QUIZ # 4 SOLUTION

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Prove that the quadratic cost function given by

$$f(x) = x^\top Q x, \;\; Q = Q^\top \succeq 0,$$

is convex.

Solution: A function $f(x)$ is convex if:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

for all $0 \leq \alpha \leq 1$. Given that $f(x) = x^\top Q x$, we apply the definition of convex function. The condition can be written as:

$$\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha)y) \geq 0.$$

Substituting for $f(x)$ into the LHS of the previous equation yields:

$$\alpha x^\top Q x + (1-\alpha) y^\top Q y - (\alpha x + (1-\alpha)y)^\top Q (\alpha x + (1-\alpha) y)$$
$$= \alpha(1-\alpha) x^\top Q x - 2\alpha(1-\alpha) x^\top Q y + \alpha(1-\alpha) y^\top Q y = \alpha(1-\alpha)(x-y)^\top Q (x-y).$$

Define $z = x - y$. We then obtain the following quadratic form:

$$\alpha(1-\alpha) z^\top Q z.$$

Since $0 \leq \alpha \leq 1$, $Q = Q^\top \succeq$, and for any z,

$$\alpha(1-\alpha) z^\top Q z \geq 0,$$

hence, the convexity definition of a function is satisfied.