Solutions of Quiz #1

THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5243
INTRODUCTION TO CYBER-PHYSICAL SYSTEMS

QUIZ # 1
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Given the following LTI dynamical system:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t), \quad \mathbf{x}_{\text{initial}} = \mathbf{x}_{t_0} \tag{1}$$

where:

- $\mathbf{x}(t)$: dynamic state-vector of the LTI system, $\mathbf{u}(t)$: control input-vector
- $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are constant matrices.

The closed-form to the above differential equation for any time-varying control input is given by:

$$\mathbf{x}(t) = e^{\boldsymbol{A}(t-t_0)}\mathbf{x}_{t_0} + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\mathbf{u}(\tau)\,d\tau. \quad \ldots \quad (*)$$

Show that the above solution is in fact a solution to the system dynamics in (1).
Hint — Leibniz Differentiation Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,\mathrm{d}x\right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x,\theta)\,\mathrm{d}x + f(b(\theta),\theta)\cdot b'(\theta) - f(a(\theta),\theta)\cdot a'(\theta)$$

Your Solution: a) Start with the initial conditions:

$x_{initial} = x_{t_0} \rightarrow$ at $t = t_0$, we have from $(*)$:

$$x(t_0) \stackrel{?}{=} \left. e^{A(t-t_0)} x_{t_0} \right]_{t=t_0} + \underbrace{\int_{t_0}^{t_0} e^{A(t-\tau)} Bu(\tau)\, d\tau}_{=0}$$

$$\rightarrow x(t_0) \stackrel{?}{=} e^{A(0)} x_{t_0} \longrightarrow \boxed{x_{t_0} = x_{t_0}} \rightarrow \text{ICs are satisfied.}$$

b) Given $x(t) = e^{A(t-t_0)} x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau \quad (*)$

$\rightarrow$ Apply Leibniz Differentiation Theorem: $\rightarrow$

$$\dot{x}(t) = A e^{A(t-t_0)} x_{t_0}^{1} + \left. e^{A(t-\tau)} Bu(\tau) \right]_{\tau=t} + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$\Rightarrow \quad \dot{x}(t) = Ae^{A(t-t_0)} x_{t_0} + Bu(t) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\,d\tau$$

$$\Rightarrow \quad \dot{x}(t) = A\left[x(t)\right] + Bu(t)$$

$\therefore$ Solution in $(*)$ satisfies (1)

c) By uniqueness theorem, $x(t)$ is the solution of the ODE given in (1).