QUIZ # 10 SOLUTIONS

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A function $\phi(x, u)$ is Globally Lipschitz (*Lipschitz Continuous*) with Lipschitz constant L if and only if:

$$\|\boldsymbol{\phi}(x_1, u) - \boldsymbol{\phi}(x_2, u)\| \leq L\|x_1 - x_2\|, \quad L \geq 0.$$

Find the Lipschitz constant for the following functions:

1. $\phi(x) = x^4$, if $x \in [-2, 2]$. You will have to use the triangular inequality.

   Hint 1: $b^4 - a^4 = (b-a)(b^3 + b^2a + ba^2 + a^3)$

   Solutions:

   Applying the definition:

   $$|f(x_2) - f(x_1)| = |x_2^4 - x_1^4|.$$

   Applying the hint, we obtain:

   $$|f(x_2) - f(x_1)| = |x_2 - x_1||x_2^3 + x_2^2x_1 + x_2x_1^2 + x_1^3| \leq .$$

   Note that,

   $$|x_2^3 + x_2^2x_1 + x_2x_1^2 + x_1^3| \leq |x_2|^3 + |x_2|^2|x_1| + |x_2||x_1|^2 + |x_1|^3 \leq 2^3 + 2^2\,2 + 2\,2^2 + 2^3 = 32.$$

   Therefore,

   $$\boxed{\|\phi(x_1, u) - \phi(x_2, u)\| \leq 32\|x_1 - x_2\|.}$$

2. $\phi(y,x) = \sqrt{y^2+x^2}$, with $x \in [-1,1]$. You should apply the definition on y here.

Hint 2: You will have to multiply by a fraction that allows you to use

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a^2 - b^2$$

Hint 3: Also, don't forget that $|a^2-b^2| = |a-b||a+b|$.

Solutions: We apply the definition and the hints to obtain:

$$\left|\sqrt{y_1^2+x^2}-\sqrt{y_2^2+x^2}\right| = \left|\sqrt{y_1^2+x^2}-\sqrt{y_2^2+x^2}\right| \frac{\left|\sqrt{y_1^2+x^2}+\sqrt{y_2^2+x^2}\right|}{\left|\sqrt{y_1^2+x^2}+\sqrt{y_2^2+x^2}\right|} = \frac{|y_1^2-y_2^2|}{\sqrt{y_1^2+x^2}+\sqrt{y_2^2+x^2}}$$

Applying Hint 3, we get:

$$\left|\sqrt{y_1^2+x^2}-\sqrt{y_2^2+x^2}\right| = \frac{|y_1+y_2|\,|y_1-y_2|}{\sqrt{y_1^2+x^2}+\sqrt{y_2^2+x^2}} \leq \frac{|y_1+y_2|\,|y_1-y_2|}{\sqrt{y_1^2}+\sqrt{y_2^2}} \leq |y_1-y_2|\,,$$

since $\max(x^2) = 1$. Thus $L = 1$.