Solutions of Quiz #2

THE UNIVERSITY OF TEXAS AT SAN ANTONIO
EE 5243
INTRODUCTION TO CYBER-PHYSICAL SYSTEMS

QUIZ # 2
Ahmad F. Taha
August 31, 2015

Given the following LTI dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x_{\text{initial}} = x_{t_0} \tag{1}$$
$$y(t) = Cx(t) \tag{2}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x(t_0) = x(1) = [0\ 1\ 1]^\top.$$

Recall that the closed-form to the above differential equation for any time-varying control input is given by:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

1. Is A nilpotent of order 2? (i.e., is $A^2 = 0$?)

Yes. $A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. Determine $e^{At}, e^{A(t-\tau)}, e^{A(t-t_0)}$, $t_0 = 1$. Recall that

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \frac{(At)^4}{4!} + \cdots$$

$$e^{At} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} t & -t & 0 \\ t & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1+t & -t & 0 \\ t & 1-t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{A(t-\tau)} = \begin{bmatrix} 1+t-\tau & -t+\tau & 0 \\ t-\tau & 1-t+\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{A(t-1)} = \begin{bmatrix} t & -t+1 & 0 \\ t-1 & 2-t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. If $u(t) = 0$, determine $x(t)$ (or the zero-input state state-response) given the provided initial conditions.

$$x(t) = e^{A(t-t_0)} x(t_0) = e^{A(t-1)} x(1)$$

$$= \begin{bmatrix} -t+1 \\ 2-t \\ 1 \end{bmatrix}$$

4. If $x(t_0) = x(1) = 0$ and $u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^+(t)$, determine $x(t)$ (or the zero-state, state-response).

$$x(t) = \int_1^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$= \int_1^t \begin{bmatrix} 1+t-\tau & -t+\tau & 0 \\ t-\tau & 1-t+\tau & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau$$

$$= \int_1^t \begin{bmatrix} 1+t-\tau \\ t-\tau \\ 1 \end{bmatrix} d\tau = \left. \begin{bmatrix} \tau + t\tau - \frac{\tau^2}{2} \\ t\tau - \frac{\tau^2}{2} \\ \tau \end{bmatrix} \right|_1^t$$

$$= \begin{bmatrix} \frac{t^2}{2} - \frac{1}{2} \\ \frac{t^2}{2} - t + \frac{1}{2} \\ t-1 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

5. Determine $y(t)$ if $u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^+(t)$ and $x(t_0) = x(1) = [0\ 1\ 1]^\top$.

$y(t) = C\,x(t)$ $\qquad$ C = given

$x(t)$ computed before

$$\rightarrow y(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\underbrace{\begin{bmatrix} -t+1 \\ 2-t \\ 1 \end{bmatrix}}_{\text{zero-input state-response}} + \underbrace{\begin{bmatrix} \frac{t^2}{2} - \frac{1}{2} \\ \frac{t^2}{2} - t + \frac{1}{2} \\ +t - 1 \end{bmatrix}}_{\text{zero-state state-response}} \right)$$

$$\rightarrow \underbrace{y(t) = \begin{bmatrix} \frac{(t-1)^2}{2} \\ t-1 \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_{\text{Complete Response}}$$

* Verify answers on MATLAB