Homework Instructions:

1. Type your solutions in the LaTeX homework template file. Otherwise, you can use any other typesetting tool or you can provide handwritten solutions, assuming everything is clear.
2. **Due date:** Friday, October, 9th, @ 5:00pm on Blackboard, **AND** drop off a copy of your solutions (slip it under the office door if I'm away).
3. **Collaboration policy:** you can collaborate with your classmates, under the assumption that everyone is required to write their own solutions. If you choose to collaborate with anyone, list their name(s).
4. You don't show your work $\Rightarrow$ You don't get credit.
5. Solutions that are unclear won't be graded.
6. Before you start with this homework assignment, make sure that you have grasped the content of Modules 05 and 06.

Problem 1 — Cost of an Infinite Horizon LQR

Prove that the total cost of the CT, LTI infinite horizon LQR problem, given by:

$$\begin{aligned} \text{minimize } J \quad &= \quad \int_0^\infty \|y(t)\|^2 \, dt \\ \text{subject to} \quad & \quad \dot{x}(t) = Ax(t) \\ & \quad y(t) = Cx(t) \end{aligned}$$

is $J = x_0^\top P x_0$ where P is the solution to the steady-state Ricatti equation, given in Module 05, and $x(0)$ is the vector of initial state conditions.

Hint: Write the cost function as a quadratic cost function in terms of $x(t)$.

Solutions:

The cost J for the above optimal control problem can be written as:

$$J = \int_0^\infty \|y(t)\|^2 \, dt = \int_0^\infty y^\top(t) y(t) \, dt = \int_0^\infty x^\top(t) C^\top C x(t) \, dt = \int_0^\infty x^\top(t) Q x(t) \, dt.$$

For the infinite horizon, CT LTI LQR problem, CARE becomes:

$$A^\top P + PA = -Q.$$

The above Lyapunov equation can be written as

$$x^\top (A^\top P + PA) x = -x^\top Q x = -x^\top C^\top C x.$$

Notice that (derivative product rule)

$$\frac{d}{dt}(x^\top P x) = x^\top P \dot{x} + \dot{x}^\top P x = x^\top P A x + x^\top A^T P x = x^\top (A^\top P + PA) x.$$

Thus, the cost J can be written as

$$
\begin{aligned}
J &= \int_0^\infty x^\top(t) Q x(t)\, dt = -\int_0^\infty \frac{d}{dt}\left(x^\top(t) P x(t)\right) dt \\
&= -x^\top(t) P x(t)\Big|_0^\infty = -\underbrace{x^\top(\infty) P x(\infty)}_{=0} + x(0)^\top P x(0) = \boxed{x(0)^\top P x(0)}
\end{aligned}
$$

Problem 2 — Infinite Horizon LQR

Compute $J = \int_0^\infty x^\top \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} x \; dt$, given that the system dynamics are given by:

$$\dot{x}(t) = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} x(t),$$

where $x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. You are supposed to solve the problem analytically using two different methods of your choice (CARE is one of them). You are not supposed to use any programming tool. You should also use the result from Problem 1.

Verify your solutions using MATLAB. Show your code.

Solutions:

Method 1 — Lyapunov Equation

From the class lecture notes, $J = \int_0^\infty x^\top \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} x \; dt = x_0^\top P x_0$, where P is the matrix solution to the Lyapunov equation:

$$PA + A^\top P + Q = 0.$$

We can find $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ analytically by solving the above matrix equation for p_1, p_2 and p_3:

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}^\top \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} = 0.$$

Note that P is symmetric and should be positive definite. The solution to the above linear system of equation yields:

$$P = \begin{bmatrix} 3 & 2 \\ 2 & 3/2 \end{bmatrix} = P^\top \succ 0.$$

The performance index, J, is equal to $\boxed{x_0^\top P x_0 = 3/2.}$

Method 2 — Solution of ODE & Integration

Check student solutions for the solution using the ODE and integration.

Problem 3 — Two Point Boundary Value Problem

In this problem, we will learn about optimal control solutions for a two point boundary value problem (TPBVP) — an optimal control problem where terminal state conditions are pre-specified. You should do research on how to solve TPBVP with fixed final and initial states.

For example, you might find Example 6-1 in `http://goo.gl/CUIwPl` useful, as it includes an example on solving TPBVP. Also, read the ***LQR Variational Solution*** section in the linked PDF.

After reading the linked PDF and going through the example, find the optimal control trajectory, $u^*(t) = [u_1^*(t)\ u_2^*(t)]^\top$, that minimizes this performance index:

$$J = \frac{1}{2}\int_0^1 \|u(t)\|^2,$$

subject to:

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} u,\ x(0) = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix},\ x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solutions:

1. To solve this TPBVP, we start by forming the Hamiltonian:
$$H(x,u,\lambda) = \frac{1}{2}u^\top u + \lambda^\top(Ax + Bu).$$
2. The optimal control can be written as:
$$u^*(t) = -B^\top \lambda(t).$$
3. The co-state equation can be written as (property 2 of the value function):
$$\dot{\lambda}(t) = -A^\top \lambda(t)$$
Hence,
$$\lambda(t) = e^{-A^\top t}\lambda(0).$$
4. Therefore,
$$u^*(t) = -B^\top e^{-A^\top t}\lambda(0).$$
5. The only unknown in the optimal control is $\lambda(0)$. In what follows, we obtain $\lambda(0)$.
6. Next, we compute $e^{-A^\top t}$:
$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix}.$$
7. We already have a closed-form solution for the LTI system (Module 3):
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$
We use this solution to obtain $\lambda(0)$, as it's the only unknown.
8. For the given initial and terminal state conditions, we obtain:
$$x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = e^{A}x(0) + \int_0^1 e^{A(1-\tau)}Bu(\tau)d\tau = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2} \end{bmatrix}\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} -$$
$$\int_0^1 \begin{bmatrix} 1 & 0 \\ 0 & e^{-2(1-\tau)} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{2\tau} \end{bmatrix}\lambda(0)d\tau.$$

9. Again, the only unknown in the above equation is $\lambda(0)$. After some computations, we obtain:

$$\boxed{\lambda(0) = \begin{bmatrix} 1 \\ \dfrac{4}{1-e^4} \end{bmatrix}.}$$

10. Thus:

$$\boxed{u^*(t) = -B^\top e^{-A^\top t}\lambda(0) = \begin{bmatrix} \dfrac{2e^{2t}}{1-e^4} - \dfrac{1}{2} \\ -\dfrac{2e^{2t}}{1-e^4} - \dfrac{1}{2} \end{bmatrix}.}$$

Problem 4 — HJB Equation for a Nonlinear System

Using the HJB-equation, and a candidate quadratic function $V(t,x)$, find the optimal control action that minimizes this performance index:

$$J = (x(1))^2 + \int_0^1 x^2(t)u^2(t)\,dt,$$

where the dynamics of the system are given by:

$$\dot{x}(t) = x(t)u(t), \;\; x(0) = 1.$$

You'll have to follow the following steps:

1. Construct the Hamiltonian
2. Obtain the optimal control
3. Apply property 1 of any value function, i.e., $\frac{\partial V}{\partial x} = \lambda^*(x,t)$
4. Substitute this optimal control and multiplier into the HJB equation
5. Formulate a candidate quadratic value function
6. Use this candidate in the HJB and obtain an ODE that relates to the value function
7. Obtain the optimal control

Solutions:

1. Find the Hamiltonian:
$$\mathcal{H} = x^2(t)u^2(t) + \lambda(x,t)x(t)u(t)$$
2. Obtain the optimal control by minimizing $\mathcal{H}$ w.r.t. $u(t)$:
$$\lambda^*(x,t)x(t) + 2x^2(t)u^*(t) = 0 \;\;\Rightarrow\;\; u^*(t) = -\frac{1}{2x(t)}\lambda^*(x,t)$$
3. Recall property 1 of any value function $V(x,t)$:
$$\frac{\partial V}{\partial x} = \lambda^*(x,t) \;\;\Rightarrow\;\; u^*(t) = -\frac{1}{2x(t)}\frac{\partial V}{\partial x}$$
4. Substitute this optimal $u^*(t)$ and $\lambda^*(x,t)$ into the HJB equation ($-V_t^*(x,t) = \min_{u\in\mathcal{U}} \mathcal{H}(x,u,\lambda^*(x,t),t)$):
$$\begin{aligned}
-V_t^*(x,t) &= \min_{u\in\mathcal{U}} \mathcal{H}(x,u,\lambda^*(x,t),t) \\
&= x^2(t)\left(-\frac{1}{2x(t)}\frac{\partial V}{\partial x}\right)^2 + \frac{\partial V}{\partial x}x(t)\left(-\frac{1}{2x(t)}\frac{\partial V}{\partial x}\right) \\
&= +\frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 \\
-V_t^*(x,t) &= -\frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 \\
-\frac{\partial V}{\partial t} &= -\frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2
\end{aligned}$$
5. Note that the boundary condition for the value function is: $V(t_f,x) = V(1,x) = x^2(t)$.

6. Consider a candidate quadratic value function: $V(t,x) = p(t)x^2(t)$.

7. Using this candidate in the LHS of HJB yields:

$$-V_t^*(x,t) = -\frac{\partial V}{\partial t} = -\dot{p}(t)x^2(t)$$

8. Using $V(t,x)$ in the RHB of HJB yields:

$$\frac{\partial V}{\partial x} = 2x(t)p(t)$$

9. Equating the two together produces:

$$-\dot{p}(t)x^2(t) = -\frac{1}{4}(2x(t)p(t))^2 \;\; \Rightarrow \;\; -\dot{p}(t)x^2(t) = -p^2(t)x^2(t)$$

10. Therefore, we have this ODE:

$$\dot{p}(t) = p^2(t), \;\; p(1) = 1.$$

11. The solution to this ODE is:

$$p(t) = \frac{1}{2-t}$$

12. Therefore,

$$\boxed{u^*(t) = -\frac{1}{2x(t)}\frac{\partial V}{\partial x} = -\frac{1}{2-t}}$$

Problem 5 — Discrete LQR Solution + Lagrangian

The discrete dynamics of an LTI system is given by:

$$x_{k+1} = Ax_k + Bu_k, \ \ k = 0, 1, \ldots, N-1.$$

Consider the optimal control problem of finding optimal control sequence, $u_0^*, \ldots, u_{N-1}^*$, given:

- Specific initial and final conditions: x_0 and x_N are fixed, and
- Cost index: $J = \dfrac{1}{2} \displaystyle\sum_{k=0}^{N-1} u_k^\top R u_k$

The objective of this problem is to transform the optimal control problem to a quadratic, static optimization problem subject to linear equality constraints.

Answer the following questions:

1. Write the cost function J as a quadratic cost function in $u = \begin{bmatrix} u_0 & u_1 & \ldots & u_{N-1} \end{bmatrix}^\top$, where you should determine the quadratic cost-matrix—it should be diagonal.
2. Given an initial and final fixed states, write the dynamics of the system as $A_u u = b_u$, where A_u and b_u should be determined in terms of A, B, x_0, x_{N-1}.
3. Formulate the optimal control problem as a quadratic program with linear equality constraints.
4. Construct the Lagrangian of the transformed optimization problem.
5. What is the optimal u^*? You have to solve a KKT-like problem for multipliers and control.

Solutions:

1. $J = \dfrac{1}{2} u^\top L u$, $L = \text{diag}(R, R, \ldots, R)$
2. From the given discrete dynamics, we have:

   $$\begin{aligned} x_1 &= Ax_0 + Bu_0 \\ x_2 &= Ax_1 + Bu_1 = A(Ax_0 + Bu_0) + Bu_1 = A^2 x_0 + ABu_0 + Bu_1 \\ x_N &= A^N x_0 + A^{N-1} Bu_0 + \ldots + Bu_{N-1}. \end{aligned}$$

   Rearranging the terms above, we obtain:

   $$A_u u = b_u, \ \text{ where } \ A_u = \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix}, b_u = x_N - A^N x_0$$

3. Problem can be posed as:

   $$\begin{aligned} \underset{u}{\text{minimize}} \quad & \frac{1}{2} u^\top L u \\ \text{subject to} \quad & A_u u = b_u \end{aligned}$$

4. Lagrangian:

   $$L(u, \lambda) = \frac{1}{2} u^\top L u + \lambda (A_u u - b_u)$$

5. Setting $\frac{\partial L}{\partial u} = 0$, we obtain $u = -L^{-1} A_u^\top \lambda$. We can plug this equality in $A_u u = b_u$ and obtain $\lambda = -\left(A_u L^{-1} A_u^\top\right)^{-1} b_u$. Thus, $\boxed{u^* = L^{-1} A_u^\top \left(A_u L^{-1} A_u^\top\right)^{-1} b_u.}$

Problem 6 — Principle of Optimality and DP

Consider the following discrete-time LQR problem:

$$\begin{aligned}
\text{minimize} \quad J &= (x_2 - 10)^2 + \frac{1}{2}\sum_{k=0}^{1}(x_k^2 + u_k^2) \\
\text{subject to} \quad x_{k+1} &= 2x_k - 3u_k \\
x(0) &= 4.
\end{aligned}$$

The final state can be anything, i.e., it is free, not fixed. The objective of this problem is to solve the above optimal control problem by invoking the Principle of Optimality and Dynamic Programming.

You should solve this problem using two different methods. Find the solution to the optimal control sequence and corresponding state-trajectories by applying dynamic programming (similar to the example on Slide 13 of Module 05)

Solutions:

Check student solutions. Solution is identical to the example in Module 05.