QUIZ # 7
Ahmad F. Taha
September 28, 2015

- Hamiltonian:
$$\mathcal{H}(x,u,\lambda^*(x,t),t) = g(x,u,t) + \lambda^*(x,t)f(x,u,t)$$
- Value function properties:
  1. $V_x(x,t) = \frac{\partial V}{\partial x} = \lambda^*(x,t)$
  2. $-V_t(x,t) = -\frac{\partial V}{\partial t} = \min_{u\in\mathcal{U}} \mathcal{H}(x,u,\lambda^*(x,t),t) = \left(\frac{\partial \mathcal{H}}{\partial x}\right)^\top$
- The **HJB Equation:**

$$\boxed{-V_t^*(x,t) = -\frac{\partial V}{\partial t} = \min_{u\in\mathcal{U}} \mathcal{H}(x,u,\lambda^*(x,t),t) = \left(\frac{\partial \mathcal{H}}{\partial x}\right)^\top}$$

For this optimal control problem,

$$\begin{array}{lcl} \text{minimize } J & = & \frac{1}{2}x_{t_f}^\top H x_{t_f} + \frac{1}{2}\int_{t_0}^{t_f}\left[x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t)\right]\,dt \\ \text{subject to} & & \dot{x}(t) = A(t)x(t) + B(t)u(t), \end{array}$$

answer the following questions:

1. Construct the **Hamiltonian**.
2. Find the optimal $u^*(t)$ in terms of $\lambda^*(x,t)$.
3. Write the **Hamiltonian** in terms of $u^*(t)$.
4. Apply the value function properties (above) for this candidate value function:
$$V^*(x,t) = \frac{1}{2}x^\top(t)P(t)x(t),\ P(t) = P^\top(t)$$
5. Based on the given, derive the Differential Riccati Equation that relates $\dot{P}(t)$ with $P(t)$, and explain how can $u^*(t)$ be obtained.

Solutions:

1. $\mathcal{H}(x,u,\lambda^*(x,t),t) = g(x,u,t) + \lambda^*(x,t)f(x,u,t)$
$$= \frac{1}{2}\left[x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t)\right] + \lambda^*(x,t)\left[A(t)x(t) + B(t)u(t)\right]$$
2. Minimum of $\mathcal{H}$ w.r.t. u:
$$\frac{\partial \mathcal{H}}{\partial u} = u(t)^\top R(t) + \lambda^*(x,t)B(t) = 0 \Rightarrow \boxed{u^*(t) = -R^{-1}(t)B(t)^\top\lambda^*(x,t)^\top}$$
Note that $\frac{\partial^2 \mathcal{H}}{\partial u^2} = R(t) \succ 0$

3. The **Hamiltonian** in terms of $u^*(t)$: $\mathcal{H}(x, u, \lambda^*(x,t), t) =$

$$\frac{1}{2}\left[x(t)^\top Q(t)x(t) + \left(R^{-1}(t)B(t)^\top \lambda^*(x,t)^\top\right)^\top R(t)\left(R^{-1}(t)B(t)^\top \lambda^*(x,t)^\top\right)\right]$$

$$+\lambda^*(x,t)\left[A(t)x(t) + B(t)R^{-1}(t)B(t)^\top \lambda^*(x,t)^\top\right]$$

$$= \frac{1}{2}x(t)^\top Q(t)x(t) + \lambda^*(x,t)A(t)x(t) - \frac{1}{2}\lambda^*(x,t)B(t)R^{-1}(t)B^\top(t)\lambda^*(x,t)^\top \quad (*)$$

4. Properties of VF (see previous slides):
   (a) $V_x^*(x,t) = \lambda^*(x,t) = x^\top(t)P(t)$
   (b) $V_t^* = \frac{1}{2}x^\top(t)\dot{P}(t)x(t) = -\min_{u\in\mathcal{U}} \mathcal{H}(x, u, \lambda^*(x,t), t) = -(*)$

5. Substitute $\lambda^*(x,t)$ into $(*)$:

$$= \frac{1}{2}x(t)^\top Q(t)x(t) + x^\top(t)P(t)A(t)x(t) - \frac{1}{2}x^\top(t)P(t)B(t)R^{-1}(t)B(t)^\top P(t)x(t)$$

$$= \frac{1}{2}x(t)^\top\left(Q(t) + P(t)A(t) + A^\top(t)P(t) - P(t)B(t)R^{-1}(t)B(t)^\top P(t)\right)x(t) \quad (**)$$

   (a) But $-V_t^*(x,t) = (*) = (**) = -\frac{1}{2}x^\top(t)\dot{P}(t)x(t)$
   (b) Hence, for $V^*(x,t) = \frac{1}{2}x^\top(t)P(t)x(t)$ to be an optimal VF, we require:
   (c) $\boxed{-\dot{P}(t) = Q(t) + P(t)A(t) + A^\top(t)P(t) - P(t)B(t)R^{-1}(t)B(t)^\top P(t)}$
   (d) Recall that $u^*(t) = -R^{-1}(t)B(t)^\top \lambda^*(x,t)^\top = -\underbrace{R^{-1}(t)B(t)^\top P(t)}_{=F(t)}x(t)$