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Unknown Input Observer Design for a Class of Nonlinear Systems: an LMI Approach

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Abstract— **A full order nonlinear unknown input observer (NUIO) for a class of Lipschitz nonlinear systems with unknown inputs is designed. A sufficient NUIO existence condition which requires solving a nonlinear matrix inequality is derived. To avoid solving the nonlinear matrix inequality directly, the existence condition is then reformulated as a new sufficient existence condition in terms of an LMI. An important advantage of this LMI based condition is that it enables us to design the proposed full order NUIO using Matlab LMI toolbox and thus makes the difficult NUIO design problem an easy task for the considered class of nonlinear systems. The new sufficient condition, when applied to linear systems, is also necessary. An example is given to show how to use the LMI approach to design the proposed NUIO, and simulation results are presented.**

I. INTRODUCTION

Design of observers for linear systems subject to unknown inputs has attracted considerable attention in the past. Many types of full order and reduced order UIOs are now available. Reduced order UIOs can be found in [1]-[4] and full order observers have been designed in [5] and [6], to name only a few. Sufficient and necessary conditions for the existence of UIOs have been established in [4], [3], [6]. The approach of [2] remains to be one of the more systematic computational approaches for the design of reduced order UIOs. However, in general design of UIOs for linear systems is still not easy or systematic compuational wise, see for example the design of UIOs proposed in [6] and [4].

Since 1990s, attempts have been made to extend the existing UIO design from linear systems to nonlinear systems. UIOs for bilinear systems were designed in [7], [8], [9]. UIOs for more general nonlinear systems were also proposed in [11], [12] and [13]. The NUIOs in these papers require construction of a state transformation to change the original nonlinear systems into canonical forms. One problem for these NUIOs is that the required state transformation only exists for a limited class of nonlinear systems. The other problem is that the construction of the state transformation, requires solving PDEs and is quite difficult. As expected, the design of NUIOs is much more difficult of a problem than the design of linear UIOs. Because the UIO design for general nonlinear systems is very difficult and no systematic design method is available, some authors have considered UIO design for a class of Lipschitz nonlinear systems. [14] first extended linear UIO design to a class of Lipschitz systems and gave an LMIs and LMEs based sufficient condition for the existence the proposed UIOs. However, how to find a solution satisfying the LMIs and LMEs is not an easy task. [15] also proposed a UIO design for fault diagnosis purpose. The difficulty here lies in solving a parametric Lyapunov equation, which is very hard because no systematic method could be used. In [18], a dynamic UIO was designed for a class of Lipschitz nonlinear systems. Although the dynamic observer introduces extra design freedom, the total order of the UIO is higher than the non-dynamic UIOs. To design their UIO, an iterative algorithm was proposed, which appears to be complicated.

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Recently, the LMI approach has become very powerful in both controller and observer design, see [16] for detailed review. LMIs can be solved very efficiently using Matlab LMI toolbox [17]. However, the power of LMI approach is not well recognized in the design of UIOs. To our knowledge, the only result trying to solve UIO design problem using LMI approach is given in [10] for a class of singular bilinear systems.

Realizing the difficulties existed in the design of NUIOs mentioned earlier and the power of LMI approaches, we use the LMI approach to solve the NUIO design problem for a class of Lipschitz nonlinear systems. Unlike [14], our conditions for the existence of the UIO only involve LMIs and do not need to solve LMIs together with LMEs.

The remainder of the paper is arranged as follows: In Section 2, we introduce the nonlinear system under consideration and formulate the NUIO design problem. In Section 3, using an LMI approach, we derive a sufficient existence condition in term of an LMI. In Section 4, an example is given to show how to use the LMI approach to design the proposed NUIO, and simulation results are presented. Finally, concluding remarks are made in the last section.

II. NONLINEAR SYSTEM DESCRIPTION AND NUIO DESIGN PROBLEM FORMULATION

We consider the following nonlinear systems

$$\begin{aligned} \dot{x} &= Ax + Bu + f(x) + Dv \\ y &= Cx, \end{aligned} \qquad (1)$$

where $x \in R^n$, $u \in R^k$,$v \in R^m$, and $y \in R^p$ are the state vector, known input vector, unknown input vector and the output vector of the systems, respectively. Without loss of generality, we assume that D is of full column rank.

$f(x)$ is any nonlinear function that satisfies the following assumption.

Assumption $A1$: For $f(x)$, there exists a positive constant γ such that

$$|f(x) - f(\hat{x})| \leq \gamma |x - \hat{x}| \tag{2}$$

for all $x, \hat{x}$.

The NUIO design problem is: Construct an observer such that it can estimate the states of the considered nonlinear systems asymptotically without any knowledge of the input v.

III. An NUIO and an LMI Based Sufficient Condition for Its Existence

In this section, we will first present the structure of the proposed full order NUIO, then derive a sufficient existence condition in term of an LMI for the NUIO, and finally will show that the sufficient condition in terms of LMI actually provides a necessary and sufficient condition for the existence of UIO for linear systems.

A. NUIO Design

Inspired by the full order UIO defined in [6], we propose an NUIO with the following structure.

$$\begin{aligned} \dot{z} &= Nz + Gu + Ly + Mf(\hat{x}) \\ \hat{x} &= z - Ey \end{aligned} \tag{3}$$

where N, G, L, M are defined as

$$\begin{aligned} N &= MA - KC,\ G = MB \\ L &= K(I + CE) - MAE \\ M &= I + EC \end{aligned} \tag{4}$$

and E and K are chosen by the designers.

Let's define the state estimation error as

$$e(t) = \hat{x} - x = z - x - Ey = z - Mx \tag{5}$$

It is easy to verify that the state estimation error satisfies

$$\begin{aligned} \dot{e}(t) &= Ne + (NM + LC - MA)x \\ &\quad + (G - MB)u + M(f(\hat{x}) - f(x)) \\ &\quad - MDv \end{aligned} \tag{6}$$

B. Sufficient Condition of the Existence of NUIO

Now we can give a sufficient condition under which the observer given by (3) and (4) is indeed an NUIO.

Theorem 1: If there exist two matrices E and K and a positive definite symmetric matrix $P > 0$ such that

$$\begin{aligned} &ECD = -D; \\ &N^T P + PN + \gamma PMM^T P + \gamma I < 0 \end{aligned} \tag{7}$$

then the observer given by (3) and (4) can make $e(t)$ tend to zero asymptotically for any initial value $e(0)$.

Proof:

Based on (4), it is easy to derive $G - MB = 0$ and $NM + LC - MA = 0$. Using these facts, (6) can be rewritten as

$$\dot{e}(t) = Ne + M(f(\hat{x}) - f(x)) - MDv \tag{8}$$

Because $ECD = -D$ implies that $MD = 0$, (8) becomes

$$\dot{e}(t) = Ne + M(f(\hat{x}) - f(x)) \tag{9}$$

Choose a Lyapunov function as $V(t) = e(t)^T Pe(t)$, then it follows from (9) and (2) that

$$\begin{aligned} \dot{V} &= e^T(N^T P + PN)e + 2e^T PM(f(\hat{x}) - f(x)) \\ &\leq e^T(N^T P + PN)e + 2\|e^T PM\| \|(f(\hat{x}) - f(x))\| \\ &\leq e^T(N^T P + PN)e + 2\|e^T PM\| \gamma \|e\| \\ &\leq e^T(N^T P + PN)e \\ &\quad + \gamma(\|e^T PM\|^2 + \|e\|^2) \\ &= e^T(N^T P + PN + \gamma PMM^T P + \gamma I)e \end{aligned} \tag{10}$$

Note that $N^T P + PN + \gamma PMM^T P + \gamma I < 0$, (10) implies that $e(t)$ tends to zero asymptotically for any initial value $e(0)$. ■

In order to design the NUIO, from Theorem 1, it is clear that we only need to find E, K and P such that (7) is satisfied. One way to find them is try to solve (7) directly. However, this is very difficult because there is no systematic way to do it. This motivates us to reformulate (7) as an LMI.

To reformulate (7) as an LMI, we first give all possible solutions of E such that $ECD = -D$. Because D is of full column rank, one necessary condition for $ECD = -D$ to have solutions is that CD is also of full column rank. If CD is of full column rank, then all possible solutions of $ECD = -D$ must have the following form

$$E = -D(CD)^+ + Y(I - (CD)(CD)^+) \tag{11}$$

where $(CD)^+ = ((CD)^T(CD))^{-1}(CD)^T$ and Y can be chosen arbitrarily. For notational convenience, let's define $U = -D(CD)^+$ and $V = I - (CD)(CD)^+$, then (11) can be rewritten as

$$E = U + YV \tag{12}$$

If we substitute E given by (12) into the matrix inequality in (7), we get

$$\begin{aligned} &((I + UC)A)^T P + P(I + UC)A \\ &+ (VCA)^T Y^T P + PY(VCA) \\ &- C^T K^T P - PKC \\ &+ \gamma(P(I + UC) + PY(VC))(P(I + UC) + PY(VC))^T \\ &+ \gamma I < 0 \end{aligned} \tag{13}$$

The problem of finding E, K and $P > 0$ is now equivalent to the problem of solving (13) for Y, K and $P > 0$. In the following lemma, we will show the matrix inequality given by (13) can be reformulated as an LMI.

Lemma 1: The matrix inequality given by (13) has a solution for Y, K and a symmetric matrix $P > 0$ if and only if the following LMI has a solution for $\bar{Y}$, $\bar{K}$ and the same P.

$$\begin{pmatrix} X & X_{12} \\ X_{12}^T & -I \end{pmatrix} < 0 \quad (14)$$

where X is defined as

$$\begin{aligned} X &= ((I+UC)A)^T P + P(I+UC)A \\ &\quad +(VCA)^T \bar{Y}^T + \bar{Y}(VCA) \\ &\quad -C^T \bar{K}^T - \bar{K}C + \gamma I \\ X_{12} &= \sqrt{\gamma}[P(I+UC) + \bar{Y}(VC)] \end{aligned} \quad (15)$$

with $Y = P^{-1}\bar{Y}$ and $K = P^{-1}\bar{K}$

Proof: It is straightforward to check that

$$\begin{aligned} &\begin{pmatrix} I & W \\ 0 & I \end{pmatrix} \begin{pmatrix} X & W \\ W^T & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ W^T & -I \end{pmatrix} \\ &= \begin{pmatrix} X + WW^T & 0 \\ 0 & -I \end{pmatrix} \end{aligned} \quad (16)$$

This implies that $\begin{pmatrix} X & W \\ W^T & -I \end{pmatrix} < 0$ is equivalent to $\begin{pmatrix} X + WW^T & 0 \\ 0 & -I \end{pmatrix} < 0$, and thus equivalent to $X + WW^T < 0$. Using this fact and by letting $W = \sqrt{\gamma}[P(I + UC) + \bar{Y}(VC)]$ the lemma is proved. ■

Now we are able to reformulate the sufficient condition given by Theorem 1 as an LMI, and the existence of a feasible solution of the LMI guarantees the existence of the NUIO given by (3) and (4).

Theorem 2: Assume that CD is of full column rank and if the LMI defined by (14) and (15) has feasible solutions for $\bar{Y}$, $\bar{K}$ and a symmetric matrix $P > 0$, the NUIO given by (3) and (4) can be designed such that the state estimation error $e(t)$ tends to zero asymptotically for any initial value of $e(0)$.

Proof: Under the conditions of the theorem, it follows from Lemma 1 that the matrix inequality given by (13) has a solution of $Y = P^{-1}\bar{Y}$, $K = P^{-1}\bar{K}$ and $P > 0$. If we let $E = U + YV$ and $N = MA - KC$, (13) can be rewritten as $N^T P + PN + \gamma PMM^T P + \gamma I < 0$. Note also that $E = U + YV$ implies $ECD = -D$, we know all the conditions required by Theorem 1 are met, the theorem is proved immediately. ■

Based on Theorem 2, we can give an NUIO design algorithm as follows.

1) Compute $U = -D(CD)^+$ and $V = I - (CD)(CD)^+$.
2) Solve the LMI defined by (14) and (15) for $\bar{Y}$, $\bar{K}$ and a symmetric matrix $P > 0$.
3) Let $Y = P^{-1}\bar{Y}$ and $K = P^{-1}\bar{K}$.
4) Using Y and K, all the observer gains can be computed as

$$\begin{aligned} E &= U + YV \\ M &= I + EC \\ N &= MA - KC, \; G = MB \\ L &= K(I + CE) - MAE \end{aligned} \quad (17)$$

Remark 1: Theorem 2 shows that the NUIO design can be carried out through solving an LMI defined by (14) and (15). To solve an LMI, we have now a very powerful Matlab LMI toolbox to use. This provides a systematic way to design the NUIOs, which overcomes the design difficulties encountered in the NUIO for a class of nonlinear Lipschitz systems.

Remark 2: Although only full order NUIO is designed, a reduced order NUIO can also be designed for nonlinear Lipschitz systems by combining the UIO design technique in [4] and the NUIO design technique in this paper.

C. UIO Design For Linear systems–A Special case

Note that (1) becomes a linear system when $f(x) = 0$, hence a UIO design for linear systems can be viewed as a special case of NUIO design for Lipschitz nonlinear systems and the results obtained for NUIO are also applicable to UIO for linear systems. In fact, for linear system, we can derive a sufficient and necessary condition given in terms of LMI based on Theorem 2.

When $f(x) = 0$, the NUIO given by (3) and (4) becomes

$$\begin{aligned} \dot{z} &= Nz + Gu + Ly \\ \hat{x} &= z - Ey \end{aligned} \quad (18)$$

where N, G, L, M are defined as

$$\begin{aligned} N &= MA - KC, \; G = MB \\ L &= K(I + CE) - MAE, \; M = I + EC \end{aligned} \quad (19)$$

and E and K are chosen by the designers.

To obtain a sufficient and necessary condition given in terms of LMI, we introduce the following LMI.

$$\begin{aligned} &((I+UC)A)^T P + P(I+UC)A \\ &+(VCA)^T \bar{Y}^T + \bar{Y}(VCA) \\ &-C^T \bar{K}^T - \bar{K}C < 0 \end{aligned} \quad (20)$$

with $Y = P^{-1}\bar{Y}$ and $K = P^{-1}\bar{K}$.

Remark 3: If we let $\gamma = 0$, it is easy to show that the LMI defined by (14) and (15) is reduced to (20).

Now, we can give a sufficient and necessary condition given in terms of LMI for the existence of UIO for linear systems.

Theorem 3: Assume that $f(x) = 0$, that is, (1) is a linear system and also that CD is of full column rank, then the UIO given by (18) and (19) exists in and only if the LMI defined by (20) has feasible solutions of $\bar{Y}$, $\bar{K}$ and a symmetric matrix $P > 0$.

Proof: The sufficient part is implied in the proof of Theorem 2. The necessity part can be proved easily using Theorem 2 in [6]. ■

Remark 4: As pointed out in [6], the design of UIO for linear systems is usually carried in a trial and error manner in the literature. The new sufficient and necessary condition provided here presents a systematic approach since it is given in terms of LMI, which can be solved efficiently using Matlab LMI toolbox.

IV. Illustrative Examples

In this section, a NUIO is designed for a nonlinear system and simulation results are presented. The following nonlinear Lipschitz system is considered.

$$\begin{aligned} \dot{x} &= Ax + Bu + f(x) + Dv \\ y &= Cx \end{aligned} \tag{21}$$

where

$$\begin{aligned} A &= \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix} \\ B = 0 \quad D &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{22}$$

and $f(x) = (\, 0.5sin(x_2) \quad 0.6cos(x_3) \quad 0\,)^T$, $v = 2sin(5t)$.

A. NUIO design

For this example, we have

$$U = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

and

$$\|f(x) - f(\hat{x})\| \leq 0.6\|x - \hat{x}\| \tag{24}$$

¿From (24), we know that we may choose $\gamma = 0.65$. Now, using the Matlab LMI toolbox, we can solve the LMI defined by (14) and (15) for $P > 0$,$\bar{Y}$, and $\bar{K}$. One feasible solution for them is found as follows.

$$P = \begin{pmatrix} 50.25 & 0 & 0 \\ 0 & 0.8992 & 0 \\ 0 & 0 & 50.25 \end{pmatrix}$$

$$\bar{Y} = \begin{pmatrix} 0 & 0 \\ 0 & 1.3874 \\ 0 & -50.25 \end{pmatrix}$$

$$\bar{K} = \begin{pmatrix} 173.5441 & 0.2944 \\ -0.8992 & -1.3874 \\ -0.2944 & -173.5441 \end{pmatrix}$$

Therefore, we can obtain Y and K as

$$Y = P^{-1}\bar{Y} = \begin{pmatrix} 0 & 0 \\ 0 & 1.543 \\ 0 & -1 \end{pmatrix}$$

and

$$K = P^{-1}\bar{K} = \begin{pmatrix} 3.4536 & 0.0059 \\ -1 & -1.543 \\ -0.0059 & 3.4536 \end{pmatrix}.$$

With Y at hand, we get

$$E = U + YV = \begin{pmatrix} -1 & 0 \\ 0 & 1.543 \\ 0 & -1 \end{pmatrix}$$

Using K and E, all NUIO gain matrices can be computed and are given below.

$$N = \begin{pmatrix} -3.4536 & 0 & -0.0059 \\ 0 & -1.543 & 0 \\ -0.0059 & 0 & -3.4536 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1.543 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 \\ -1 & 0.8378 \\ 0 & 0 \end{pmatrix} \quad G = 0$$

With N, M, L, E, the NUIO given by (3) can be easily constructed as

$$\begin{aligned} \dot{z} &= Nz + Ly + Mf(\hat{x}) \\ \hat{x} &= z - Ey \end{aligned} \tag{25}$$

B. Simulation results

To see the effectiveness of the proposed NUIO, simulation results are presented in Figures 1 and 2. Figure 1 shows the result for initial conditions given by

$$x(0) = 0, z(0) = (\, 0.3 \quad 0.3 \quad 0.3\,)^T$$

while Figure 2 shows the result for initial conditions given by

$$x(0) = 0, z(0) = (\, 3 \quad 3 \quad 3\,)^T$$

Based on Figures 1 and 2, it can seen that the observer performs as expected and the state estimation errors for the two different initial values do tend to zero asymptotically as expected.

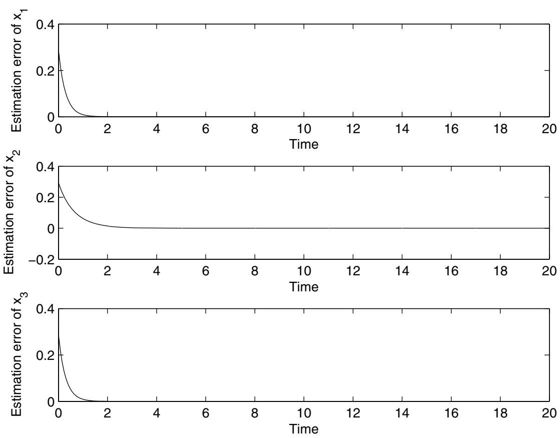

Fig. 1. State estimation error

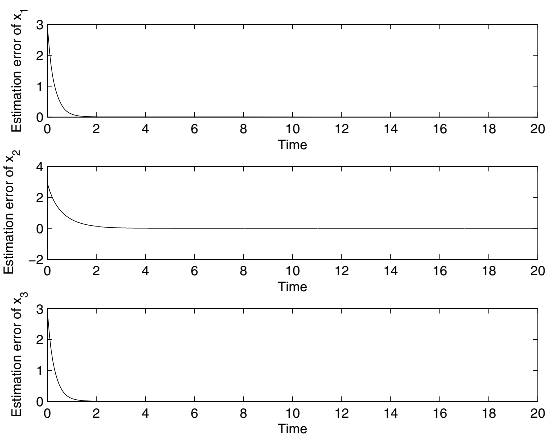

Fig. 2. State estimation error

V. CONCLUSIONS

A full order nonlinear unknown input observer (NUIO) was proposed and designed for a class of Lipschitz nonlinear systems with unknown inputs. Sufficient existence conditions were derived for the NUIO. The main advantage of the LMI based condition is that it enables us to design the proposed full order NUIO using Matlab LMI toolbox and thus makes the difficult NUIO design problem an easy task for the considered class of nonlinear systems. The new sufficient condition, when applied to linear systems, is also shown to be necessary. An example was given to show how to use the LMI approach to design an NUIO, and simulation results show that the designed NUIO can indeed make the state estimation error to asymptotically converge to zero regardless of the unknown inputs. How to extend LMI based NUIO design technique to more general nonlinear systems is a subject future research.

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