QUIZ # 8 SOLUTIONS

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Solve the following problems, given the augmented MPC dynamics,

$$\begin{aligned} x_a(k+1) &= \Phi_a x_a(k) + \Gamma_a \Delta u(k) \\ y(k) &= C_a x_a(k),\ x_a \in \mathbb{R}^{n+p}, \Gamma_a \in \mathbb{R}^{n+p\times m}, C_a \in \mathbb{R}^{p\times n+p}. \end{aligned}$$

1. For a predicted horizon N_p, derive an equation that relates the predicted outputs to $x_a(k)$ and the MPC variables Δu. That is, derive matrices W, Z in this equation:

$$Y = \begin{bmatrix} y(k+1|k) \\ y(k+2|k) \\ \cdots \\ y(k+N_p|k) \end{bmatrix} = W x_a(k) + Z \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix} = W x_a(k) + Z\Delta U.$$

2. Derive the optimal ΔU^* if the given cost function is (without constraints):

$$J(\Delta U) = \frac{1}{2}(r-Y)^\top Q(r-Y) + \frac{1}{2}\Delta U^\top R \Delta U,\ \ Q = Q^\top \succ 0, R = R^T \succ 0.$$

3. Suppose that you're given the following constraints on the rate of change of the control action:

$$u^{\min} \leq \Delta U \leq u^{\max}.$$

Write the corresponding optimization problem in the following form:

$$\begin{aligned} &\text{minimize} && J(\Delta U) \\ &\text{subject to} && g(\Delta U) \leq 0\ , \end{aligned}$$

where $g(\Delta U)$ is a linear set of constraints.

Solutions:

1. Problem 1—

   - We can write the **predicted future state variables as**:

$$
\begin{aligned}
x_a(k+1|k) &= \Phi_a x_a(k) + \Gamma_a \Delta u(k) \\
x_a(k+2|k) &= \Phi_a x_a(k+1|k) + \Gamma_a \Delta u(k+1) = \Phi_a^2 x_a(k) + \Phi_a \Gamma_a \Delta u(k) + \Gamma_a \Delta u(k+1) \\
\dots &= \dots \\
x_a(k+N_p|k) &= \Phi_a^{N_p} x_a(k) + \Phi_a^{N_p-1}\Gamma_a \Delta u(k) + \dots + \Gamma_a \Delta u(k+N_p-1)
\end{aligned}
$$

   - Also, we can write the predicted outputs as:

$$
\underbrace{C_a \begin{bmatrix} x_a(k+1|k) \\ x_a(k+2|k) \\ \vdots \\ x_a(k+N_p|k) \end{bmatrix}}_{Y} = \underbrace{C_a \begin{bmatrix} \Phi_a \\ \Phi_a^2 \\ \vdots \\ \Phi_a^{N_p} \end{bmatrix}}_{W} x_a(k) + \underbrace{C_a \begin{bmatrix} \Gamma_a & & & \\ \Phi_a \Gamma_a & \Gamma_a & & \\ \vdots & & \ddots & \\ \Phi_a^{N_p-1}\Gamma_a & \dots & \Phi_a\Gamma_a & \Gamma_a \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix}}_{\Delta U}
$$

   - Hence, we obtain:

$$
\boxed{Y = \begin{bmatrix} y^\top(k+1|k) & y^\top(k+2|k) & \dots & y^\top(k+N_p|k) \end{bmatrix}^\top = W x_a(k) + Z \Delta U}
$$

2. Problem 2—

   - This is an unconstrained optimization problem $\Rightarrow$ it's easy to find ΔU^*
   - Setting $\dfrac{\partial J}{\partial \Delta U} = 0 \Rightarrow \boxed{\Delta U^* = (R + Z^\top Q Z)^{-1} Z^\top Q (r - W x_a)}$
   - Note that SONC are satisfied as $\dfrac{\partial^2 J}{\partial \Delta U^2} = R + Z^\top Q Z \succ 0$

3. Problem 3—

   - If $\Delta u^{\min} \leq \Delta u(k) \leq \Delta u^{\max}$, then:

$$
\begin{bmatrix} -I_m \\ I_m \end{bmatrix} \Delta u(k) \leq \begin{bmatrix} -\Delta u^{\min} \\ \Delta u^{\max} \end{bmatrix}
$$

   - For a prediction horizon N_p, we have:

$$
\begin{bmatrix}
-I_m & O & \dots & O & O \\
I_m & O & \dots & O & O \\
O & -I_m & \dots & O & O \\
O & I_m & \dots & O & O \\
\vdots & & & & \vdots \\
O & O & \dots & O & -I_m \\
O & O & \dots & O & I_m
\end{bmatrix}
\underbrace{\begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix}}_{\Delta U}
\leq
\begin{bmatrix} -\Delta u^{\min} \\ \Delta u^{\max} \\ -\Delta u^{\min} \\ \Delta u^{\max} \\ \vdots \\ -\Delta u^{\min} \\ \Delta u^{\max} \end{bmatrix}
$$

- Hence, the problem can be written as:

$$
\begin{aligned}
\text{minimize} \quad & J(\Delta U) = \frac{1}{2}(r - Y)^\top Q (r - Y) + \frac{1}{2}\Delta U^\top R \Delta U \\
\text{subject to} \quad & g(\Delta U) = \begin{bmatrix} -I_m & O & \dots & O & O \\ I_m & O & \dots & O & O \\ O & -I_m & \dots & O & O \\ O & I_m & \dots & O & O \\ \vdots & & & & \vdots \\ O & O & \dots & O & -I_m \\ O & O & \dots & O & I_m \end{bmatrix} \underbrace{\begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \Delta u(k+N_p-1) \end{bmatrix}}_{\Delta U} - \begin{bmatrix} -\Delta u^{\min} \\ \Delta u^{\max} \\ -\Delta u^{\min} \\ \Delta u^{\max} \\ \vdots \\ -\Delta u^{\min} \\ \Delta u^{\max} \end{bmatrix} \leq 0 \, ,
\end{aligned}
$$