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Full-order and reduced-order observers for one-sided Lipschitz nonlinear systems using Riccati equations

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ABSTRACT

This paper aims to design full-order and reduced-order observers for one-sided Lipschitz nonlinear systems. The system under consideration is an extension of its known Lipschitz counterpart and possesses inherent advantages with respect to conservativeness. For such system, we first develop a novel Riccati equation approach to design a full-order observer, for which rigorous mathematical analysis is performed. Consequently, we show that the conditions under which a full-order observer exists also guarantee the existence of a reduced-order observer. A design method for the reduced-order observer that is dependent on the solution of the Riccati equation is then presented. The proposed conditions are easily and numerically tractable via standard numerical software. Furthermore, it is theoretically proven that the obtained conditions are less conservative than some existing ones in recent literature. The effectiveness of the proposed observers is illustrated via a simulative example.

1. Introduction

During the past decades, state estimation or observers have been widely used in system control and many other areas such as energy systems, fault detection and isolation, chaos-based secure communication, etc. [1–6]. Recent efforts are focused on the observer design of nonlinear systems [7,8]. Generally, there are two basic approaches for nonlinear observer design. The first approach is based on a nonlinear coordinate transformation by which the error dynamics admits a linear form, thus the design of state observer can be carried out by using linear techniques [1,9]. Another approach does not need the transformation and the design is directly based on the original system [2,3,7,8]. The latter is also called the directly designing method, which has been popularly used in the existing references. In practice, many real systems are global Lipschitz, or at least locally Lipschitz, which has motivated the increasing attention in designing observers for Lipschitz systems. Thau [10] first obtained a sufficient condition ensuring asymptotical stability of the observer error dynamics. Rajamani [11] derived the existence conditions of the full-order observers for Lipschitz systems. Zhu and Han [12] showed that the conditions given in [11] also guarantee the existence of a reduce-order observer. Robust observer for Lipschitz nonlinear systems subject to disturbances was designed in [13]. In [14–16], observer synthesis conditions for Lipschitz discrete-time systems and Lipschitz descriptor systems were investigated, respectively.

Most of the above results are dealt with the Lipschitz nonlinearities. However, a major limitation in the existing results is that they usually work only for the small Lipschitz constant. When the Lipschitz constant becomes large, most of the existing

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results fail to provide a solution (see, e.g., [17,18]). In mathematics, in order to overcome this drawback, the Lipschitz continuity has been generalized to one-sided Lipschitz continuity [19,20]. For many problems, the one-sided Lipschitz constant is significantly smaller than the usual Lipschitz constant, which makes it much more suitable for estimating the influence of nonlinear part (see, e.g., [15,17,19–22]).

Recently, the superiority of the one-sided Lipschitz continuity has received increasing attention in nonlinear observer design. Hu [23] first proposed asymptotical stability conditions of the error dynamics for one-sided Lipschitz systems. Reduced-order observers for such systems were given in Xu et al. [24]. Improved results on this topic could be found in [25,26]. In these references, however, the results focused on analyzing the stability of the error dynamics for a given observer, and the design problem was left as an open problem. Very recently, Abbaszadeh and Marquez [17] addressed the observer design problem for one-sided Lipschitz nonlinear systems. They provided a solution via solving a class of nonlinear matrix inequalities (NMIs). Some linear algebra techniques were employed in [17] to obtain the NMI condition. The authors also converted the NMI condition into a linear matrix inequality (LMI) for the sake of simple calculation.

In this paper, inspired by [17], we revisit the problem of observer design for one-sided Lipschitz nonlinear systems. We address both full-order and reduced-order observers design problems. A new sufficient condition ensuring the existence of full-order state observers is presented by using a novel Riccati equation approach. The condition can be easily solved via standard numerical software. Furthermore, we show that the conditions under which a full-order observer exists also guarantee the existence of a reduced-order observer. A design method for the reduced-order observer that is dependent on the solution of the Riccati equation is then presented. Moreover, we prove that the proposed conditions are less conservative than those given in [17].

The rest of the paper is organized as follows. Section 2 introduces the preliminaries. In Section 3, we present a Riccati-type condition that ensures the asymptotical stability of the full-order observer error dynamics system. The reduced-order observer design of the one-sided Lipschitz nonlinear systems is given in Section 4. The efficiency of the approach is shown through an illustrative example in Section 5. Finally, Section 6 draws the conclusions.

Notations: $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space. $\mathbb{R}^{m\times n}$ represents the set of all $m\times n$ real matrices. $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^n$, i.e., given $x,y\in\mathbb{R}^n$, then $\langle x,y\rangle = x^Ty$, where x^T is the transpose of $x\in\mathbb{R}^n$. $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. For a symmetric matrix S, $S>0$ ($S<0$) means that the matrix is positive definite (negative definite). $\lambda_{\min}(S)$ and $\lambda_{\max}(S)$ are the minimum and maximum eigenvalues of the symmetric matrix S, respectively. $\sigma_{\max}(S)$ denotes the maximum singular value of the matrix S. In symmetric block matrices, an asterisk '$*$' represents a term induced by symmetry. I is an identity matrix with appropriate dimension.

2. Preliminaries

Consider a class of nonlinear dynamical systems described by

$$\begin{cases} \dot{x}(t) = Ax(t) + \Phi(x,u) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^m$ is the input, $y\in\mathbb{R}^p$ is the output, $A\in\mathbb{R}^{n\times n}$, and $C\in\mathbb{R}^{p\times n}$. Recall that the nonlinear function $\Phi(x,u)$ is said to be *locally Lipschitz* in a region D including the origin with respect to x, uniformly in u, if there exists a constant $\gamma>0$ satisfying

$$\|\Phi(x_1,u^*) - \Phi(x_2,u^*)\| \leqslant \gamma\|x_1-x_2\|, \quad \forall x_1,x_2\in D, \tag{2}$$

where u^* is any admissible control and γ is called the *Lipschitz constant*. If the condition (2) is valid everywhere in $\mathbb{R}^n$, then the function $\Phi(x,u)$ is said to be *globally Lipschitz*. $\Phi(x,u)$ is said to be *one-sided Lipschitz* if there exist $\rho\in\mathbb{R}$ such that $\forall x_1,x_2\in D$

$$\langle \Phi(x_1,u^*) - \Phi(x_2,u^*), x_1-x_2\rangle \leqslant \rho\|x_1-x_2\|^2, \tag{3}$$

where $\rho\in\mathbb{R}$ is called the *one-sided Lipschitz constant* [17]. By definition, it is easy to see that any Lipschitz function is also one-sided Lipschitz [14]. Next concept is quadratic inner-boundedness of the function $\Phi(x,u)$, i.e., for $\forall x_1,x_2\in\tilde{D}$, if there exist $\delta,\varphi\in\mathbb{R}$ such that

$$(\Phi(x_1,u)-\Phi(x_2,u))^T(\Phi(x_1,u)-\Phi(x_2,u)) \leqslant \delta\|x_1-x_2\|^2 + \varphi\langle x_1-x_2, \Phi(x_1,u)-\Phi(x_2,u)\rangle, \tag{4}$$

then $\Phi(x,u)$ is called *quadratic inner-boundedness* in the region $\tilde{D}$. It is easy to see that the Lipschitz continuity implies quadratic inner-boundedness. However, the converse is not true [17]. Note that φ in (4) can be any real number. In fact, if φ is restricted to be positive, then Φ must be Lipschitz.

As usual, we consider the following full-order Luenberger-like observer for system (1)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \Phi(\hat{x},u) + L(y - C\hat{x}). \tag{5}$$

Let $e(t) = x(t) - \hat{x}(t)$. Then the error dynamics of the observer are given by

$$\dot{e}(t) = (A-LC)e(t) + \Phi - \hat{\Phi}, \tag{6}$$

where $\Phi \triangleq \Phi(x,u)$ and $\hat{\Phi}\triangleq\Phi(\hat{x},u)$. From (4), we know that the following inequality holds for the estimation error

$$(\Phi - \hat{\Phi})^T(\Phi - \hat{\Phi}) \leqslant \delta\|e(t)\|^2 + \varphi e^T(t)(\Phi - \hat{\Phi}). \tag{7}$$

Our design goal is to find an observer gain L such that the error dynamics (6) is asymptotically stable assuming that Φ satisfies (3) and (4). Recently, the authors in [17] proposed an LMI solution for this full-order observer design problem. We list it as follows.

Proposition 1 [17]. *Suppose that system (1) satisfies the conditions (3) and (4) with constants ρ, δ and φ, and the observer holds the form of (5). Then the error dynamic is asymptotically stable if there exist a matrix L and positive scalars $\alpha > 0$ and $0 < \lambda < 1$ such that the following matrix inequalities problem is feasible:*

$$\begin{bmatrix} \frac{1}{2\alpha}(\lambda - \xi)I & (A - LC)^T \\ A - LC & \frac{1}{2\alpha}(\lambda - \xi)I \end{bmatrix} > 0, \tag{8}$$

$$\varphi + 2\alpha > 0, \quad \lambda > 1 - \frac{1}{\alpha^2}, \tag{9}$$

where $\xi = (\delta + 1) + \rho(\varphi + 2\alpha)$.

Lemma 1 [27]. *Let $f(\cdot)$ be a given polynomial. If λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$, then $f(\lambda)$ is an eigenvalue of the matrix $f(A)$.*

Lemma 2 [28]. *If X and Y are $n \times n$ symmetric matrices such that $X \geqslant 0$ and $x^TYx > 0$ for all nonzero $x \in \mathbb{R}^n$ satisfying $x^TXx = 0$, then there exists a constant $\sigma > 0$ such that the matrix $Y + \sigma X$ is positive definite.*

Lemma 3 (The Schur complement lemma, see [29]). *For a given matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$ with $S_{11}^T = S_{11}$ and $S_{22}^T = S_{22}$, then the following conditions are equivalent:*

(1) $S < 0$,
(2) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,
(3) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

3. Full-order observer design

This section investigates the full-order observer design for one-sided Lipschitz nonlinear systems by using a Riccati equation approach. A Riccati-type sufficient condition is proposed to guarantee the asymptotical stability of the observer error dynamics. We also show that the obtained conditions are less conservative than some existing ones in recent literature.

Theorem 1. *Suppose that system (1) satisfies the conditions (3) and (4) with constants ρ, δ and φ, and the observer holds the form of (5). Then the observer error dynamics is asymptotically stable if there exist scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\sigma > 0$ such that the following Riccati inequality has a symmetric positive definite solution P:*

$$A^TP + PA + (\varepsilon_1\rho + \varepsilon_2\delta)I + \frac{1}{\varepsilon_2}\left(P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2}I\right)^2 - \sigma C^TC < 0. \tag{10}$$

The observer gain can then be chosen as

$$L = \frac{\sigma}{2}P^{-1}C^T. \tag{11}$$

Proof. For the error dynamics (6), consider the Lyapunov function $V(t) = e^T(t)Pe(t)$. Then

$$\dot{V}(t) = e^T(t)\left[(A - LC)^TP + P(A - LC)\right]e(t) + 2e^T(t)P(\Phi - \hat{\Phi}) = \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix}^T \begin{bmatrix} (A - LC)^TP + P(A - LC) & P \\ P & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix}, \tag{12}$$

where $\tilde{\Phi} = \Phi - \hat{\Phi}$.

From (3), we get $\rho e^T(t)e(t) - e^T(t)\tilde{\Phi} \geqslant 0$. Therefore, for any positive scalar ε_1,

$$\varepsilon_1 \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix}^T \begin{bmatrix} \rho I & -\frac{I}{2} \\ * & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix} \geqslant 0. \tag{13}$$

Similarly, from (7), we have

$$\varepsilon_2 \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix}^T \begin{bmatrix} \delta I & \frac{\varphi I}{2} \\ * & -I \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix} \geqslant 0, \tag{14}$$

where ε_2 is a positive scalar. Then, adding the terms on the left-hand sides of (13) and (14) to the right-hand side of (12) yields

$$\dot{V}(t) \leqslant \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix}^T \begin{bmatrix} (A-LC)^T P + P(A-LC) + (\varepsilon_1\rho + \varepsilon_2\delta)I & P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I \\ * & -\varepsilon_2 I \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{\Phi} \end{bmatrix}. \tag{15}$$

Let $L = \frac{\sigma}{2} P^{-1} C^T$. Then, (10) can be rewritten as

$$(A-LC)^T P + P(A-LC) + (\varepsilon_1\rho + \varepsilon_2\delta)I + \frac{1}{\varepsilon_2}\left(P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I\right)^2 < 0. \tag{16}$$

Using the Schur complement lemma, i.e., Lemma 3, the condition (16) is equivalent to

$$\begin{bmatrix} (A-LC)^T P + P(A-LC) + (\varepsilon_1\rho + \varepsilon_2\delta)I & P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I \\ * & -\varepsilon_2 I \end{bmatrix} < 0. \tag{17}$$

Thus, it follows from (15) that $\dot{V}(t) < 0$ for all $e(t) \neq 0$, which implies that the observer error dynamics is asymptotically stable. This completes the proof. □

Remark 1. Theorem 1 presents a Riccati-type sufficient condition to design observers for one-sided Lipschitz nonlinear systems. Note that (10) can be modified as follows:

$$A^T P + PA + (\varepsilon_1\rho + \varepsilon_2\delta)I + \frac{1}{\varepsilon_2}\left(P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I\right)^2 - \sigma C^T C = -\mu I < 0, \tag{18}$$

where $\mu > 0$, μ is a small scalar. For given ε_1, ε_2, σ, and μ, the following Riccati equality

$$A^T P + PA + (\varepsilon_1\rho + \varepsilon_2\delta)I + \frac{1}{\varepsilon_2}\left(P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I\right)^2 - \sigma C^T C = -\mu I. \tag{19}$$

can be easily solved via using MATLAB. Thus, one can find the solution of the Riccati inequality (10). On the other hand, by using Lemma 3, (10) is equivalent to $P > 0$ and

$$\begin{bmatrix} A^T P + PA + (\varepsilon_1\rho + \varepsilon_2\delta)I - \sigma C^T C & P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I \\ * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{20}$$

which can be solved using the MATLAB LMI toolbox. Therefore, the condition (10) is easily tractable via standard numerical software.

Remark 2. The relationship between Proposition 1 and Theorem 1 can be established by the following Theorem 2, which indicates that the latter is less conservative than the former.

Theorem 2. *Assume that there exist a matrix L and positive scalars $\alpha > 0$ and $0 < \lambda < 1$ such that the inequalities (8) and (9) hold. Let $P = I$, $\varepsilon_1 = \frac{\varphi + 2\alpha}{\alpha}$ and $\varepsilon_2 = \frac{1}{\alpha}$. Then there exists a constant $\sigma > 0$ together with the matrix P and the scalars ε_1 and ε_2 satisfy the inequality (10).*

Proof. Assume that L, $\alpha > 0$, and $0 < \lambda < 1$ satisfy (8) and (9). By using Lemma 3, (8) is then equivalent to

$$\frac{\lambda - \xi}{2\alpha} I - \frac{2\alpha}{\lambda - \xi}(A-LC)^T(A-LC) > 0, \tag{21}$$

i.e.,

$$(A-LC)^T(A-LC) \leqslant \frac{(\lambda - \xi)^2}{4\alpha^2} I, \tag{22}$$

where $\xi = (\delta + 1) + \rho(\varphi + 2\alpha)$. From (22), we have

$$\sigma_{\max}(A-LC) \leqslant \frac{\lambda - \xi}{2\alpha}. \tag{23}$$

Then, by using Fan's theorem [27], we get

$$\lambda_{\max}\left[A - LC + (A - LC)^T\right] \leqslant 2\sigma_{\max}(A - LC) \leqslant \frac{\lambda - \xi}{\alpha}. \tag{24}$$

Denote

$$\Pi = (A - LC)^T P + P(A - LC) + (\varepsilon_1\rho + \varepsilon_2\delta)I + \frac{1}{\varepsilon_2}\left(P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2}I\right)^2. \tag{25}$$

Let $P = I$, $\varepsilon_1 = \frac{\varphi + 2\alpha}{\alpha}$ and $\varepsilon_2 = \frac{1}{\alpha}$. Then, it follows from (24) and (16) that

$$\begin{aligned}\Pi &= A - LC + (A - LC)^T + \frac{\rho(\varphi + 2\alpha) + \delta}{\alpha}I \leqslant \lambda_{\max}\left[A - LC + (A - LC)^T\right]I + \frac{\rho(\varphi + 2\alpha) + \delta}{\alpha}I \\ &\leqslant \frac{\lambda - \xi}{\alpha}I + \frac{\rho(\varphi + 2\alpha) + \delta}{\alpha}I = \frac{\lambda - 1}{\alpha}I.\end{aligned} \tag{26}$$

Note that $\alpha > 0$ and $0 < \lambda < 1$. Thus, we have $\Pi < 0$, which can be rewritten as

$$A^T P + PA + (\varepsilon_1\rho + \varepsilon_2\delta)I + \frac{1}{\varepsilon_2}\left(P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2}I\right)^2 - C^T L^T P - PLC < 0. \tag{27}$$

Inequality (27) implies for all nonzero $x \in \mathbb{R}^n$ such that $x^T C^T C x = 0$, i.e., $Cx = 0$, we must have

$$x^T\left[A^T P + PA + (\varepsilon_1\rho + \varepsilon_2\delta)I + \frac{1}{\varepsilon_2}\left(P + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2}I\right)^2\right]x < 0. \tag{28}$$

Therefore, it follows from Lemma 2 that there exists a constant $\sigma > 0$ such that (10) is satisfied, which completes the proof of Theorem 3. □

Remark 3. Note that Proposition 1 and Theorem 1 are derived via different techniques. Now Theorem 2 shows that if there exists a solution for (8) and (9) in Proposition 1, then there exists a solution for (10) in Theorem 1 as well. On the other hand, one can give some examples to show that it is possible that the conditions (8) and (9) in Proposition 1 have no solution, but (10) is still feasible (see, e.g., the example in Section 5). This observation, together with Theorem 2, implies that the sufficient conditions in our paper are less conservative than those in [17].

4. Reduced-order observer design

A reduced-order observer estimates only partial states that are independent of the output of system. Hence, it has a lower dimension than that of the full-order observer. This implies that reduced-order observer can be constructed with fewer integrators and the whole control system will be simpler. This section presents a reduced-order observer for the one-sided Lipschitz nonlinear systems. It is shown that the conditions under which a full-order observer exists also guarantee the existence of a reduced-order observer.

In this section, we decompose A and P into block matrices as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, \tag{29}$$

where $A_{11}, P_1 \in \mathbb{R}^{p\times p}$ and $A_{22}, P_3 \in \mathbb{R}^{(n-p)\times(n-p)}$. For simplicity, we let $C = [I_p \;\; 0]$. The reduced-order observer of system (1) can be designed as follows:

$$\begin{cases} \dot{\hat{z}}_2 = (A_{22} + KA_{12})\hat{z}_2 + [K(A_{11} - A_{12}K) + A_{21} - A_{22}K]y + (K \quad I_{n-p})\Phi\left(\begin{pmatrix} y \\ \hat{z}_2 - Ky \end{pmatrix}, u\right) \\ \hat{z}_1 = \hat{x}_1 = y \\ \hat{x}_2 = \hat{z}_2 - Ky \end{cases} \tag{30}$$

where $K = P_3^{-1}P_2^T \in \mathbb{R}^{(n-p)\times p}$.

Theorem 3. *Let $C = [I_p \;\; 0]$. Assume that system (1) satisfies the conditions (3) and (4) with constants ρ, δ and φ. If there exist $P > 0$ and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\sigma > 0$ such that the inequality (10) is satisfied, then (30) is a reduced-order observer for system (1).*

Proof. Let P be a positive definite solution of (10). Let $\Omega = (A_{22} + KA_{12})^T P_3 + P_3(A_{22} + KA_{12})$, where $K = P_3^{-1}P_2^T \in \mathbb{R}^{(n-p)\times p}$. Then, from (10), we have

$$\Omega + (\varepsilon_1\rho + \varepsilon_2\delta)I_{n-p} + \frac{P_2^T P_2}{\varepsilon_2} + \frac{1}{\varepsilon_2}\left(P_3 + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2}I_{n-p}\right)^2 < 0. \tag{31}$$

Take a coordinate transformation of $z = Tx$, where $T = \begin{bmatrix} I_p & 0 \\ K & I_{n-p} \end{bmatrix}$. Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1 = y \in \mathbb{R}^p$ and $z_2 \in \mathbb{R}^{n-p}$. Then, from (1), z_2 satisfies the following equation:

$$\dot{z}_2 = (A_{22} + KA_{12})z_2 + [K(A_{11} - A_{12}K) + A_{21} - A_{22}K]y + (K \quad I_{n-p})\Phi\left(\begin{pmatrix} y \\ z_2 - Ky \end{pmatrix}, u\right) \tag{32}$$

Subtracting the first equation of (30) from (32), the error $\tilde{z}_2 = z_2 - \hat{z}_2$ is then governed by

$$\dot{\tilde{z}}_2 = (A_{22} + KA_{12})\tilde{z}_2 + (K \quad I_{n-p})\Delta\Phi, \tag{33}$$

where

$$\Delta\Phi = \Phi\left(\begin{pmatrix} y \\ z_2 - Ky \end{pmatrix}, u\right) - \Phi\left(\begin{pmatrix} y \\ \hat{z}_2 - Ky \end{pmatrix}, u\right). \tag{34}$$

Consider the Lyapunov function candidate

$$V_2(t) = \tilde{z}_2^T P_3 \tilde{z}_2. \tag{35}$$

Then its time derivative along the trajectories of (33) is

$$\begin{aligned}\dot{V}_2(t) &= \tilde{z}_2^T\left[(A_{22} + KA_{12})^T P_3 + P_3(A_{22} + KA_{12})\right]\tilde{z}_2 + 2\tilde{z}_2^T P_3(K \quad I_{n-p})\Delta\Phi = \tilde{z}_2^T\left[(A_{22} + KA_{12})^T P_3 + P_3(A_{22} + KA_{12})\right]\tilde{z}_2 + 2\tilde{z}_2^T(P_2^T \quad P_3)\begin{bmatrix} \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix}^T \begin{bmatrix} (A_{22} + KA_{12})^T P_3 + P_3(A_{22} + KA_{12}) & P_2^T & P_3 \\ P_2 & 0 & 0 \\ P_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix}. \end{aligned} \tag{36}$$

where $\Delta\Phi_1 \in \mathbb{R}^p$, $\Delta\Phi_2 \in \mathbb{R}^{n-p}$. Using the one-sided Lipschitz condition (3), we have

$$\left\langle \Delta\Phi, \begin{pmatrix} 0 \\ \tilde{z}_2 \end{pmatrix} \right\rangle \leqslant \rho \left\| \begin{pmatrix} 0 \\ \tilde{z}_2 \end{pmatrix} \right\|^2, \tag{37}$$

The above inequality implies that $\Delta\Phi_2^T \tilde{z}_2 \leqslant \rho \tilde{z}_2^T \tilde{z}_2$. Therefore, for any positive scalar ε_1, we have

$$\varepsilon_1 \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix}^T \begin{bmatrix} \rho I_{n-p} & 0 & -\frac{I_{n-p}}{2} \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix} \geqslant 0. \tag{38}$$

On the other hand, from the condition (4) of quadratic inner-boundedness, we get

$$\Delta\Phi^T \Delta\Phi \leqslant \delta \left\| \begin{pmatrix} 0 \\ \tilde{z}_2 \end{pmatrix} \right\|^2 + \varphi \left\langle \begin{pmatrix} 0 \\ \tilde{z}_2 \end{pmatrix}, \Delta\Phi \right\rangle, \tag{39}$$

which implies that

$$\Delta\Phi_1^T \Delta\Phi_1 + \Delta\Phi_2^T \Delta\Phi_2 \leqslant \delta \tilde{z}_2^T \tilde{z}_2 + \varphi \tilde{z}_2^T \Delta\Phi_2, \tag{40}$$

Thus, for any positive scalar ε_2, we have

$$\varepsilon_2 \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix}^T \begin{bmatrix} \delta I_{n-p} & 0 & \frac{\varphi I_{n-p}}{2} \\ * & -I_p & 0 \\ * & * & -I_{n-p} \end{bmatrix} \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix} \geqslant 0. \tag{41}$$

Then, adding the left terms of (38) and (41) to the right-hand side of (36) yields

$$\dot{V}_2(t) \leqslant \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix}^T \Xi \begin{bmatrix} \tilde{z}_2 \\ \Delta\Phi_1 \\ \Delta\Phi_2 \end{bmatrix}, \tag{42}$$

where

$$\Xi = \begin{bmatrix} (A_{22} + KA_{12})^T P_3 + P_3(A_{22} + KA_{12}) + (\varepsilon_1\rho + \varepsilon_2\delta)I_{n-p} & P_2^T & P_3 + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I_{n-p} \\ P_2 & -\varepsilon_2 I_p & 0 \\ P_3 + \frac{\varphi\varepsilon_2 - \varepsilon_1}{2} I_{n-p} & 0 & -\varepsilon_2 I_{n-p} \end{bmatrix}. \tag{43}$$

Using Lemma 3, the condition (31) is equivalent to $\Xi < 0$. Thus, it follows from (42) that $\dot{V}_2(t) < 0$ for all $\tilde{z}_2 \neq 0$. Therefore, according to the standard Lyapunov stability theory, the error dynamics (33) is asymptotically stable. This completes the proof. □

Remark 4. It should be pointed out that in Theorem 3 the assumption of C being of the form of $[I_p\ 0]$ does not lose the generality. In fact, if C is full rank, then there exists a coordinate transformation T such that $CT^{-1} = [I_p\ 0]$.

5. Simulation study

In this section, we illustrate the proposed design through a numerical example. For convenience, we borrow the example illustrated in [17]. Consider a dynamical nonlinear system described by (1) with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \Phi(x) = \begin{bmatrix} -x_1(x_1^2 + x_2^2) \\ -x_2(x_1^2 + x_2^2) \end{bmatrix}, \quad C = [1 \quad 0]. \tag{44}$$

The above equation can be used to describe the motion of a moving object. Note that $y = x_1$, i.e., only the state x_1 is available. So our goal is to design an observer to estimate x_2. We first consider the full-order observer. From [17], we know that $\Phi(x)$ is globally one-sided Lipschitz with $\rho = 0$. The system is locally Lipschitz and on any set $D = \{x \in \mathbb{R}^2 : \|x\| \leqslant r\}$, the Lipschitz constant l is $3r^2$. Consider the set $\tilde{D} = \{x \in \mathbb{R}^2 : \|x\| \leqslant r\}$. Let

$$r = \min\left(\sqrt{-\frac{\varphi}{4}}, \sqrt[4]{\delta + \frac{\varphi^2}{4}}\right), \quad \varphi < 0, \ \delta + \frac{\varphi^2}{4} > 0.$$

Then one can verify the quadratically inner-bounded property of $\Phi(x)$ in $\tilde{D}$ [17]. As the system is globally one-sided Lipschitz, i.e., $D = \mathbb{R}^2$, $D \cap \tilde{D} = \tilde{D}$. Note that the region $\tilde{D}$ can be made arbitrarily large by choosing appropriate values for γ and β.

Now let us consider the solvability of the conditions (8) and (9) of Proposition 1. Denote $\theta = \frac{\lambda - \xi}{2\alpha}$. Let $L = [l_1\ l_2]^T$. Then (8) can be rewritten as

$$\begin{bmatrix} \theta & 0 & 1 - l_1 & -1 - l_2 \\ 0 & \theta & 1 & 1 \\ 1 - l_1 & 1 & \theta & 0 \\ -1 - l_2 & 1 & 0 & \theta \end{bmatrix} > 0. \tag{45}$$

It follows from (45) that

$$\begin{bmatrix} \theta & 1 \\ 1 & \theta \end{bmatrix} > 0.$$

Hence, $\theta > 1$, i.e., $\lambda - \xi > 2\alpha$. Note that $\xi = \delta + 1$, $\varphi + 2\alpha > 0$ and $0 < \lambda < 1$. We then have $\delta < \lambda - 2\alpha - 1 < -2\alpha < \varphi$. Given $\varphi = -100$, by applying Proposition 1, one has to assume $\delta < -100$. However, we need not to assume $\delta < -100$ while applying the condition (10) in Theorem 1. In fact, if take $\delta = -99$, $\varphi = -100$ and solve the LMI (20), we get

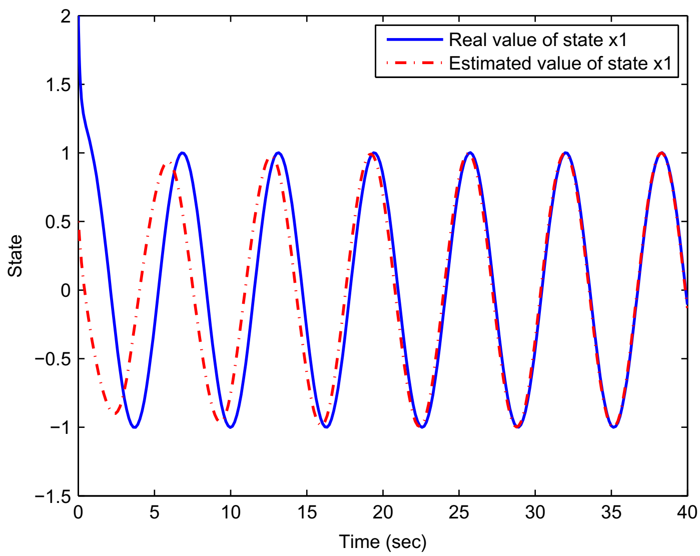

Fig. 1. The simulation for state x_1 by the full-order observer (5).

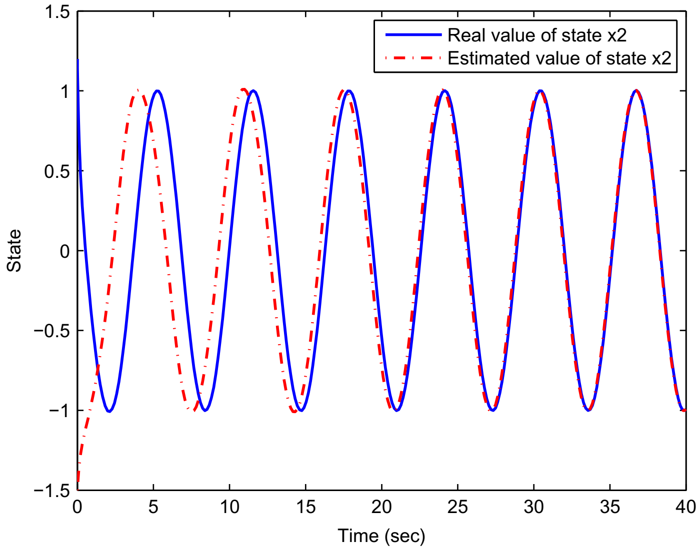

Fig. 2. The simulation for state x_2 by the full-order observer (5).

$$P = \begin{bmatrix} 1236.2 & -23.1 \\ -23.1 & 1218.5 \end{bmatrix}, \quad \sigma = 557.1775.$$

Hence

$$L = \frac{\sigma}{2} P^{-1} C^T = \begin{bmatrix} 0.2254 \\ 0.0043 \end{bmatrix}.$$

Then we can use (5) to design a full-order observer to estimate the state. For simulation, we set initial conditions as follows: $x(0) = (2\ 1.2)^T, \hat{x}(0) = (0.5\ \ -1.5)^T$. Figs. 1 and 2 show the trajectories of $x_1(t)$ and $x_2(t)$ and their estimates, respectively. The trajectories of error dynamics are displayed in Fig. 3. The simulation results verify the effectiveness of the proposed design.

The reduced-order observer can also be designed by using Theorem 3. According to (30), the reduced-order observer is given by

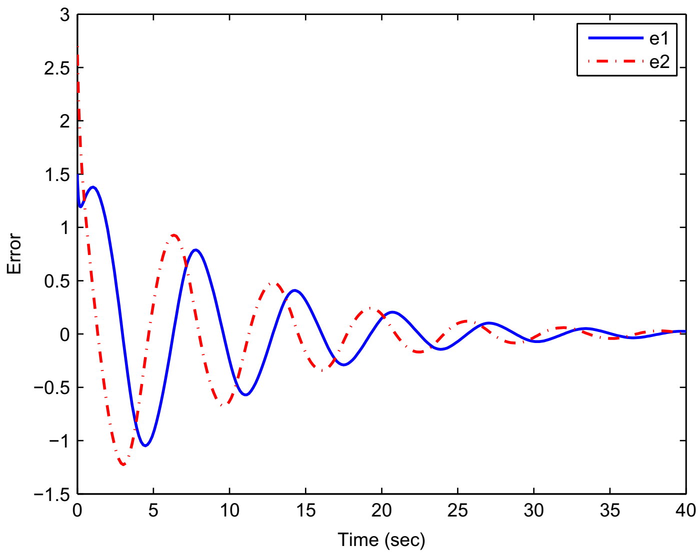

Fig. 3. The simulation for the error dynamics of the full-order observer (5).

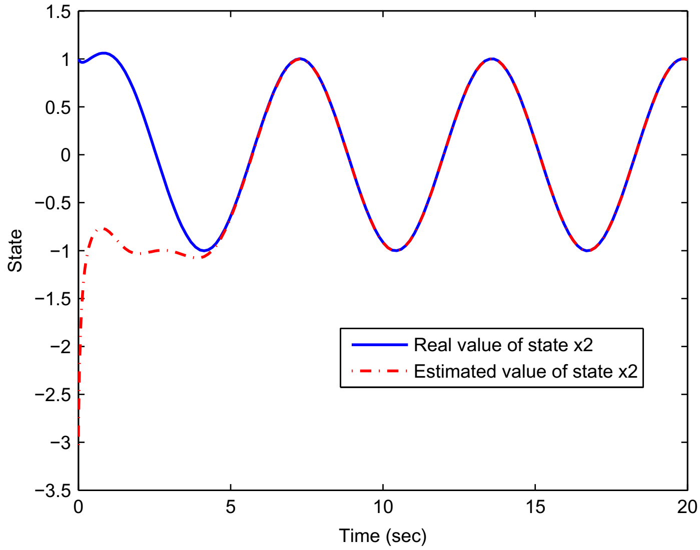

Fig. 4. The simulation for state x_2 by the reduce-order observer (46).

$$\begin{cases} \dot{\hat{z}}_2 = 0.981\hat{z}_2 - 1.0004y - \hat{z}_2\left[y^2 + (\hat{z}_2 + 0.019y)^2\right] \\ \hat{x}_1 = y \\ \hat{x}_2 = \hat{z}_2 + 0.019y \end{cases} \tag{46}$$

The simulation for state x_2 by the reduce-order observer (46) is presented in Fig. 4, where the initial conditions are $x(0) = (-1.5\ 1)^T$ and $\hat{x}_2(0) = -3$. From the figure, we know that the effect of state trajectory tracking is satisfactory.

6. Conclusions

We have addressed the design problems of full-order as well as reduced-order state observers for one-sided Lipschitz nonlinear systems by using a novel Riccati equation method. A Riccati-type condition is presented to ensure the existence of a full-order observer. The same condition also guarantees the existence of a reduce-order observer. It has been shown that the obtained conditions are less conservative than some existing ones in recent literature. A simulative example is included to illustrate the effectiveness of the proposed observers.

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