_p x_p + B_p^{(1)} u_1 + B_p^{(2)} u_2$$
$$y = C_p x_p$$

* **Assumption:** unknown input u_2 is bounded, i.e., $\|u_2\| \leq \rho$

- SMO Dynamics [Hui & Żak, 2005]:

$$\begin{aligned} \dot{\hat{x}}_p &= A_p \hat{x}_p + B_p^{(1)} u_1 + L(y - \hat{y}) - B_p^{(2)} E(\hat{y}, y, \eta) \\ \hat{y} &= C_p \hat{x}_p, \end{aligned}$$

- u_1 and y: readily available signals for the SMO
- $E(\cdot)$ is defined as (η is SMO gain):

$$E(\hat{y}, y, \eta) = \begin{cases} \eta \dfrac{F(\hat{y} - y)}{\|F(\hat{y} - y)\|_2}, & \text{if } \; F(\hat{y} - y) \neq 0 \\ 0, & \text{if } \; F(\hat{y} - y) = 0. \end{cases}$$

- **SMO design objective**: find matrices F and L

SMO Design

- $F \in \mathbb{R}^{m_2 \times p}$ satisfies: $\boxed{FC_p = (B_p^{(2)})^\top P}$
- L is chosen to guarantee the asymptotic stability of $A_p - LC_p$
- Thus, for $Q = Q^\top \succ 0$, there is a unique $P = P^\top \succ 0$ such that P satisfies:

$$\boxed{(A_p - LC_p)^\top P + P(A_p - LC_p) = -Q,} \quad \boxed{P = P^\top \succ 0}$$

- $E(\cdot)$ guarantees that $e(t)$ is insensitive to the unknown input $u_2(t)$ and the estimation error converges asymptotically to zero

* If for the chosen Q, no matrix F satisfies the above equality, another matrix Q can be chosen
* A design algorithm (to find matrices F, L, P) is presented in [Hui & Żak, 2005]

LMI Solution for the SMO Design

- The SMO design problem boils down to solving matrix equalities
- *Can we solve the matrix design problem using LMIs?* **Yes!**
- We have two (nonlinear) matrix equations in terms of P, F, L:

$$
\begin{aligned}
(A_p - LC_p)^\top P + P(A_p - LC_p) &= -Q \\
P &= P^\top \\
FC_p &= (B_p^{(2)})^\top P
\end{aligned}
$$

* LMI trick: set $Y = PL$, rewrite above system of **linear** matrix equations as:

$$
\begin{aligned}
A_p^\top P + PA_p - C_p^\top Y^\top - YC_p &= -Q \\
P &= P^\top \\
FC_p &= (B_p^{(2)})^\top P
\end{aligned}
$$

SMO Design Using CVX

Sample CVX code:

```
cvx_clear
cvx_begin sdp quiet

variable P(n, n) symmetric
variable Y(n, p)
variable F(m2, p)


minimize(0)

subject to

Ap'*P + P*Ap - Y*Cp - Cp'*Y' <= 0
F*Cp-Bp2'*P==0;
P >= 0

cvx_end

L = P\Y;
```

Numerical Example

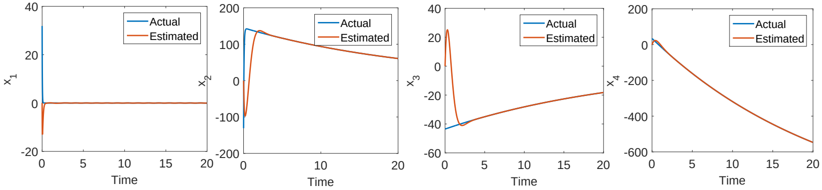
- Linearized dynamics of a power system:

$$A = \begin{bmatrix} -41 & 0 & 0 & 0 \\ 27.67 & -16.67 & -55.33 & 0 \\ 0 & 0.01 & -0.01 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 333.33 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C^\top = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- Solving for P, L, F using CVX, we obtain:

$$L = \begin{bmatrix} -28.22 & -0.12 \\ 12.23 & -39.15 \\ -0 & 5.92 \\ 0.05 & 3.76 \end{bmatrix}, F = \begin{bmatrix} 4.89 & 0.42 \end{bmatrix}, P = \begin{bmatrix} 2.45 & 0 & 0 & 0.21 \\ 0 & 0.19 & 0.36 & 0.36 \\ 0 & 0.36 & 11.43 & -15.01 \\ 0.21 & 0.36 & -15.01 & 43.62 \end{bmatrix}$$

- After simulating the observer, we obtain:



Dynamic Observers for NL Systems — Architecture # 1

- **Question:** What if system dynamics are nonlinear?
- **Answer:** Use deterministic estimators for nonlinear systems
- System dynamics:

$$\dot{x} = \underbrace{Ax + B_1 u_1}_{\text{linear terms}} + \underbrace{\phi(x,u)}_{\text{nonlinearities}} + \underbrace{B_2 u_2}_{\text{unknown inputs}}$$

- Nonlinear term in the dynamics $\phi(x,u)$ is:
  - Globally Lipschitz (*Lipschitz Continuous*):
    $$\|\phi(x,u) - \phi(z,u)\| \leq L\|x - z\|, \quad L \geq 0$$
  - One-sided Lipschitz:
    $$\langle \phi(x,u) - \phi(z,u), x - z \rangle \leq k_1 \|x - z\|^2$$
  - Quadratically inner-bounded:

$$(\phi(x,u) - \phi(z,u))^\top (\phi(x,u) - \phi(z,u)) \leq k_2\|x-z\|^2 + k_3 \langle \phi(x,u) - \phi(z,u), x - z \rangle$$

    * Lipschitz continuity $\Rightarrow$ quadratic inner-boundedness
    * Example: if $\phi(x) = \sin(x)$, then $L = 1$

Finding Lipschitz Constants — Examples

- **Example 1:** if $\phi(x) = x^2$, what is the Lipschitz constant L if x is defined on the interval $[-2, 2]$?
  - **Solution:** applying the definition, we have:

$$\|\phi(x_2) - \phi(x_1)\| = |x_2^2 - x_1^2| = |x_2 - x_1||x_2 + x_1| \leq 4|x_2 - x_1| \quad \Rightarrow \boxed{L = 4}$$

- **Example 2:** find L if $\phi(x) = \begin{bmatrix} ax_1 + bx_2 \\ 1 - \cos(cx_1) \end{bmatrix}$, $x \in \mathbb{R}^2_+$ and $a, b, c \in \mathbb{R}_+$
  - **Solution:**

$$\begin{aligned}
\|\phi(x) - \phi(z)\| &= \left\| \begin{bmatrix} ax_1 + bx_2 \\ 1 - \cos(cx_1) \end{bmatrix} - \begin{bmatrix} az_1 + bz_2 \\ 1 - \cos(cz_1) \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} a(x_1 - z_1) + b(x_2 - z_2) \\ \cos(cz_1) - \cos(cx_1) \end{bmatrix} \right\| = \left\| \begin{bmatrix} a(x_1 - z_1) + b(x_2 - z_2) \\ -2\sin(0.5c(z_1 + x_1))\sin(0.5c(z_1 - x_1)) \end{bmatrix} \right\| \\
&\leq \left\| \begin{bmatrix} a(x_1 - z_1) + b(x_2 - z_2) \\ 2\sin(0.5c(x_1 - z_1)) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a(x_1 - z_1) + b(x_2 - z_2) \\ c(x_1 - z_1) \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \begin{bmatrix} x_1 - z_1 \\ x_2 - z_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} (x - z) \right\| \leq \left\| \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \right\| \|x - z\| \\
&\leq \sqrt{2} \left\| \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \right\|_\infty \|(x - z)\| \quad \Rightarrow \quad \boxed{L = \sqrt{2}\max(a + b, c)}
\end{aligned}$$

Observer Design

- Plant dynamics under unknown inputs:

$$\begin{aligned} \dot{x} &= Ax + B_1 u_1 + \phi(x,u) + B_2 u_2 \\ y &= Cx \end{aligned}$$

- Observer dynamics [Zhang et al., 2012]:

$$\dot{\hat{x}} = A\hat{x} + B_1 u_1 + \phi(\hat{x},u) + L(y - C\hat{x})$$

- Matrix-gain L determined as follows:
  1. Given k_1, k_2, k_3, solve this LMI for $\epsilon_1, \epsilon_2, \sigma$ and $P = P^\top \succ \boldsymbol{O}$:

$$\begin{bmatrix} A^\top P + PA + (\epsilon_1 k_1 + \epsilon_2 k_2) I_n - \sigma C^\top C & P + \dfrac{k_3 \epsilon_2 - \epsilon_1}{2} I_n \\ \left(P + \dfrac{k_3 \epsilon_2 - \epsilon_1}{2} I_n\right)^\top & -\epsilon_2 I_n \end{bmatrix} < 0$$

  2. Compute observer gain L:

$$L = \frac{\sigma}{2} P^{-1} C^\top$$

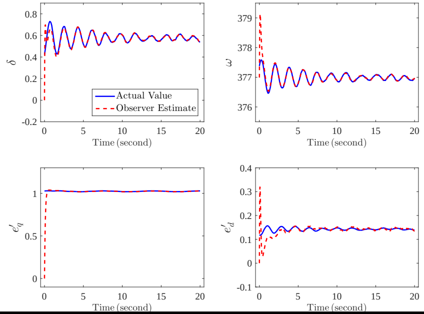
- **Extension**: reduced-order DSE
- Read [Zhang et al., 2012] to understand the derivation of the above LMI

Simulation Example

- Nonlinear power system, consider Lipschitz parameters: $\rho = \varphi = \mu = 1$
- Using CVX, we solve for $P, \epsilon_1, \epsilon_2$ and σ:
  $\epsilon_1 = 0.0122, \epsilon_2 = 0.0144, \sigma = 6.424,$
- Then, the observer gain-matrix $\boldsymbol{L}$ is computed:

$$\boldsymbol{P} = \begin{bmatrix} 0.4894 & -0.017 & 0.062 & -0.46 \\ -0.01 & 0.005 & 0 & 0.006 \\ 0.062 & 0 & 0.77 & 0.02 \\ -0.46 & 0.006 & 0.02 & 0.49 \end{bmatrix}, \boldsymbol{L} = \frac{\sigma}{2}\boldsymbol{P}^{-1}\boldsymbol{C}^\top = \begin{bmatrix} -6.02 & 15.93 & 31.86 & 12.04 \\ -15.74 & 42.50 & 85.02 & 31.503 \\ 4.20 & 0.06 & 0.12 & -8.46 \\ -3.11 & 8.69 & 17.39 & 6.23 \end{bmatrix}$$

- Given L, plot the observer response given random estimator initial conditions:



Dynamic Observer for NL Systems — Architecture # 2

- Here, we introduce an observer design for a specific class of nonlinear systems with unknown inputs
- Observer design based on the methods presented in [Chen & Saif, 2006]
- Observer design assumes:
  1. B_2 is full-column rank
  2. Nonlinear function is Lipschitz
- The design problem is formulated as an SDP

Observer Design for NL Systems

- System dynamics:

$$\begin{aligned}\dot{x} &= Ax + B_1u_1 + \phi(x) + B_2u_2\\ y &= Cx\end{aligned}$$

- Proposed observer dynamics:

$$\begin{aligned}\dot{z} &= Nz + Gu + Ly + M\phi(\hat{x})\\ \hat{x} &= z - Ey\end{aligned}$$

* Matrices E, K, N, G, L and M are obtained from the matrix equalities that ensure the asymptotic stability of estimation error

* Lipschitz constant γ: $\|\phi(x_1) - \phi(x_2)\| \leq \gamma\|x_1 - x_2\|$

- Authors in [Chen & Saif, 2006] develop matrix equations that guarantee $e = x - \hat{x}$ converges to zero
- *Can you re-derive the equations?* Design matrix parameters s.t. $e \to 0$
- **Read [Chen & Saif, 2006] to understand the design algorithm**

Observer Design Algorithm for NL Systems

Algorithm 1 Observer with Unknown Input Design Algorithm

1: **given** parameters: A, B_1, B_2, C and γ (the Lipschitz constant)

2: **compute** matrices $U, V, \bar{A}$ and $\bar{B}_1$:

$$U = -B_2(CB_2)^\dagger \qquad V = I - (CB_2)(CB_2)^\dagger$$
$$\bar{A} = (I + UC)A \qquad \bar{B}_1 = VCA$$

3: **find** matrices $\bar{Y}, \bar{K}$ and a symmetric positive definite matrix P that are a solution for this LMI:

$$\begin{bmatrix} \mathbf{\Psi}_{11} & \mathbf{\Psi}_{12} \\ \mathbf{\Psi}_{12}^\top & I_{2n} \end{bmatrix} < 0$$

where

$$\mathbf{\Psi}_{11} = \bar{A}^\top P + P\bar{A} + \bar{B_1}^\top \bar{Y}^\top \bar{Y}\bar{B}_1 - C^\top \bar{K}^\top - \bar{K}C + \gamma I,$$
$$\mathbf{\Psi}_{12} = \sqrt{\gamma}\left(P(I + UC) + \bar{Y}(VC)\right)$$

4: **obtain** matrices Y and K and the observer parameters N, G, L and M:

$$\begin{aligned} Y &= P^{-1}\bar{Y}\ ,\ K = P^{-1}\bar{K} \\ E &= U + YV,\ M = I + EC \\ N &= MA - KC,\ G = MB_1 \\ L &= K(I + CE) - MAE \end{aligned}$$

5: **simulate** the UIO given the computed matrices

SMO Design Using CVX

```
[p n]  = size(C); [n m1] = size(B1); [n m2] = size(B2);
U = -B2*pinv(C*B2); V = eye(length(C*B2))-(C*B2)*pinv(C*B2);

cvx_begin sdp quiet
variable P(n,n) symmetric
variable Ybar(n,p)
variable Kbar(n,p)

minimize(1)

subject to

P >= 0;

-[((eye(n)+U*C)*A)'*P + P*((eye(n)+U*C)*A) + ...
(V*C*A)'*Ybar' + Ybar*(V*C*A) - C'*Kbar' - Kbar*C + ...
gamma*eye(length(Kbar*C)) , sqrt(gamma)*(P*(eye(n)+U*C)+Ybar*(V*C));
(sqrt(gamma)*(P*(eye(n)+U*C)+Ybar*(V*C)))' , -eye(n)] >= 0;

cvx_end

Y = inv(P)*Ybar; K = inv(P)*Kbar;
E = U+Y*V; M = eye(n)+E*C;
N = M*A-K*C; G = M*B1;
L = K*(eye(p)+C*E)-M*A*E;
```

Numerical Example [Chen & Saif, 2006]

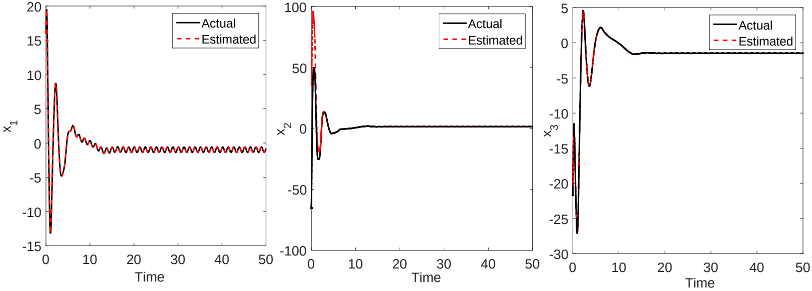
- Consider this dynamical system:

$$A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, B_1 = 0, B_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}^\top, \phi = \begin{bmatrix} 0.5\sin(x_2) \\ 0.6\cos(x_3) \\ 0 \end{bmatrix}, u_2 = 2\sin(5t)$$

- Applying the algorithm, we obtain:

$$U = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 50.25 & 0 & 0 \\ 0 & 0.89 & 0 \\ 0 & 0 & 50.25 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 0 & 1.3874 \\ 0 & -50.25 \end{bmatrix}$$

- Compute matrices K, E, M, N, G, L and simulate the observer
- Converging estimates:



Comparison between DSE Techniques

Functionality/Characteristic	**Kalman Filter Derivatives**			
	EKF	UKF	CKF	Observer
System's Nonlinearities	✗	✓	✓	✓
Feasibility	—	—	—	✗
Tolerance to Different Initial Conditions	✗	✗	✗	✓
Tolerance to Unknown Inputs	✗	✗	✗	✓
Tolerance to Cyber-Attacks	✗	✗	✗	✓
Tolerance to Process & Measurement Noise	✓	✓	✓	✓
Guaranteed Convergence	—	—	—	✓
Numerical Stability	—	—	—	✓
Computational Complexity	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$

Questions And Suggestions?

Thank You!

Please visit

engineering.utsa.edu/~taha

IFF you want to know more ☺

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