HOMEWORK # 4
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Homework Instructions:

1. Type your solutions in the LaTeX homework template file. Otherwise, you can use any other typesetting tool or you can provide handwritten solutions, assuming everything is clear.
2. **Due date:** Wednesday, September 30th, @ 5:00pm on Blackboard, **AND** drop off a copy of your solutions (slip it under the office door if I'm away).
3. **Collaboration policy:** you can collaborate with your classmates, under the assumption that everyone is required to write their own solutions. If you choose to collaborate with anyone, list their name(s).
4. You don't show your work ⇒ You don't get credit.
5. Solutions that are unclear won't be graded.
6. Before you start with this homework assignment, make sure that you have grasped the content of Module 04.

Problem 1 — Convexity Property

A function $f : \mathbb{R}^n \to R$ is continuously differentiable. Also, assume that $f(x)$ **is concave on a convex set** $\mathcal{X}$. Given the aforementioned properties of $f(x)$, prove that for all $x_1, x_2 \in \mathcal{X}$, $f(x)$ satisfies this property:

$$f(x_2) \leq f(x_1) + Df(x_1)(x_2 - x_1).$$

Hint: *Back to basics—what is the basic definition of a derivative?*

Problem 2 — Convexity of a Disc

Show that the set Ω given by $\Omega = \{y \in \mathbb{R}^2; \|y\|^2 \leq 4\}$ is convex, where $\|y\|^2 = y^\top y$.

Hint: Show that if $z = \beta x + (1-\beta)y$, then $\|z\|^2 \leq 4$. You might find the submultiplicative matrix-vector property to be useful too.

Problem 3 — Minimizing a Function

Given a multivariable function $f(x)$, many optimization solvers use the following algorithm to solve $\min_x f(x)$:

1. Choose an initial guess, $x^{(0)}$
2. Choose an initial real, symmetric positive definite matrix $H^{(0)}$
3. Compute $d^{(k)} = -H^{(k)}\nabla_x f(x^{(k)})$
4. Find $\beta^{(k)} = \arg\min_\beta f(x^{(k)} + \beta^{(k)} d^{(k)}),\ \beta \geq 0$
5. Compute $x^{(k+1)} = x^{(k)} + \beta^{(k)} d^{(k)}$

For this problem, we assume that the given function is a typical quadratic function ($x \in \mathbb{R}^n$), as follows:

$$f(x) = \frac{1}{2}x^\top Qx - x^\top b + c, \quad Q = Q^\top \succ 0.$$

Answer the following questions:

1. Find $f(x^{(k)} + \beta^{(k)}d^{(k)})$ for the given quadratic function.
2. Obtain $\nabla_x f(x^{(k)})$ for $f(x)$.
3. Using the chain rule, and given that $\beta^{(k)} = \arg\min_\beta f(x^{(k)} + \beta^{(k)}d^{(k)})$, find a closed form solution for $\beta^{(k)}$ in terms of the given matrices ($H^{(k)}, \nabla f(x^{(k)}), d^{(k)}, Q$).
4. Since it is required that $\beta^{(k)} \geq 0$, provide a sufficient condition related to $H^{(k)}$ that guarantees the aforementioned condition on $\beta^{(k)}$.

Problem 4 — KKT Conditions, 1

Using the KKT conditions discussed in class, obtain all the candidate strict local minima for the following nonlinear optimization problem:

$$\begin{aligned} \max \quad & -x_1^2 - 2x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \geq 3 \\ & x_2 - x_1^2 \geq 1 \end{aligned}$$

There are many cases to consider. Make sure that you don't miss any.

After solving the problem analytically, code the problem on `NEOS` solver (http://www.neos-server.org/neos/solvers/index.html), using any solver of your choice and any modeling language (GAMS, AMPL, ...). **Show your code and outputs.**

Problem 5 — KKT Conditions, 2

Using the KKT conditions discussed in class, obtain all the candidate strict local minima for the following nonlinear optimization problem:

$$\begin{aligned} \min \quad & x_1 + x_2^2 \\ \text{subject to} \quad & x_1 - x_2 = 5 \\ & x_1^2 + 9x_2^2 \leq 36 \end{aligned}$$

There are many cases to consider. Make sure that you don't miss any.

After solving the problem analytically, code the problem on `NEOS` solver, using any solver of your choice. **Show your code and outputs.**

Problem 6 — Convexity Range

For the following function, find the set of values for β such that the function is convex.

$$f(x, y, z) = x^2 + y^2 + 5z^2 - 2xz + 2\beta xy + 4yz$$

Problem 7 — CVX Programming

The objective of this problem is to get you started with CVX—the convex optimization solver on MATLAB. Do the following:

1. Watch this CVX introductory video: https://www.youtube.com/watch?v=N2b_B4TNfUM
2. Download and install CVX on your machine: http://cvxr.com/cvx/download/
3. Read the first few pages of the CVX User's Guide: http://web.cvxr.com/cvx/doc/
4. Solve Problems 4 and 5 using CVX. **Show your code and outputs.**

Problem 8 — Solving LMIs using CVX

Using CVX, solve the following LMI for P:

$$\begin{aligned} A^\top P + PA &< 0 \\ B^\top P + PB &< 0 \\ P = P^\top &> 0.1I, \text{ where:} \\ A &= \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \\ B &= \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Show your code and outputs.

What happens if you try to solve the same LMI when $B = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$?

Justify the results.